Notes on principal bundles and classifying spaces

Stephen A. Mitchell

August 2001

1 Introduction

Consider a real n-plane bundle ξ with Euclidean metric. Associated to ξ are a number of auxiliary bundles: disc bundle, sphere bundle, projective bundle, k-frame bundle, etc. Here "bundle" simply means a local product with the indicated fibre. In each case one can show, by easy but repetitive arguments, that the projection map in question is indeed a local product; furthermore, the transition functions are always linear in the sense that they are induced in an obvious way from the linear transition functions of ξ. It turns out that all of this data can be subsumed in a single object: the "principal $O(n)$-bundle" P_ξ, which is just the bundle of orthonormal n-frames. The fact that the transition functions of the various associated bundles are linear can then be formalized in the notion "fibre bundle with structure group $O(n)$". If we do not want to consider a Euclidean metric, there is an analogous notion of principal $GL_n\mathbb{R}$-bundle; this is the bundle of linearly independent n-frames.

More generally, if G is any topological group, a principal G-bundle is a locally trivial free G-space with orbit space B (see below for the precise definition). For example, if G is discrete then a principal G-bundle with connected total space is the same thing as a regular covering map with G as group of deck transformations. Under mild hypotheses there exists a *classifying space* BG, such that isomorphism classes of principal G-bundles over X are in natural bijective correspondence with $[X, BG]$. The correspondence is given by pulling back a *universal* principal G-bundle over BG. When G is discrete, BG is an Eilenberg-Maclane space of type $(G, 1)$. When G is either $GL_n\mathbb{R}$ or $O(n)$, BG is homotopy equivalent to the infinite Grassmanian $G_n\mathbb{R}^\infty$. The homotopy classification theorem for vector bundles then emerges as a special case of the homotopy classification theorem for principal bundles.

As these examples begin to suggest, the concept *principal bundle* acts as a powerful unifying force in algebraic topology. Classifying spaces also play a central role; indeed, much of the research in homotopy theory over the last fifty years involves analyzing the homotopy-type of BG for interesting groups G. There are also many applications in differential geometry, involving connections, curvature, etc. In these notes we will study principal bundles and classifying spaces from the homotopy-theoretic point of view.

2 Definitions and basic properties

Let G be a topological group. A *left G-space* is a space X equipped with a continuous left G-action $G \times X \longrightarrow X$. If X and Y are G-spaces, a *G-equivariant map* is a map $\phi : X \longrightarrow Y$ such that $\phi(gx) = g\phi(x)$ for all $g \in G, x \in X$. Synonymous terms include *equivariant* (if the group G is understood) and *G-map* (for short). This makes left G-spaces into a category. A *G-homotopy* (or G-equivariant homotopy, or equivariant homotopy) between G-maps ϕ, ψ is a homotopy $F : X \times I \longrightarrow Y$ in the usual sense, with the added condition that F be G-equivariant (here G acts trivially on the I coordinate). This yields the G-homotopy category of left G-spaces. Similar definitions apply to right G-spaces.

Now let B be a topological space. Suppose that P is a right G-space equipped with a G-map $\pi : P \longrightarrow B$, where G acts trivially on B (in other words, π factors uniquely through the orbit space P/G). We say that (P, π) is a *principal G-bundle* over B if π satisfies the following local triviality condition:

B has a covering by open sets U such that there exist G-equivariant homeomorphisms $\phi_U : \pi^{-1}U \longrightarrow U \times G$ commuting in the diagram

$$\begin{tikzcd} \pi^{-1}U \arrow[r, "\phi_U"] \arrow[d] & U \times G \arrow[dl] \\ U & \end{tikzcd}$$

Here $U \times G$ has the right G-action $(u, g)h = (u, gh)$. Note this condition implies that G acts freely on P, and that π factors through a homeomorphism $\overline{\pi} : P/G \longrightarrow B$ (thus B "is" the orbit space of P). Summarizing: *A principal G-bundle over B consists of a locally trivial free G-space with orbit space B.*

A *morphism* of principal bundles over B is an equivariant map $\sigma : P \longrightarrow Q$. This makes the collection of all principal G-bundles over B into a category. The set of isomorphism classes of principal G-bundles over B will be denote $\mathcal{P}_G B$. A principal G-bundle is *trivial* if it is isomorphic to the product principal bundle $B \times G \longrightarrow B$. Every principal bundle is locally trivial, by definition.

Note that (P, π) is in particular a local product over B with fibre G. To be a principal G-bundle, however, is a far stronger condition. Here are two striking and important properties that illustrate this claim:

Proposition 2.1 *Any morphism of principal G-bundles is an isomorphism.*

Proof: Let $\sigma : P \longrightarrow Q$ be a morphism. Suppose first that $P = Q = B \times G$. Then $\sigma(b, g) = (b, f(b)g)$ for some function $f : B \longrightarrow G$; clearly f is continuous. Hence σ is an isomorphism with $\sigma^{-1}(b, g) = (b, f(b)^{-1}g)$. This proves the proposition in the case when P and Q are trivial. Since every principal bundle is locally trivial, the general case follows immediately.

Proposition 2.2 *A principal G-bundle $\pi : P \longrightarrow B$ is trivial if and only if it admits a section.*

Proof: If P is trivial, then there is a section; this much is trivially true for any local product. Conversely, suppose $s : B \longrightarrow P$ is a section. Then the map $\phi : B \times G \longrightarrow P$ given by $\phi(b, g) = s(b)g$ is a morphism of principal bundles, and is therefore an isomorphism by Proposition 2.1.

The difference between a principal G-bundle and a run-of-the-mill local product with fibre G can be illustrated further in terms of transition functions. Suppose $\pi : E \longrightarrow B$ is a local product with fibre G, and U,V are open sets over which π is trivial, with $U \cap V$ nonempty. Comparing the two trivializations leads to a homeomorphism $(U \cap V) \times G \longrightarrow (U \cap V) \times G$ of the form $(x, g) \mapsto (x, \phi(x)g)$, where the *transition function* ϕ is a map from $U \cap V$ into the set of homeomorphisms from G to itself. In a principal G-bundle, each $\phi(x)$ is left translation by an element of G, and $\phi : U \cap V \longrightarrow G$ is continuous.

Given a principal G-bundle P over B and a map $f : B' \longrightarrow B$, we can form the pullback $P' \equiv f^*P \equiv B' \times_B P$; the pullback inherits a natural structure of principal G-bundle over B' from P. The reader should note the following two simple and purely categorical facts: First, if Q is a principal G-bundle over B', then bundle maps $Q \longrightarrow f^*P$ are in bijective correspondence with commutative squares

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & P \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

in which the top arrow is a G-equivariant map. Second, sections of the pullback bundle f^*P are in bijective correspondence with lifts in the diagram

$$\begin{tikzcd} & P \arrow[d] \\ B' \arrow[r, "f"'] \arrow[ur, dotted] & B \end{tikzcd}$$

We conclude this section with an interesting special case of the pullback construction. We can pull P back over itself:

$$\begin{tikzcd} P \times_B P \arrow[r, "\pi''"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"'] & B \end{tikzcd}$$

Here π' is projection on the lefthand factor, and defines a principal G-bundle structure in which the G-action on $P \times_B P$ is on the righthand factor.

Proposition 2.3 *$\pi' : P \times_B P \longrightarrow P$ is a trivial principal G-bundle over P.*

Proof: The diagonal map $P \longrightarrow P \times_B P$ is a section; now apply Proposition 2.2.

Note that the trivialization obtained is the map $P \times G \longrightarrow P \times_B P$ given by $(p, g) \mapsto (p, pg)$. By symmetry, a similar result holds for π'', with the roles of the left and right factors reversed.

Pulling P back over itself might seem a strange thing to do, and indeed we will not use this construction in these notes. In some contexts, however, the isomorphism $P \times G \longrightarrow P \times_B P$ is taken essentially as the definition of a principal bundle. This point of view is especially important in algebraic geometry, where the coarseness of the Zariski topology makes local triviality too stringent a condition to impose.

3 Balanced products and fibre bundles with structure group

Note that any left G-action on a space X can be converted to right action—and vice-versa—by setting $xg = g^{-1}x$, $x \in X$.

If W is a right G-space and X is a left G-space, the *balanced product* $W \times_G X$ is the quotient space $W \times X/\sim$, where $(wg, x) \sim (w, gx)$. Equivalently, we can simply convert X to a right G-space as above, and take the orbit space of the diagonal action $(w, x)g = (wg, g^{-1}x)$; thus $W \times_G X = (W \times X)/G$. The following special cases should be noted:

(i) If $X = *$ is a point, $W \times_G * = W/G$.

