

MATH 658, Spring 2000 - The topology algebraic varieties

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Norman Steenrod once said that a topologist is “*a person who sees no difference between a donut and a mug*”. For this course, the fact that the surface of a donut happens to be an algebraic variety (elliptic curve to be precise) will be particularly relevant.

In general, the algebraic varieties are sets of solutions of *polynomial* equations and their structure has many special features. I will be interested mainly in their topological properties. The class of algebraic varieties may seem small, special and thus uninteresting. By the end of this course I hope to convince you of the contrary, and present a few interesting features of these special topological spaces.¹ The course will consist of three parts.

1. **Picard-Lefschetz theory**
2. **Singularities of curves and surfaces and their resolutions.**
3. **The Milnor fiber, the link and the vanishing cohomology of a singularity.**

The *Picard-Lefschetz theory* is the complex analytic counterpart of the versatile Morse theory. Historically, Picard-Lefschetz theory came first, at the dawn of the 20th century, and had a dramatic impact on the development of modern algebraic topology. The recent years have witnessed an increased the interest in this old theory due mainly to the surprising results of S. Donaldson on the topology of symplectic manifolds.

Roughly speaking, Morse theory describes a procedure of reconstructing a smooth manifold M assuming we know a smooth function $f : M \rightarrow \mathbb{R}$ and its behavior near its critical points. Picard-Lefschetz theory proposes a general reconstruction process for a *complex analytic* manifold M knowing a *holomorphic* function $f : M \rightarrow \mathbb{C}$ and its behavior near its critical points. Picard-Lefschetz theory requires that all critical points of $f : M \rightarrow \mathbb{C}$ are nondegenerate, just as Morse theory does. For this part of the course I plan to follow closely the modern approach in [?].

Naturally, one can ask what happens if we relax the above requirement and allow *degenerate* critical points as well. In this case one is lead to the study of singularities of hypersurfaces

$$X = P^{-1}(0) \subset \mathbb{C}^n$$

where $P = P(z_1, \dots, z_n)$ is a polynomial in n -complex variables such that $0 \in X$ is a singular point. I will discuss two methods for studying the singularities of such hypersurfaces. Both are based on the same principle: investigate the singular object X by “approximating” it with a smooth one.

The first “approximation” technique is the so called *resolution of singularities* and I will discuss it only in low dimensions $n = 2, 3$. The case $n = 2$ leads to beautiful connections with *knot theory*, while the case $n = 3$ leads to interesting and important examples of 3-manifolds. My presentation will be inspired from [?, ?].

Another, simpler way of approximating X is by deforming the defining equation $P = 0$ to $P = \varepsilon$. This process is called *smoothing* and the resulting hypersurface

$$X_\varepsilon := \{\vec{z} \in \mathbb{C}^n; P(\vec{z}) = \varepsilon\}$$

¹Even if you don't plan to attend this class you ought to read the captivating brief historical survey “*Singularities*” by Alan Durfee, xxx-Archive math.AG/9801123, to learn about some exciting piece of mathematics.

is called the *Milnor fiber* of the singularity. The odd-dimensional manifold $N_\varepsilon := X\{|z| = \varepsilon\}$ is called the *link* of the singularity. When $n = 2$, so that X is a complex curve, the link is a one-dimensional submanifold of the sphere S^3 , a.k.a. *knot*. The knots obtained as links of singularities are called *algebraic*.

The topologies of the Milnor fiber and the link of a singularity are nontrivially constrained by the *complex geometry* of the singularity. For example, J. Milnor has shown that amongst these links one can detect manifolds which “talk like a sphere, walk like a sphere” but they are not quite spheres: they are *homeomorphic* but *not diffeomorphic* to spheres.

As $\varepsilon \rightarrow 0$, some of the cycles of X_ε collapse. These are called *vanishing cycles* and the subgroup they generate is called *the vanishing (co)homology* of the singularity. My presentation will be inspired from [?, ?, ?]. If time permits, I plan to say a few words about the connection between the asymptotics of certain oscillatory integrals and vanishing cohomology.

To enjoy this class you need to be familiar with basic facts of algebraic topology, especially singular (co)homology and *very little* analysis. I believe the things I will talk about will be of interest to topologists, algebraic and/or differential geometers alike. The algebraic varieties represent an excellent testing ground for many topological questions and ought to belong to the bag of examples of any topologist or geometer.

The class will be example-oriented because, as E.C. Zeeman once put it, “a good example is worth ten theorems”. This is quite an eclectic subject and there will be many special interesting things I won’t be able to cover in class. That is why, part of the requirement of this class will be to present a one hour informal talk on a subject of your choice from a list I will submit in class. In compiling this list I will try to satisfy all the backgrounds in class. E.g. a student pursuing a degree in algebraic geometry will be able to choose an elementary topic in this field which will be relevant to all of us in class.

Notes for this course are available (as PDF files) from my homepage

<http://www.nd.edu/~lnicolae/>

References

- [1] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko: *Singularities of Differentiable Maps. Volume II*, Monographs in Mathematics, vol. 83, Birkhäuser, 1988.
- [2] E. Brieskorn, H. Knörrer: *Plane Algebraic Curves*, Birkhäuser, 1986.
- [3] A. Dimca: *Singularities and Topology of Hypersurfaces*, Universitext, Springer Verlag, 1992.
- [4] K. Lamotke: *The topology of complex projective varieties after S. Lefschetz*, Topology, **20**(1981), 15-51.
- [5] H.B. Laufer: *Normal Two-Dimensional Singularities*, Annals of Mathematics Studies, vol. **71**, Princeton University Press, 1971.
- [6] J. Milnor: *Singular Points of Complex Hypersurfaces*, Annals of Math. Studies, vol. 61, Princeton Univ. Press, 1968.

TENTATIVE LIST OF TOPICS FOR MATH 658

I. Complex manifolds -Definitions and Examples

1. The projective spaces.
2. Complex submanifolds: implicit function theorem, critical and regular points, critical values.
3. Algebraic manifolds.
4. "Baby" Lefschetz theory: Riemann-Hurwitz theorem and the Plücker formula.
5. The tautological line bundle and the blow-up construction

II. Lefschetz theory

1. Linear systems, pencils.
2. Lefschetz pencils, examples.
3. Vanishing cycles. Weak and Hard Lefschetz theorems.
4. Morse lemma and Picard-Lefschetz formulæ
5. Deligne's interpretation of the Hard Lefschetz theorem.

III. Invariants of critical points

1. Mather-Tougeron sufficiency theorem.
2. Milnor number
3. Morsification and Milnor fibrations
4. Monodromy.
5. Resolutions of singularity (curves and surfaces only)