11.3 (4)

Let \( L = \lim_{x \to x_0} f(x) \)

Take \( \varepsilon = M - L \)

Then, by def of limit:

\[
f(x) \approx L \quad \text{for} \quad x \approx x_0 \quad (M - L)
\]

\[
\Rightarrow f(x) < L + (M - L) = M
\]

\[
\quad \text{for} \quad x \approx x_0
\]

11.4 (2)

Want to show:

\[
\begin{align*}
9 < x_0 < b \\
(f(a) < f(x_0))
\end{align*}
\]

\[
\begin{align*}
9 < x_0 < b \\
f(x_0) < f(b)
\end{align*}
\]

\[
\text{I} \quad \text{II} \quad \text{III}
\]

and \((f(a) < f(b))\)
Suppose $a < x_0 < b$

Pick $a < x_1 < x_0$.

We know $f(x_1) < f(x_0)$

and for $a < x < x_1$, $f(x) < f(x_1)$

By limit theorem then

$$f(a) = \lim_{x \to a^+} f(x) \leq f(x_1) < f(x)$$

pt of pr II

Suppose $a < x_0 < b$

Choose $x_0 < x_1 < b \Rightarrow f(x_0) < f(x_1)$

For $x_1 < x < b \Rightarrow f(x_1) < f(x)$

$$f(b) = \lim_{x \to b^-} f(x) \geq f(x_1) > f(x_0)$$

pt of pr II
pf of part III

Pick \( x_0 \in (a, b) \)

\[
f(c) < f(x_0) < f(b)
\]

Let \( x \) be irrational.

Choose a sequence of rational numbers \( \{x_n\} \) such that \( x_n \to x \).

Such a sequence exists: for instance, if \( x \) has decimal expansion

\[
x = a_1a_2a_3 \ldots a_k b_1b_2b_3 \ldots
\]

then one could take

\[
x_n = a_1a_2 \ldots a_kb_1b_2b_3 \ldots b_n
\]

(finite decimal)

By the sequential continuity this

\[
f(x_n) \to f(x)
\]

\[
\Rightarrow f(x) = 0
\]
(b) Again, let \( \{x_n\} \) be irrational and suppose \( x_n \to x \) with \( x_n \in \mathbb{Q} \) for all \( n \).

\[
f(x_n) \leq g(x_n)
\]

and \( f(x_n) \to f(x) \) \quad \text{[Sequential cont. Thm.]} \n
\[
g(x_n) \to g(x)
\]

\[
\implies f(x) \leq g(x)
\]

(limit theorem)

\[\text{However, consider}\]

\[
f(x) = 0
\]
\[
g(x) = (x - \pi)^2
\]

\[g(x) > f(x) \text{ for all } x \neq \pi \text{ (in fact all } x \neq \pi)\]

but \( g(\pi) = f(\pi) \).
Suppose \( f(x) \in \mathbb{Q} \) for all \( x \),

\( f \) is continuous,

but \( f \) is not constant.

Then there exist \( a < b \)

such that \( f(a) \neq f(b) \).

The exists an irrational number \( y \)

between \( f(a) \) and \( f(b) \).

IVT \( \Rightarrow \) there exists \( x \in [a, b] \) such that \( f(x) = y \) \& \( \mathbb{Q} \)

\( \rightarrow \leftrightarrow \)

So \( f \) must be constant.
12.1(5) If $\gamma = 1$:

$$y^2 \cos(0) - e^0 = 0,$$ and we're done.

Assume $\gamma > 1$.

$$\Rightarrow y^2 \cos(0) - e^0 > 0$$

Also, $y^2 \cos(\pi/2) - e^{\pi/2} = -e^{\pi/2} < 0$

$$f(x) = y^2 \cos(x) - e^x$$ is continuous in $x$,

IVT $\Rightarrow$ there is $x_0 \in [0, \pi/2]$ such that

$$f(x_0) = y^2 \cos(x_0) - e^{x_0} = 0$$

Since $f(\pi/2) < 0$, $x_0 < \pi/2$.

12-2:

Since $f$ has no repeats or value,

$$f(a) < f(b)$$
Let \( a \leq x' < x'' \leq b \)

Suppose \( f(x') \geq f(x'') \)

By assumption, since \( f(x') \neq f(x'') \)

\[ f(a) < f(x'') < f(x') \]

\[ \text{IUT } \Rightarrow \text{ there exists } x_0 \in [a, b] \]

such that \( f(x_0) = f(x'') \)

but since \( x'' > x' \), \( x_0 \neq x' \).

This contradicts the fact that \( f \) never repeats a value.

Thus, \( f(x') < f(x'') \).