(ii) If $X = G$ with the left translation action, the right action of G on itself makes $W \times_G G$ into a right G-space, and the action map $W \times G \longrightarrow W$ induces a G-equivariant homeomorphism $W \times_G G \xrightarrow{\cong} W$.

Let G,H be topological groups. A *(G,H)-space* is a space Y equipped with a left G-action and right H-action, such that the two actions commute: $(gy)h = g(yh)$. Note that if Y is a (G,H)-space and X is a right G-space, $X \times_G Y$ receives a right H-action defined by $[x, y]h = [x, yh]$; similarly $Y \times_H Z$ has a left G-action if Z is a left H-space.

Proposition 3.1 *The balanced product is associative up to natural isomorphism: Let X be a right G-space, Y a (G,H)-space, and Z a left H-space. Then there is a natural homeomorphism*

$$(X \times_G Y) \times_H Z \cong X \times_G (Y \times_H Z)$$

Thus we can write $X \times_G Y \times_H Z$ without fear of ambiguity. The proof is left as an exercise. The homeomorphism in question takes an equivalence class $[x, y, z]$ on the left to the equivalence class $[x, y, z]$ on the right; the only problem is to show that this map and its inverse are continuous. Here it is important to note the following trivial but useful lemma:

Lemma 3.2 *Let X be any G-space, $\pi : X \longrightarrow X/G$ the quotient map. Then π is an open map.*

One reason this lemma is so useful is that while a product of quotient maps need not be a quotient map, a product of open maps is open and therefore a quotient map. This fact is needed in the proof of the proposition.

Note that if H is a subgroup of G, then G can be regarded as a $(G, H)-space$. Combining the second example above with the proposition, we find that the symbol $\times_G G$ (or $G\times_G$) can be "cancelled" whenever it occurs.

Corollary 3.3 *Suppose X is a right G-space, Y a left H-space, where H is a subgroup of G. Then $X \times_G G \times_H Y \cong X \times_H Y$.*

On the righthand side, X is regarded as an H-space by restricting the G-action. The proof is trivial. Taking Y to be a point and applying the first example above, we have the important special case:

Corollary 3.4 *Suppose X is a right G-space and H is a subgroup of G. Then*

$$X \times_G (G/H) \cong X/H$$

In the situation of this last corollary, suppose that $X \longrightarrow X/G$ is a principal G-bundle. We can ask whether $X \longrightarrow X/H$ is a principal H-bundle. In general it is not, even when X/G a point: For example, take G to be the additive group of real numbers, $X = G$ acting on itself by translation, and $H = \mathbb{Q}$. Then $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Q}$ is not a principal $\mathbb{Q}$-bundle—for it were locally trivial, then since $\mathbb{R}/\mathbb{Q}$ has the trivial topology it would have to be globally trivial, which is clearly absurd.

We eliminate this pathology according to the usual custom; that is, with a definition. Call a subgroup H of G *admissible* if the quotient map $G \longrightarrow G/H$ is a principal H-bundle. For example, any subgroup of a discrete group is admissible. More interesting examples can be found below.

Proposition 3.5 *Suppose $P \longrightarrow B$ is a principal G-bundle, and let H be an admissible subgroup of G. Then the quotient map $P \longrightarrow P/H$ is a principal H-bundle.*

Proof: For any subgroup H, we have $P/H = P \times_G (G/H)$. Our quotient map $P \longrightarrow P/H$ can then be identified with $P \times_G G \longrightarrow P \times_G (G/H)$. The proposition now follows easily from the fact that H is admissible.

Note that for fixed W, $X \mapsto W \times_G X$ is a functor, and similarly in the other variable. Now suppose $\pi : P \longrightarrow B$ is a principal G-bundle and F is a left G-space. The unique map $F \longrightarrow *$ is of course G-equivariant, and so induces a map $P \times_G F \longrightarrow P \times_G * = B$. It is easy to check that this map is a local product with fibre F. A local product of this form is called a *fibre bundle with fibre F and structure group G*, and denoted (P, p, B, F, G) if we want to display all the ingredients explicitly. For example, an n-dimensional real vector bundle is a fibre bundle with fibre $\mathbb{R}^n$ and structure group $GL_n\mathbb{R}$.

Once again, note that this is a much stronger condition than merely requiring a local product structure. The point is that the transition functions for the local product are required to take values in G. More precisely, suppose $p : E \longrightarrow B$ is a local product with fibre F, and U, V are intersecting open sets over which p is trivial. Then over $U \cap V$, we have two different trivializations, and comparing these leads to an automorphism of $(U \cap V) \times F$ of the form $(x, y) \mapsto (x, f(x)(y))$. Here f is a map from $U \cap V$ to the set of self-homeomorphisms of F. We cannot discuss the continuity of f without first topologizing this set, and in any event this would embroil us in point-set topological difficulties. If, on the other hand, p is the projection of a fibre bundle with structure group G, then the transition functions f can be interpreted as continuous maps into G. The automorphism above can be then be written $(x, y) \mapsto (x, f(x)y)$.

If (P, p, B, F, G) is a fibre bundle, and $f : X \longrightarrow B$ is a continuous map, the pullback f^*E is again a fibre bundle over X, with the same fibre and structure group. To make sense of this, one has to know that the two different ways of forming the pullback are really the same.

Proposition 3.6 *Let $p : E \longrightarrow B$ be a fibre bundle with fibre F and structure group G. Let $f : X \longrightarrow B$ be a map. Then there is a natural homeomorphism $f^*(P \times_G F) \cong (f^*P) \times_G F$.*

The proof is left as an exercise. Note there is a notational pitfall: Using another standard notation for pullbacks, the conclusion can be written as $X \times_Y (P \times_G F) \cong (X \times_Y P) \times_G F$. But the roles of Y and G as subscripts here are totally different! In general, the pullback $X \times_Y Z$ is a *subspace* of $X \times Z$, whereas a balanced product $P \times_G F$ is a *quotient* of $P \times F$. As long as one does not confuse the two constructions, the exercise is straightforward.

We will see many examples of fibre bundles with structure group in the next two sections.

4 Vector bundles and principal GL_n-bundles

Let $\xi = (E, p, B)$ be a real vector bundle of dimension n. The associated principal $GL_n\mathbb{R}$-bundle can be defined in two equivalent ways. The first is perhaps more vivid and intuitive, while the second displays the group action more clearly. We emphasize that the two definitions are really little more than mild paraphrases of one another, based on the following trivial fact: If W is a real vector space of dimension n, let V_nW denote the space of n-frames in W—that is, bases $(v_1, ..., v_n)$—and let $Iso\,(\mathbb{R}^n, W)$ denote the space of linear isomorphisms $f : \mathbb{R}^n \xrightarrow{\cong} W$. Then there is a natural homeomorphism $Iso\,(\mathbb{R}^n, W) \xrightarrow{\cong} V_nW$ given by $f \mapsto (f(e_1), ..., f(e_n))$.

In the first definition, we set $P = P_\xi = V_n\xi$, where $V_n\xi$ denotes the n-frame bundle; that is, the set of pairs $(b, \underline{v})$ with $b \in B$ and $\underline{v} = (v_1, ..., v_n)$ an n-frame in E_b. We topologize P as a subspace of the Whitney sum $nE = E \oplus ... \oplus E$ (n copies). Note that $GL_n\mathbb{R}$ acts on the right of nE by the rule $(v_1, ..., v_n)A = (v_1', ..., v_n')$, where $v_j' = \sum_i v_iA_{ij}$. It is not hard to check that this action is free, and that it makes P into a principal $GL_n\mathbb{R}$-bundle over B. But our second description of P makes this check even easier.

Consider the vector bundle $\underline{Hom}\,(\epsilon^n, E)$, where ϵ^n denotes the n-dimensional product bundle $B \times \mathbb{R}^n$, and recall that it can be identified with the n-fold Whitney sum $E \oplus ... \oplus E$.

Now $GL_n\mathbb{R}$ acts on the left of ϵ^n by $g(b,v) = (b, gv)$, inducing a *right* action on $\underline{Hom}\,(\epsilon^n, E)$. As our second definition of P_ξ we take the subbundle $\underline{Iso}\,(\epsilon^n, E)$ of $\underline{Hom}\,(\epsilon^n, E)$ for which the fibre over $b \in B$ is the space of all isomorphisms $\mathbb{R}^n \longrightarrow E_b$.

(*Warning:* For vector bundles E, E', do not confuse the *bundle* $\underline{Hom}(E, E')$ with the *vector space* $Hom(E, E')$. The latter is the space of global sections of the former. Similarly, do not confuse the bundle $\underline{Iso}(E, E')$ with the set of isomorphisms $Iso(E, E')$. Again, the latter is the space of global sections of the former—and indeed $Iso(E, E')$ is often the empty set.)

