18.100A final exam, spring 2007

You may use your book, but nothing else. Cite major theorems that you use.

1. Show that if \( \{a_n\} \) is an increasing sequence, and that \( a_n \to L \), then 
   \[ \sup\{a_n\} = L. \]

2. Using the definition of integrability, show that the function
   \[ f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \]
   is integrable on the interval \([-1, 1]\).

3. Which of the following are uniformly continuous?
   \[ (a) \quad f(x) = 1/x, \quad x \in (1, \infty) \]
   \[ (b) \quad f(x) = 1/x, \quad x \in (0, 1) \]
   [For each of the above, if you claim it is uniformly continuous, prove it. If you don’t think it is, just give me an intuitive explanation.]

4. Prove that the function
   \[ g(x) = \int_0^x e^{-t^2} dt \]
   is continuous.

5. Write down the Taylor series for the function \( g(x) \) of problem 4. What is the radius of convergence of this power series? [Hint: the Taylor series for \( e^{-t^2} \) can be easily deduced by substituting in \( y = -t^2 \) in the Taylor series for \( e^y \).

6. Which of the following has uniform convergence? Justify your answers.
   \[ (a): \] the series of functions
   \[ \sum \frac{\sin(x)}{n^2} \]
   \[ (b): \] the sequence of functions
   \[ f_n(x) = \sqrt[2]{\frac{x}{n}} \]

7. Suppose that \( f \) and \( g \) are continuous functions on the interval \([0, 1]\). Show there is a point \( x \in [0, 1] \) which minimizes the vertical distance between the graphs of the functions. (In other words, show that there is a point \( x \) which minimizes the distance between \( f(x) \) and \( g(x) \).)

8. Suppose that \( f \) is differentiable on \((-\infty, \infty)\), and that \( f'(x) > 1 \) for all \( x \). Suppose furthermore that \( f(0) = 0 \). Show that
   \[ f(x) > x \]
   for \( x > 0 \).