Because there are only very few terms in the sequence and only finitely many values, there must exist an \( i \) such that

\[ a_i = x_n \quad \text{for only many } n. \]

Let \( \varepsilon > 0 \)

Then \( |a_i - x_n| = 0 < \varepsilon \) for only many \( n \).

\[ \Rightarrow a_i \text{ is a cluster point of } \{x_n\} \]

6.2 (2)\]

(a) Note that \(-1 \leq \cos (a_n) \leq 1\)

\[ \Rightarrow 0 \leq \cos^2 (a_n) \leq 1 \]

\[ \Rightarrow \{\cos^2 (a_n)\} \text{ bounded} \]

\[ \Rightarrow b_n \text{ has a convergent subsequence} \]
(b) \( b_n \) does not always have a convergent subsequence.

For example, take \( a_n = -1 + \frac{1}{n} \)

then \( \frac{a_n}{1+a_n} = \frac{-1 + \frac{1}{n}}{\frac{1}{n}} = -n + 1 \)

The sequence \( \{-n+1\} \) has no convergent subsequences.

(c) Since \( 0 \leq |a_n| \)

\[ \Rightarrow 1 \leq |1 + |a_n|| \]

\[ \Rightarrow \frac{1}{|1 + |a_n||} \leq 1 \quad \text{also} \quad \frac{1}{1 + |a_n|} \geq 0 \]

So \( \{b_n\} \) is bounded.

\[ \Rightarrow b_n \text{ has a convergent subsequence.} \]
(6.4(1))
Suppose \( a_n \to L \).

we wish to show \( \{a_n\} \) is Cauchy.

Let \( \varepsilon > 0 \).

Since \( a_n \to L \),

\[ a_n \approx \frac{\varepsilon}{2} L \quad \text{for} \quad n \to 0 \]

in particular, for \( n, m \to 0 \)

\[ a_n \approx \frac{\varepsilon}{2} L \]
\[ a_m \approx \frac{\varepsilon}{2} L \]

\[ \Rightarrow a_n \approx \frac{\varepsilon}{2} a_m \quad \text{for} \quad n, m \to 0. \]

---

(6.5(1))

(a) \( \sup = 1 \quad \max = 1 \)
    \[ \inf = -1 \quad \min = -1 \]

(b) \( \sup = \frac{1}{2} \quad \max = \frac{1}{2} \)
    \[ \inf = -1 \quad \min = -1 \]
    \[ -1, \frac{1}{2}, -\frac{3}{4} \]
(c) \( \sup = \frac{5}{4} \quad \max = \frac{5}{4} \)

\begin{align*}
\inf = -1 & \quad \min = \text{d.n.e.} \\
1 + 0 & \\
\frac{1}{2} - 1 & \\
\frac{1}{3} + 0 & \\
\frac{1}{4} + 1 & \\
\frac{1}{5} + 0 & \\
\frac{1}{6} - 1 &
\end{align*}

(d) \( \sup = \frac{1}{2} \quad \max = \frac{1}{2} \)

\begin{align*}
\inf = 0 & \quad \min = \text{d.n.e.} \\
\frac{1}{2} & \\
\frac{2}{2^2} = \frac{1}{2} & \\
\frac{3}{2^3} = \frac{3}{8} & \\
\frac{4}{2^4} = \frac{1}{4} &
\end{align*}
Let \( M = \sup S \)
and \( N = \inf T \)

Claim: \( N \) is an upper bound for \( S \).

Proof:

1. Let \( s \leq S \)
2. \( s \leq t \) for all \( t \)
3. \( \Rightarrow S \) is a lower bound
4. \( \Rightarrow s \leq N \)

\[ B > \sup^{-2}, \quad M \leq N \]