Clearly $GL_n\mathbb{R}$ acts freely on P. Using the local triviality of E, one easily checks that $P \longrightarrow B$ is a principal $GL_n\mathbb{R}$-bundle over B. Moreover, the natural map $\underline{Iso}\,(\epsilon^n, \xi) \longrightarrow V_n\xi$ given by $(b, f : \mathbb{R}^n \xrightarrow{\cong} E_b) \mapsto (b, (f(e_1), ..., f(e_n)))$ is an equivariant homeomorphism. Hence this definition of P agrees with the previous one.

Proposition 4.1 *For any space B, there is a natural bijection*

$$\phi : \mathcal{P}_{GL_n\mathbb{R}}B \xrightarrow{\cong} Vect_n^{\mathbb{R}}B$$

given by $P \mapsto P \times_{GL_n\mathbb{R}} \mathbb{R}^n$. The inverse ψ is given by $\xi \mapsto P_\xi$, as defined above. Similarly,

$$\mathcal{P}_{GL_n\mathbb{C}}B \xrightarrow{\cong} Vect_n^{\mathbb{C}}B$$

Proof: It is clear that both constructions are well-defined on isomorphism classes. Now suppose ξ is an n-plane bundle. Then there is an evident map

$$\sigma : \underline{Iso}\,(\epsilon^n, \xi) \times_{GL_n\mathbb{R}} \mathbb{R}^n \longrightarrow E(\xi)$$

defined by $\sigma([b, f, v]) = (b, f(v))$. Here $f : \mathbb{R}^n \xrightarrow{\cong} E_b$, and $v \in \mathbb{R}^n$. (We are also following our standard practice of including b in the notation to specify the fibre, even though this is redundant.) Note that σ is well-defined, since if $g \in GL_n\mathbb{R}$ then $f(gv) = (fg)(v)$. To see that σ is continuous, observe that over a trivializing neighborhood U for ξ, σ has the form

$$U \times GL_n\mathbb{R} \times_{GL_n\mathbb{R}} \mathbb{R}^n \xrightarrow{\cong} U \times \mathbb{R}^n$$

Since σ is clearly a linear isomorphism on fibres, σ is an isomorphism of vector bundles. This shows $\phi\psi$ is the identity.

Now suppose we start with a principal $GL_n\mathbb{R}$-bundle P, and let $\xi = P \times_{GL_n\mathbb{R}} \mathbb{R}^n$. To show that $\psi\phi$ is the identity, we will show that P is isomorphic to $V_n\xi$. By Proposition 2.1, it is enough to construct an equivariant map $\tau : P \longrightarrow V_n\xi$. Let $\tau(x) = ([x, e_1], ..., [x, e_n])$. Then τ is clearly continuous. To see that τ is equivariant, we identify $V_n\xi$ with $\underline{Iso}\,(\epsilon^n, \xi)$ as above. Then $\tau(x) = (b, f_x)$, where $b = [x]$ and $f_x : \mathbb{R}^n \longrightarrow E_b$ maps e_i to $[x, e_i]$. Hence if $g \in GL_n\mathbb{R}$, f_{xg} maps e_i to $[xg, e_i] = [x, ge_i]$. In other words, $\tau(xg) = (\tau(x))g$, as required.

If the vector bundle $\xi = (E, p, B)$ has a Euclidean metric, we can define an associated principal $O(n)$-bundle in an analogous way. The analogue of our first definition is to take $P = V_n^O\xi$, the bundle of orthonormal n-frames. The analogue of the second definition is to take $P = \underline{Isom}(\epsilon^n, \xi)$, the bundle whose fibre over $b \in B$ is the space of isometries $\mathbb{R}^n \cong E_b$. As before, it is easy to see that these two definitions are equivalent. Similarly, a complex n-plane bundle with Hermitian metric has an associated principal $U(n)$-bundle. The following lemma will be proved later, at least in the case when B is a CW-complex.

Lemma 4.2 *Let ξ be a real (resp. complex) n-plane bundle with two Euclidean (resp. Hermitian) metrics β_0, β_1. Then the corresponding principal $O(n)$-bundles (resp. $U(n)$-bundles) P_0, P_1 are isomorphic.*

Assuming the lemma, the method of proof of Proposition 4.1 yields:

Proposition 4.3 *For any paracompact Hausdorff space B, there are natural bijections*

$$\phi : \mathcal{P}_{O(n)}B \xrightarrow{\cong} Vect_n^{\mathbb{R}}B$$

and

$$\mathcal{P}_{U(n)}B \xrightarrow{\cong} Vect_n^{\mathbb{C}}B$$

Example 1: Let $\gamma = \gamma^1_{n,\mathbb{R}}$, the canonical line bundle over $\mathbb{R}P^n$. The associated principal $O(1)$-bundle P_γ is just the usual covering map $S^n \longrightarrow \mathbb{R}P^n$, with $O(1) \cong \mathbb{Z}/2$ acting via the antipodal involution. Hence $S^n \times_{\mathbb{Z}/2} \mathbb{R}^1 \cong E(\gamma)$.

Example 2: Let $\gamma = \gamma^1_{n,\mathbb{C}}$, the canonical line bundle over $\mathbb{C}P^n$. The associated principal $U(1)$-bundle P_γ is just the usual quotient map $S^{2n+1} \longrightarrow \mathbb{C}P^n$, with $U(1) \cong S^1$ acting in the usual way via complex multiplication. Hence $S^{2n+1} \times_{S^1} \mathbb{C}^1 \cong E(\gamma)$.

Example 3: Let $\tau = \tau_{S^n}$ denote the tangent bundle of S^n. Then the associated principal $O(n)$-bundle P_τ is the space of pairs $(x, \underline{v})$ with $x \in S^n$ and $\underline{v} = (v_1, ..., v_n)$ an orthornormal n-frame perpendicular to x. But this is just $V_{n+1}\mathbb{R}^{n+1} = O(n+1)$. Hence $P \longrightarrow S^n$ can be indentified with the standard quotient map $O(n+1) \longrightarrow S^n$ that exhibits S^n as the homogeneous space $O(n+1)/O(n)$.

Example 4. Using the principal bundle, the reader can now construct a plethora of fibre bundles associated to a vector bundle. To give just one example, let E be a complex n-plane bundle over B. The associated projective bundle $q : P(E) \longrightarrow B$ can be constructed in *ad hoc* fashion as the set of pairs (b, L) with $b \in B$ and L a line through the origin in E_b. We then give $P(E)$ the unique topology compatible with the trivializations $q^{-1}U \longrightarrow U \times \mathbb{C}P^{n-1}$ inherited from E. But we now have a more systematic construction of such bundles: Let $P = \underline{Iso}(\epsilon^n, E)$ denote the principal $GL_n\mathbb{C}$-bundle associated to E. Using the natural action of $GL_n\mathbb{C}$ on $\mathbb{C}P^{n-1}$, we form the associated fibre bundle

$$P \times_{GL_n\mathbb{C}} \mathbb{C}P^{n-1} \longrightarrow B$$

To see that this new bundle agrees with the old one, we use the argument of Example 4: Map (b, f, L) to $(b, f(L))$. Here f is an isomorphism $\mathbb{C}^n \longrightarrow E_b$, and $L \in \mathbb{C}P^{n-1}$. Again this map is well-defined on the balanced product, and yields a homeomorphism of spaces over B.

Example 5. A k-dimensional *distribution* on a smooth n-manifold M, as defined in any text on manifolds, is the same thing as a section of the bundle $G_k\tau_M$ whose fibre at $x \in M$ is the Grassmanian of k-planes in the tangent space at x. With principal bundles in hand, we can identify this bundle as

$$G_k\tau_M = P_{\tau_M} \times_{GL_n\mathbb{R}} G_k\mathbb{R}^n$$

Example 6. The reader should experiment with further variations on this theme: sphere bundles, disc bundles, k-frame bundles, etc.

5 More examples

Example 1: If G is discrete, a principal G-bundle with connected total space P is the same thing as a regular covering map with G as group of deck transformations.

To see this, first observe that any local product with discrete fibre and connected total space is a covering (covering spaces have connected total spaces, by definition). So if $\pi : P \longrightarrow B$ is a principal G-bundle with G discrete and P connected, π is at least a covering map. On the other hand, a covering map is *regular* if and only if its automorphism group A acts transitively—and hence simply transitively—on the fibre over a basepoint b_0. Here we have G acting as a group of automorphisms of π, so that $G \subset A$, and the action is simply transitive on fibres by the definition of principal bundle. This forces $G = A$: For if $a \in A$, choose any $x \in P$. Then there is an element $g \in G$ such that $xg = xa$ (for the sake of consistency, we let A act on the right). But a covering space automorphism (or "deck transformation") is uniquely determined by its value at a single point; hence $g = a$.

Conversely, if $\pi : P \longrightarrow B$ is a regular covering with group G, it is almost immediate from the definitions that π is a principal G-bundle.

If B has a universal cover (this is always the case if B is locally 1-connected), so that covering spaces are classified by conjugacy classes of subgroups of the fundamental group of B, the regular covering spaces are precisely those corresponding to normal subgroups of $\pi_1 B$.[1] In this case we can interpret all coverings as fibre bundles associated to the universal cover $\tilde{B} \longrightarrow B$, where $\tilde{B}$ is naturally a principal *left* $\pi_1 B$-bundle. Let $q : E \longrightarrow B$ be any covering space, and let S denote the fibre over a chosen basepoint. Then from basic covering space theory we have a natural right $\pi_1 B$ action on S, and if we choose $s \in S$ with isotropy group H, then we can identify E with $\tilde{B}/H$. By Corollary 3.4, this latter space in turn can be identified with $S \times_{\pi_1 B} \tilde{B}$, the associated bundle with fibre S.

Example 2: Suppose G is a Lie group, $H \subset G$ a closed subgroup. Then H is admissible; that is, the natural map $p : G \longrightarrow G/H$ is a principal H-bundle. (See [Brocker-tom Dieck], Theorem 4.3 p. 33.) In fact, p is a *smooth* principal bundle, meaning that all maps occuring in the definition can be taken smooth. Recall also that if M is a homogeneous space of G, then M is equivariantly diffeomorphic to G/H, where H is the isotropy group of any point of M (*loc. cit.*, 4.6). This leads to a variety of interesting examples. For example,

[1] So an alternative and probably better term is "normal covering". Yet another term is "Galois covering", drawing on the extraordinarily close analogy between covering space theory and Galois theory.

$S^n \cong O(n+1)/O(n)$ exhibits S^n as the base space of a principal O(n)-bundle. In fact this is the principal $O(n)$-bundle associated to the tangent bundle of S^n, as discussed in the previous section.

Example 3: Suppose G is a Lie group acting smoothly and freely on the smooth manifold M. In general, the orbit space M/G can be very badly behaved, and $M \longrightarrow M/G$ need not be a principal bundle. (Consider, for example, the $\mathbb{R}$-action on the torus given by the flow associated to a vector field of "irrational slope".) But a very useful result asserts that if G is *compact*, then M/G is a smooth manifold and $M \longrightarrow M/G$ is a smooth principal G-bundle. (*loc. cit.*, exercise 3, p. 40.) This result holds even for noncompact G, provided that the action is *proper* (*loc. cit.*).

For example, the natural projection $V_n^O\mathbb{R}^{n+k} \longrightarrow G_n\mathbb{R}^{n+k}$ from the orthonormal n-frames to the Grassmannian is a principal $O(n)$-bundle—in fact, it is the principal bundle associated to the vector bundle γ^n_{n+k} as in the previous section.

Example 4. Suppose $P \longrightarrow B$ is a principal G-bundle, and $\theta : G \longrightarrow GL_n\mathbb{C}$ is a continuous representation of G. Then $\mathbb{C}^n$ receives a left G-space structure, and the balanced product $P \times_G \mathbb{C}^n$ is evidently a vector bundle over B. This yields a functor from complex representations of G to complex vector bundles over B that commutes with direct sum and tensor product. For example, suppose $P \longrightarrow B$ is the usual quotient map $S^{2n+1} \longrightarrow \mathbb{C}P^n$. This is the principal S^1-bundle associated to the canonical line bundle $\gamma = \gamma^1_n$. If θ is the representation $S^1 \longrightarrow GL_1\mathbb{C}$ given by $z \mapsto z^k$, $k \in \mathbb{Z}$, the corresponding vector bundle is the k-th tensor power $\otimes^k\gamma$.

Of course similar remarks apply to real representations and real vector bundles.

6 Equivariant maps as sections of bundles

The next proposition will be the key to the homotopy classification Theorem 7.4 below. To motivate it, we consider a simple example. Suppose we want to construct sections (necessarily with zeros) of the canonical line bundle γ over $\mathbb{R}P^n$. We could start with a function $f : S^n \longrightarrow \mathbb{R}$ that is $\mathbb{Z}/2$-equivariant in the sense that $f(-x) = -f(x)$. For example, any function on the northern hemisphere that vanishes on the equator extends uniquely to an equivariant function. Now recall that $E(\gamma) = \{(L, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \in L\}$. We then define $g : S^n \longrightarrow E(\gamma)$ by $g(x) = ([x], f(x)x)$, and observe that g factors through the desired section s:

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "p"'] & E(\gamma) \\ \mathbb{R}P^n \arrow[ur, "s"'] & \end{tikzcd}$$

In fact every section arises in this way; we can also use sections to construct equivariant maps. For if s is given, we can set $g = sp$. Then $g(x) = ([x], f(x)x)$ for some function

f satisfying $f(-x)(-x) = f(x)(x)$, or $f(-x) = -f(x)$. One can easily check that f is continuous.

Recalling that $E(\gamma) \cong S^n \times_{\mathbb{Z}/2} \mathbb{R}$, with $\mathbb{Z}/2$ acting on $\mathbb{R}$ by the sign representation, we conclude that there is a bijective correspondence between the set of $\mathbb{Z}/2$-equivariant maps $S^n \longrightarrow \mathbb{R}$ and the set of sections of the bundle $S^n \times_{\mathbb{Z}/2} \mathbb{R} \longrightarrow S^n/(\mathbb{Z}/2)$. In this form our example admits a vast generalization.

Let $\pi : P \longrightarrow B$ be a principal G-bundle, X a right G-space, and $f : P \longrightarrow X$ a G-equivariant map. Then the map $P \longrightarrow P \times X$ given by $p \mapsto (p, f(x))$ is also G-equivariant, and passing to G-orbits yields a map $s = s_f : B \longrightarrow P \times_G X$ that is in fact a section of the fibre bundle map $q : P \times_G X \longrightarrow B$. Write $Hom_G(P, X)$ for the set of G-equivariant maps $P \longrightarrow X$, and $\Gamma(P \times_G X \longrightarrow B)$ for the set of sections of q.

Proposition 6.1 *Let P be a principal G-bundle over B, X a right G-space. Then there is a natural bijection* $\phi : Hom_G(P, X) \longrightarrow \Gamma(P \times_G X \longrightarrow B)$ *given by* $f \mapsto s_f$.

The point of this result is that sections of a bundle are usually easier to construct and study than equivariant maps.

Proof: If $P \cong B \times G$ is a trivial bundle, then $Hom_G(B \times G, X) = Hom\,(B, X)$, the set of continuous maps $B \longrightarrow X$. Similarly,

$$\Gamma(B \times G \times_G X \longrightarrow B) = \Gamma(B \times X \longrightarrow B) = Hom(B, X)$$

This proves the proposition for trivial bundles. The reader can then show directly that ϕ is bijective in the general case. Alternatively, here is a slick categorical method for passing from local to global information: Let $\{U_i\}$ be an open cover of B with $P_i \equiv P \mid U_i$ trivial. There is an evident commutative diagram of sets

$$\begin{tikzcd}
Hom_G(P,X) \arrow[r] \arrow[d, "\phi"] & \prod Hom_G(P_i,X) \arrow[r] \arrow[d, "\phi'"] & \prod Hom_G(P_i \cap P_j, X) \arrow[d, "\phi''"] \\
\Gamma(P \times_G X \longrightarrow B) \arrow[r] & \prod \Gamma(P_i \times_G X \longrightarrow U_i) \arrow[r] & \prod \Gamma((P_i \cap P_j) \times_G X \longrightarrow U_i \cap U_j)
\end{tikzcd}$$

with exact rows. (Given maps of sets $f, g : B \longrightarrow C$, the *equalizer* of the two maps is $\{b \in B : f(b) = g(b)\}$. Saying that the rows of the above diagram are *exact* means that the first arrow is an isomorphism onto the equalizer of the second two.) A diagram chase reminiscent of the 5-lemma shows that for any such diagram with ϕ' bijective and ϕ'' injective, ϕ must be bijective.

7 Homotopy classification and universal bundles

In this section we usually assume that the base space is a CW-complex. Many of the results below hold under the weaker assumption that the base space is paracompact (for an even more general setting, see [Dold]), but the proofs are completely different. The CW-approach involves techniques that are better suited to later homotopy-theoretic developments.

We also assume various facts about Serre fibrations; see my *Notes on Serre fibrations* for details.

Proposition 7.1 *Let X be an arbitrary space, P a principal G-bundle over X. Suppose that B is a CW-complex and that $f, g : B \longrightarrow X$ are homotopic maps. Then the pullbacks f^*P, g^*P are isomorphic as principal G-bundles over B.*

Let $F : B \times I \longrightarrow X$ be a homotopy from f to g. By considering the pullback F^*P, we reduce at once to proving the following lemma:

Lemma 7.2 *Let $Q \longrightarrow B \times I$ be a principal G-bundle, Q_0 its restriction to $B \times 0$. Then Q is isomorphic to $Q_0 \times I$. In particular Q_0 is isomorphic to Q_1.*

Proof: By Proposition 2.1, it is enough to construct a morphism $Q \longrightarrow Q_0 \times I$. By Proposition 6.1, this is equivalent to constructing a section s of $Q \times_G (Q_0 \times I) \longrightarrow B \times I$. But by Proposition 6.1 again, we have a section s_0 on B_0. By a general property of Serre fibrations, any section defined over B_0 extends to a section over $B \times I$. This completes the proof of the Lemma, and of Proposition 7.1.

In other words, $B \mapsto \mathcal{P}_G(B)$ is a homotopy functor from CW-complexes to sets. Thus a homotopy equivalence induces a bijection on $\mathcal{P}_G(-)$. In particular:

Corollary 7.3 *If B is contractible, every principal G-bundle over B is trivial.*

Remark: In view of Examples 4 and 5 above, Proposition 7.1 also implies that real and complex vector bundles have the stated homotopy invariance property under pullback.

As another corollary, we prove Lemma 4.2 in the case of a CW-base. Suppose β_0, β_1 are Euclidean metrics on the real n-plane bundle $\xi = (E, p, B)$. In other words, β_0, β_1 are sections of the bundle $Sym^+\xi$ whose fibre at $b \in B$ is the space of inner products on E_b. (This bundle can in turn be constructed as the balanced product $P_\xi \times_{GL_n\mathbb{R}} Sym^+\mathbb{R}^n$, but we will not make use of this construction.) Since the space of inner products on a vector space is a convex subset of the space of all bilinear forms, the homotopy $\beta_t = (1-t)\beta_0 + t\beta_1$ is a homotopy through Euclidean metrics from β_0 to β_1. More precisely, β defines a Euclidean metric on $\pi^*\xi$, where $\pi : B \times I \longrightarrow B$ is the projection. Let P denote the principal $O(n)$-bundle associated to P. Applying Proposition 7.1 to the inclusions $i_0, i_1 : B \longrightarrow B \times I$, we conclude that $P_0 \cong P_1$ as desired. The Hermitian case is proved the same way.

Now recall that a space X is said to be *weakly contractible* if $X \longrightarrow *$ is a weak equivalence; that is, for all $n \geq 0$, every map $S^n \longrightarrow X$ extends to a map $D^{n+1} \longrightarrow X$. Every contractible space X is weakly contractible, and by Whitehead's theorem every weakly contractible CW-complex is contractible.

Theorem 7.4 *Suppose $P \longrightarrow B$ is a principal G-bundle with P weakly contractible. Then for all CW-complexes X, the map $\phi : [X, B] \longrightarrow \mathcal{P}_G X$ given by $f \mapsto f^*P$ is bijective.*

We then call B a *classifying space* for G, and P a *universal G-bundle.* We will see below that the converse of Theorem 7.4 holds also.

Proof: Suppose P is weakly contractible. We first show ϕ is onto. Let $Q \longrightarrow B$ be a principal G-bundle. Then the Serre fibration $Q \times_G P \longrightarrow B$ has weakly contractible fibre and therefore admits a section. By Proposition 6.1, this section determines a G-equivariant map $\tilde{f} : Q \longrightarrow P$. Let $f : B \longrightarrow X$ denote the induced map on orbit spaces. Then $Q \cong f^*P$, as desired.

Now suppose given maps $f_0, f_1 : B \longrightarrow X$ and an isomorphism $\psi : f_0^*P \xrightarrow{\cong} f_1^*P$. Let Q denote the principal G-bundle $(f_0^*P) \times I$ over $B \times I$, and consider the local product $\rho : Q \times_G P \longrightarrow B \times I$. The equivariant maps $Q_0 = f_0^*P \longrightarrow P$ and $Q_1 = f_0^*P \xrightarrow{\psi} f_1^*P \longrightarrow P$ define a section of ρ over $B \times 0 \cup B \times 1$. Since the fibre is weakly contractible, this section extends over all of $B \times I$, and so determines a G-map $Q \longrightarrow P$. Passing to orbit spaces yields a homotopy $B \times I \longrightarrow X$ from f_0 to f_1. This shows ϕ is injective.

Proposition 7.5 *Suppose a universal G-bundle $P \longrightarrow B$ exists. Then*
a) B can be taken to be a CW-complex;
b) a CW-classifying space B is unique up to canonical homotopy equivalence;
c) P is unique up to G-homotopy equivalence.

Proof: For any space B, we can choose a CW-complex B' and a weak equivalence $g : B' \longrightarrow B$. Using the exact homotopy sequence of a Serre fibration and the 5-lemma, it follows that g^*P is also weakly contractible. This proves (a). Then (b) follows from Yoneda's lemma, since B represents the functor $\mathcal{P}_G(-)$ from the homotopy category of CW-complexes to Sets. Part (c) is left as an exercise.

For this theory to be of any use, we need to know classifying spaces exist. The following theorem is due to [Milnor].

Theorem 7.6 *Let G be any topological group. Then there exists a classifying space for G.*

The customary notation is to write BG for "the" classifying space of G, and EG for the universal bundle over BG. Bear in mind, however, that in these notes BG is well-defined only up to homotopy-equivalence, and similarly for EG. A particular choice of BG will be called a *model* for BG. Milnor constructs explicit, functorial models for BG.

Remark: Our definition of classifying space and universal bundle is weaker than the definition one commonly finds in the literature (cf. [May]). The stronger definition requires EG to be contractible instead of just weakly contractible, but also requires the bundles in question to be "numerable". See the note below on [Dold].

Remark: Using Theorem 7.6, we can prove the converse of Theorem 7.4. Suppose $P' \longrightarrow B'$ is a principal G-bundle having the property that $[-, B']$ classifies principal G-bundles on the homotopy category of CW-complexes. We want to show that P' is weakly contractible. Let $P \longrightarrow B$ be as in Theorem 7.6. Then P and P' are G-homotopy equivalent by part (c) of Proposition 7.5. In particular they are homotopy equivalent, and therefore P' is weakly contractible.

We will prove Milnor's theorem in some important special cases. Consider first the case $G = GL_n\mathbb{R}$. Recall that the infinite Grassmannian $G_n\mathbb{R}^\infty$ is a CW-complex, and that the "infinite Stiefel manifold" $V_n\mathbb{R}^\infty$ is a local product over $G_n\mathbb{R}^\infty$ with fibre $GL_n\mathbb{R}$. In fact the natural map $q : V_n\mathbb{R}^\infty \longrightarrow G_n\mathbb{R}^\infty$ is a principal $GL_n\mathbb{R}$-bundle.

Theorem 7.7 *$q : V_n\mathbb{R}^\infty \longrightarrow G_n\mathbb{R}^\infty$ is a universal $GL_n\mathbb{R}$-bundle, and hence $G_n\mathbb{R}^\infty$ is a classifying space for $GL_n\mathbb{R}$.*

Proof: We have to show that $V_n\mathbb{R}^\infty$ is weakly contractible. The Gramm-Schmidt process shows that the subspace $V_n^O\mathbb{R}^\infty$ of orthonormal frames is a deformation retract of $V_n\mathbb{R}^\infty$, and we will show instead (this is only a matter of convenience) that the orthonormal frames are weakly contractible. Since $\pi_i V_n^O\mathbb{R}^\infty$ is the direct limit of the $\pi_i V_n^O\mathbb{R}^{n+k}$ as $k \longrightarrow \infty$, it is enough to show:

Lemma 7.8 $\pi_i V_n^O\mathbb{R}^{n+k} = 0$ *for* $i < k$.

The proof of the lemma is an easy induction, using the long exact sequence of the evident fibration $p : V_n^O\mathbb{R}^{n+k} \longrightarrow S^{n+k-1}$. Note that the fibre is $V_{n-1}^O\mathbb{R}^{n+k-1}$. Note also that we now have an easy way of showing that p is a local product, and hence a Serre fibration: p is just the $(n-1)$-frame bundle associated to the principal $O(n+k-1)$-bundle of the tangent bundle of S^{n+k-1}.

This completes the proof of the theorem.

Thus $BGL_n\mathbb{R} \cong G_n\mathbb{R}^\infty$. By a similar argument, we find that $V_n^O\mathbb{R}^\infty \longrightarrow G_n\mathbb{R}^\infty$ is a universal $O(n)$-bundle (V_n^O denotes orthonormal n-frames). Hence $BO(n) \cong G_n\mathbb{R}^\infty$. Similarly, $BGL_n\mathbb{C} \cong G_n\mathbb{C}^\infty \cong BU(n)$. Note these spaces are also classifying spaces for vector bundles (cf. example 4 above). So we have the corollary:

Corollary 7.9 *For all CW-complexes X, the map $[X, G_n\mathbb{R}^\infty] \longrightarrow Vect_n^{\mathbb{R}}X$ given by $f \mapsto f^*\gamma^n$ is bijective. The analogous result with $\mathbb{R}$ replaced by $\mathbb{C}$ also holds.*

We can now easily prove the following special case of Milnor's theorem.

Theorem 7.10 *Let G be a Lie group that embeds as a closed subgroup of some $GL_n\mathbb{R}$. Then a classifying space BG exists.*

Proof: Take $EG = V_n\mathbb{R}^\infty$, regarded as a right G-space, and $BG = V_n\mathbb{R}^\infty/G$. By Proposition 3.5, $EG \longrightarrow BG$ is a principal G-bundle, and EG is weakly contractible as shown above.

Remark: Not every Lie group can be so embedded. For example, it is known that the universal covering group of $SL_3\mathbb{R}$ admits no faithful representations (this is far from obvious!).

On the other hand, every compact Lie group G embeds as a closed subgroup of $GL_n\mathbb{R}$ for some n ([Brocker-tom Dieck], p. 136).

Suppose now that G is a discrete group. Recall that there exists a CW-complex $K(G,1)$ such that $\pi_1 K(G,1) = G$ and all other homotopy groups of $K(G,1)$ vanish; these properties characterize $K(G,1)$ up to homotopy equivalence.

Theorem 7.11 *Let G be a discrete group. Then any $K(G,1)$ is a classifying space for G.*

Proof: Let EG be the universal cover of $K(G,1)$. Then $p : EG \longrightarrow K(G,1)$ is a principal G-bundle, and p induces an isomorphism on π_n for $n > 1$. Since EG is also simply-connected, it is weakly contractible (in fact contractible, since EG is also a CW-complex).

This proves Milnor's theorem for discrete groups. For example, $B\mathbb{Z} = S^1$ and $E\mathbb{Z} = \mathbb{R}$. It should be remarked that Milnor's general construction is not difficult; the interested reader should consult the original paper [Milnor].

8 Induced maps of classifying spaces

We assume that a choice of CW-classifying space BG has been fixed for each topological group G.

Let $\theta : H \longrightarrow G$ be a homomorphism of topological groups. Then G receives the structure of left H-space: $h \cdot g = \theta(h)g$. Combined with the action of G on itself by right translation, this makes G a $(H,G) -$ *space*. If $P \longrightarrow B$ is a principal H-bundle, we can then form the balanced product $P \times_H G$. If it is necessary to display θ to avoid confusion, we write this as $P \times_{H,\theta} G \longrightarrow B$. One easily checks that the natural right G-action on $P \times_{H,\theta} G$ is free, and that $P \times_{H,\theta} G \longrightarrow B$ is a principal G-bundle. Thus we have a natural transformation $P_\theta : \mathcal{P}_H B \longrightarrow \mathcal{P}_G B$.

It follows from Yoneda's lemma that there is a unique homotopy class $B\theta : BH \longrightarrow BG$ inducing P_θ. In fact $B\theta$ is just a classifying map for the principal G-bundle $EH \times_{H,\theta} G \longrightarrow BH$.

Proposition 8.1 *With this definition of $B\theta$, $G \mapsto BG$ is a functor from the category of topological groups to the homotopy category of CW-complexes.*

Proof: If $H = G$ and θ is the identity, $B\theta$ classifies $EG \times_G G \cong EG$ and hence is the identity (as homotopy class). Now suppose we are given a composite homomorphism

$$H \xrightarrow{\theta} G \xrightarrow{\tau} K$$

We must show that $B(\tau \circ \theta) = B(\tau) \circ B(\theta)$. Equivalently, we must show that $\mathcal{P}_{\tau\theta} = \mathcal{P}_\tau \mathcal{P}_\theta$. Let P be a principal H-bundle over a space B. Then

$$(P \times_{H,\theta} G) \times_{G,\tau} K \cong P \times_{H,\theta} (G \times_{G,\tau} K) \cong P \times_{H,\tau\theta} K$$

This proves the proposition.

As it stands, our definition of $B\theta$ applies only to CW-models for BH, BG. If we have models BH, BG that are not necessarily CW, and a map $f : BH \longrightarrow BG$ covered by a bundle map $EH \times_{H,\theta} G \longrightarrow EG$, we will call f a *model* for $B\theta$.

Remark: As noted earlier, Milnor's construction of BG gives a functor to the category of topological spaces, as opposed to merely the homotopy category (see also [May], p. 126 and p. 196). There is also a strictly functorial way to replace any space by a CW-complex; applying this construction to Milnor's BG yields a strictly functorial, CW-version of BG that avoids the fussing over CW-models that our approach requires. But this alternative comes with its own baggage, namely the machinery of simplicial sets, and we prefer to avoid it here.

8.1 Examples of induced maps

Suppose we are given a map $f : BH \longrightarrow BG$ and wish to identify it as a model for $B\theta$ for some homomorphism $\theta : H \longrightarrow G$. Proceeding directly from the definition, what we must show is that there is a bundle map $\tilde{f}$:

$$\begin{tikzcd}
EH \times_{H,\theta} G \arrow[r, "\tilde{f}"] \arrow[d] & EG \arrow[d] \\
BH \arrow[r, "f"'] & BG
\end{tikzcd}$$

When G is one of the classical groups $O(n)$, $U(n)$, $Sp(n)$ (or $GL_n\mathbb{R}$, $GL_n\mathbb{C}$, $GL_n\mathbb{H}$), this condition can be reformulated in terms of vector bundles. Taking $G = U(n)$ to illustrate, suppose we are given a homomorphism $\theta : H \longrightarrow U(n)$.

Proposition 8.2 *$B\theta$ classifies the vector bundle $EH\times_{H,\theta}\mathbb{C}^n$ over BH. The analogous result holds for real or quaternionic vector bundles.*

Proof: The principal bundle associated to $EH \times_{H,\theta} \mathbb{C}^n$ is just $EH \times_{H,\theta} U(n)$, so this is immediate.

Example 1. Let $i : O(n) \longrightarrow U(n)$ denote the inclusion. The universal vector bundle γ_n over $BO(n)$ has total space $EO(n)\times_{O(n)}\mathbb{R}^n$, and hence its complexification is $EO(n)\times_{O(n)}\mathbb{C}^n$. Thus $Bi : BO(n) \longrightarrow BU(n)$ classifies $\gamma_n \otimes \mathbb{C}$. If an explicit geometric model is desired, one can take the natural map $G_n\mathbb{R}^\infty \longrightarrow G_n\mathbb{C}^\infty$.

The reader should study in similar fashion the induced maps associated to the inclusions $U(n) \subset O(2n)$, $U(n) \subset Sp(n)$, $Sp(n) \subset U(2n)$.

Example 2. Consider the automorphism σ of $U(n)$ given by complex conjugation: $\sigma(A) = \overline{A}$. Then $B\sigma : BU(n) \longrightarrow BU(n)$ classifies the conjugate $\overline{\gamma}_{n,\mathbb{C}}$ over the universal complex vector bundle $\gamma_{n,\mathbb{C}}$. This is clear because $E(\gamma_{n,\mathbb{C}}) = EU(n) \times_{U(n)} \mathbb{C}^n$ and hence $E(\overline{\gamma}_{n,\mathbb{C}}) = EU(n) \times_{U(n),\sigma} \mathbb{C}^n$. Here an explicit geometric model is given by complex conjugation on $G_n\mathbb{C}^\infty$.

Example 3. Consider the determinant $det : U(n) \longrightarrow S^1$. Then $Bdet : BU(n) \longrightarrow \mathbb{C}P^\infty$ classifies the n-th exterior power $\wedge^n \gamma_{n,\mathbb{C}}$. Once again this is clear from Proposition 8.2: $E(\wedge^n \gamma_{n,\mathbb{C}}) = EU(n) \times_{U(n)} \wedge^n \mathbb{C}^n$, where the action of $U(n)$ on $\wedge^n \mathbb{C}^n$ is multiplication by the determinant.

Exercise. Suppose $\theta : G \longrightarrow G$ is an *inner* automorphism of G. Show that $B\theta$ is homotopic to the identity.

8.2 The induced map of an inclusion

Suppose $i : H \longrightarrow G$ is the inclusion of an admissible subgroup. Then there is an alternative way to think about the induced map Bi: Recall that since H is admissible (i.e., $G \longrightarrow G/H$ is a principal H-bundle), $EG \longrightarrow EG/H$ is a universal principal H-bundle and so EG/H is a classifying space for H. Furthermore, there is a map of principal G-bundles

$$\begin{tikzcd}
EG \times_H G \arrow[r, "\tilde{\pi}"] \arrow[d] & EG \arrow[d] \\
BH = EG/H \arrow[r, "\pi"'] & EG/G = BG
\end{tikzcd}$$

where $\tilde{\pi}[e, g] = eg$. Recalling that $EG/H = EG \times_G (G/H)$, we have the following result:

Proposition 8.3 *If $i : H \longrightarrow G$ is inclusion of an admissible subgroup, then $\pi : EG \times_G (G/H) \longrightarrow BG$ is a model for Bi.*

Example: Consider $j : U(n) \subset O(2n)$ and the induced map $Bj : BU(n) \longrightarrow BO(2n)$. By the method of Proposition 8.2, we know that Bj classifies the underlying real vector bundle of $\gamma_{n,\mathbb{C}}$. We also have an obvious geometric model: the natural embedding $G_n\mathbb{C}^\infty \subset G_{2n}\mathbb{R}^\infty$. Proposition 8.3, on the other hand, gives a very different model of Bj. We are now thinking of $BU(n)$ as $EO(2n) \times_{O(2n)} (O(2n)/U(n))$, or equivalently as the space of pairs (W, J) with $W \in G_{2n}\mathbb{R}^\infty$ and J a complex structure on W. Furthermore, our model for Bi is now a local product with fibre $O(2n)/U(n))$.

Examples of this type will be studied further below.

9 Products of classifying spaces

Proposition 9.1 *Let G,H be topological groups. Then the natural homotopy class*

$$B(G \times H) \longrightarrow BG \times_k BH$$

is a homotopy equivalence.

Remark: Recall that the notation $\times_k$ means that the product topology is to be replaced by the associated compactly-generated topology, which in this case is also the CW-topology. In general this topology is strictly finer than the product topology, although if BG and BH have countably many cells then the two topologies agree. In any case, for arbitrary spaces X, Y the natural map $X \times_k Y \longrightarrow X \times Y$ is a weak equivalence; for our purposes, therefore, the distinction is not very important. Note, however, that $X \times_k Y$ is the categorical product in the full subcategory of compactly-generated Hausdorff spaces.

Proof: It is readily checked that if $P \longrightarrow X$ is a principal G-bundle and $Q \longrightarrow Y$ is a principal H-bundle, then $P \times Q \longrightarrow X \times Y$ is a principal $G \times H$-bundle. In particular, this is true for $EG \times EH \longrightarrow BG \times BH$. Since a product of weakly contractible spaces is weakly contractible, it follows from Theorem 7.4 that $BG \times BH$ is a classifying space for $G \times H$. It may not be a CW-complex, but this deficiency is easily remedied as above: We retopologize $BG \times BH$ with the compactly-generated topology, and pull back $EG \times EH$ along the canonical map $BG \times_k BH \longrightarrow BG \times BH$.

Remark: It follows that there must be a natural transformation $\mathcal{P}_G B \times \mathcal{P}_H B \longrightarrow \mathcal{P}_{G\times H}(B)$ inverse to $(\mathcal{P}_{\pi_G}, \mathcal{P}_{\pi_H})$, and indeed one can see this explicitly as follows: Suppose given a principal G-bundle $Q \longrightarrow B$ and a principal H-bundle $R \longrightarrow B$. Then $Q \times R \longrightarrow B \times B$ is a principal $G \times H$-bundle over $B \times B$. Setting $P = \Delta^*(Q \times R)$, where Δ is the diagonal, we obtain a principal $G \times H$-bundle over B.

10 Change of structure group

Let $E \longrightarrow B$ be a vector bundle. Some typical questions we might ask about E are: Does it admit a nonvanishing section? Is it trivial? Does it decompose in some nontrivial way as a direct sum of two subbundles? If it is a real vector bundle, is orientable? Does it admit a complex structure? A Euclidean metric? If it is complex, is it the complexification of some real vector bundle? Does it admit a Hermitian metric?

Using principal bundles and classifying spaces, these questions can be formulated and studied in a uniform and very elegant way. Suppose $P \longrightarrow B$ is a principal G-bundle, and H is a subgroup of G. We say that P is *induced from an H-bundle* if there exists a principal H-bundle Q and an isomorphism $Q \times_H G \cong P$. Writing $i : H \subset G$ for the inclusion, this is just $\mathcal{P}_i$, a special case of the functor $\mathcal{P}_\theta$ defined in the previous section. For example, if H is the trivial subgroup this just means that P is a trivial bundle. If $E \longrightarrow B$ is a fibre bundle with fibre F and structure group G, we say that the structure group of the bundle can be *reduced* to H if the associated principal G-bundle is induced from a principal H-bundle.

We assume for the rest of this section that the base space B is a CW-complex.

Theorem 10.1 *Suppose H is an admissible subgroup of G (for example, G is a Lie group and H is a closed subgroup). Then the following are equivalent:*

a) P is induced from an H-bundle

b) $P \times_G (G/H) \longrightarrow B$ admits a section

c) The classifying map f of P lifts to BH, up to homotopy:

$$\begin{tikzcd} & BH \arrow[d] \\ B \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] & BG \end{tikzcd}$$

Proof: (a) $\Rightarrow$ (b): Suppose $P \cong Q \times_H G$ for some principal H-bundle Q. Then

$$P \times_G (G/H) = P \times_G G \times_H * = Q \times_H G \times_G G \times_H * = Q \times_H G \times_H * = Q \times_H (G/H)$$

The identity coset in G/H is an H-fixed point, and so defines an H-equivariant map $* \longrightarrow G/H$. Applying the functor $Q \times_H (-)$ to this map yields the desired section $B \longrightarrow P \times_G (G/H)$.

(b) $\Rightarrow$ (c): Since H is admissible, we can take $EG/H = EG \times_G (G/H)$ as a model for BH. With this model, we claim there is a strict lift (as opposed to a lift up to homotopy) in the diagram

$$\begin{tikzcd} & EG \times_G (G/H) \arrow[d] \\ B \arrow[ur, dotted] \arrow[r, "f"'] & BG \end{tikzcd}$$

where f is a classifying map for P. For a lift as indicated in the diagram is the same thing as a section of the pullback $f^*(EG \times_G (G/H))$, and this pullback is precisely $P \times_G (G/H)$.

(c) $\Rightarrow$ (a): If the lift $\tilde{f}$ exists, we simply take Q to be the pullback $\tilde{f}^* EH$. It is easy to check that this works.

Note: We only used the assumption that H is admissible in the implication (b) $\Rightarrow$ (c).

Example 1: Let $G = GL_n\mathbb{R}$, $H = O(n)$. Then G/H is homeomorphic to the group of upper triangular matrices with positive diagonal entries, and so is contractible. Hence, using criterion (b), any $GL_n\mathbb{R}$-bundle is induced from an $O(n)$-bundle. In other words, every real vector bundle admits a Euclidean metric. Similarly, every $GL_n\mathbb{C}$-bundle is induced from a $U(n)$-bundle; equivalently, every complex vector bundle admits a Hermitian metric.

Example 2: Let $\xi = (E, p, B)$ be a real n-plane bundle with Euclidean metric. Then E is orientable if and only if the structure group of ξ can be reduced from $O(n)$ to $SO(n)$. Note that in criterion (b), $G/H = O(n)/SO(n) = \{\pm 1\}$ is the set of orientations of $\mathbb{R}^n$, so $P \times_{O(n)} (O(n)/SO(n))$ is just the usual orientation bundle.

Example 3: Let $\xi = (E, p, B)$ be a real $2n$-plane bundle. We can ask whether or not E admits a complex structure—in other words, whether or not E is the underlying real vector bundle of some complex vector bundle. This amounts to reducing the structure group from $O(2n)$ to $U(n)$. Since $O(2n)/U(n)$ is the space of orthogonal complex structure maps on $\mathbb{R}^{2n}$, criterion (b) says that there is a complex structure map J on E (i.e., a bundle map such that $J^2 = -I$). Criterion (c) says that the classifying map for ξ can be lifted from $G_{2n}\mathbb{R}^\infty$ to $G_n\mathbb{C}^\infty$:

$$\begin{tikzcd} & G_n\mathbb{C}^\infty \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & G_{2n}\mathbb{R}^\infty \end{tikzcd}$$

Note this gives a simple necessary condition for the existence of a complex structure: All the odd Stiefel-Whitney classes of E must vanish.

It is also possible to consider all complex structure maps, instead of just the orthogonal ones. Then $O(2n)/U(n)$ should be replaced by $GL_{2n}\mathbb{R}/GL_n\mathbb{C}$. Topologically, the distinction is not significant because the inclusion $O(2n)/U(n) \subset GL_{2n}\mathbb{R}/GL_n\mathbb{C}$ is a homotopy equivalence.

Example 4: Let $\xi = (E, p, B)$ be a complex n-plane bundle. We can ask whether or not E is the complexification of a real n-plane bundle τ. This amounts to reducing the structure group from $U(n)$ to $O(n)$. Recall that $U(n)/O(n)$ is precisely the *Lagrangian Grassmannian* of orthogonally *totally real* subspaces of $\mathbb{C}^n$; that is, n-dimensional real subspaces $V \subset \mathbb{C}^n$ such that V is orthogonal to iV. Hence criterion (b) amounts to picking out a suitable totally real subspace in each fibre, yielding the desired real n-plane bundle as a real sub-bundle of ξ. Alternatively, one can identify $U(n)/O(n)$ with the space of conjugate-linear involutions on $\mathbb{C}^n$; then criterion (b) provides a conjugate linear bundle involution of E, whose fixed point set is the desired real sub-bundle.

Criterion (c) says that the classifying map for ξ can be lifted from $G_n\mathbb{C}^\infty$ to $G_n\mathbb{R}^\infty$:

$$\begin{tikzcd} & G_n\mathbb{R}^\infty \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & G_n\mathbb{C}^\infty \end{tikzcd}$$

Again, one can deduce a necessary condition on characteristic classes for the existence of τ. This time, however, it is easier to proceed directly without using classifying spaces: If a complex vector bundle is the complexification of a real vector bundle, then its odd Chern classes have order two (exercise, or see Milnor's *Characteristic Classes*).

It is also possible to consider all totally real subspaces; that is, n-dimensional real subspaces $V \subset \mathbb{C}^n$ such that V and iV are independent ($V + iV = \mathbb{C}^n$). Then $U(n)/O(n)$ should be replaced by $GL_n\mathbb{C}/GL_n\mathbb{R}$. Topologically, the distinction is not significant because the inclusion $U(n)/O(n) \subset GL_n\mathbb{C}/GL_n\mathbb{R}$ is a homotopy equivalence.

Example 5: Our definition of "reduction of structure group" assumed that H was a subgroup of G. More generally, one can consider an arbitrary homomorphism of topological groups $\theta : H \longrightarrow G$ and then define induced bundles, reduction of structure group and so on using the functor $\mathcal{P}_\theta$ discussed in the previous section (although condition (b) of the theorem no longer applies). As an interesting example, let $Spin(n) \longrightarrow SO(n)$ denote the double cover of $SO(n)$. Recall that for $n > 2$, this is the universal cover. Then an oriented n-plane bundle ξ admits a *spin structure* if the structure group of ξ can be reduced ("lifted" would be a better term here) to $Spin(n)$. In the next section we will show that ξ admits a spin structure if and only if its second Stiefel-Whitney class vanishes (recall that orientability is equivalent to the vanishing of the first Stiefel-Whitney class).

Exercise: Express the following two problems as instances of the "reduction of structure group" problem, and interpret conditions (b) and (c) of the theorem:

(i) Given a real or complex n-plane bundle ξ, and $r + s = n$, does ξ split as the Whitney sum of an r-plane bundle and an s-plane bundle?

(ii) Given a real or complex n-plane bundle ξ, and $k \leq n$, does ξ admit k sections that are everywhere linearly independent?

Remark: Theorem 9.1 gives conditions for the *existence* of a structure group reduction. One can go further and ask for a classification of all structure group reductions (from G to a specified H) of a given bundle. For example, if a real vector bundle ξ admits a complex structure, we would like to classify the distinct complex structures on ξ. Here on has to make precise what is meant by "distinct"; that is, one has to specify the right equivalence relation on the set of all complex structures on ξ. We will not pursue this question here, but further investigation would make an interesting project for the reader. Roughly speaking, the idea is to show that the following sets of equivalence classes are the same (see Theorem 9.1): (i) isomorphism classes of pairs (Q, θ) with Q a principal H-bundle and $\theta : Q \times_H G \longrightarrow P$ an isomorphism; (ii) homotopy classes of sections of $P \times_G (G/H)$; and (iii) homotopy classes of lifts $\tilde{f}$, where $\tilde{f}$ is as in part (c) of Theorem 9.1. Part of the problem, of course, is to make all this precise.

11 Homotopical properties of classifying spaces

This section requires more background in homotopy theory than the previous sections.

Theorem 11.1 *Let G be any topological group. Then G is weakly equivalent to the loop space ΩBG.*

There are two ways to prove this theorem; both are instructive, and are left as exercises. The first is to show directly that G and ΩBG represent the same functor, namely $X \mapsto \mathcal{P}_G(S^1 \wedge X_+)$, on the homotopy category of CW-complexes. The second is to prove a much more general result: If $p : E \longrightarrow B$ is a pointed Serre fibration and E is weakly contractible, then the fibre F is weakly equivalent to ΩB. Alternatively, see Theorem 11.3 below.

Corollary 11.2 *For all $n \geq 1$, $\pi_n BG \cong \pi_{n-1} G$.*

This implies, for example, that a path-connected group has a simply-connected classifying space.

We can generalize Theorem 11.1 as follows:

Theorem 11.3 *Let G be any topological group, H an admissible subgroup. Then the homotopy-fibre of $BH \longrightarrow BG$ is G/H, up to weak equivalence.*

Theorem 11.1 is the case when H is the trivial subgroup.

Proof: We have already seen that if $i : H \longrightarrow G$ is the inclusion, then as a model for Bi we can use $EG \times_G (G/H) \longrightarrow BG$. This map is a Serre fibration with fibre G/H, and therefore G/H is weakly equivalent to the homotopy-fibre L_π. (See *Notes on Serre fibrations.*)

Thus we have a fibre sequence $G/H \xrightarrow{j} BH \longrightarrow BG$, where j classifies the principal H-bundle $G \longrightarrow G/H$.

Example 1. In one of the standard inductive calculations of $H^*(BO(n); \mathbb{Z}/2)$, a key lemma identifies $BO(n) \cong G_n\mathbb{R}^\infty$ as the sphere bundle of γ_{n+1} over $BO(n+1)$, up to homotopy equivalence (then one can use the Gysin sequence). Here this lemma appears as the fibre sequence

$$S^n = O(n+1)/O(n) \xrightarrow{j} BO(n) \longrightarrow BO(n+1),$$

where j classifies the tangent bundle. Similar remarks apply to $U(n) \subset U(n+1)$ and $Sp(n) \subset Sp(n+1)$.

Example 2. Let $T^n \subset U(n)$ denote the diagonal matrices. Then there is a fibre sequence

$$U(n)/T^n \xrightarrow{j} BT^n \xrightarrow{Bi} BU(n),$$

where $BT^n \cong (CP^\infty)^n$, $U(n)/T^n$ is manifold of complete flags in $\mathbb{C}^n$, and j classifies the evident n-tuple of complex line bundles over the flag manifold.

Example 3. In the fibre sequence

$$U(n)/O(n) \xrightarrow{j} BO(n) \longrightarrow BU(n),$$

we can identify $U(n)/O(n)$ with the *Lagrangian Grassmanian* of totally real subspaces of $\mathbb{C}^n$. Then j can be taken as the natural embedding $U(n)/O(n) \subset G_n\mathbb{R}^{2n} \subset G_n\mathbb{R}^\infty$. Note that j classifies the universal example of an n-plane bundle with trivial complexification.

Example 4. In the fibre sequence

$$Sp(n)/U(n) \xrightarrow{j} BU(n) \longrightarrow BSp(n),$$

we can identify $Sp(n)/U(n)$ with the Grassmanian of maximal isotropic subspaces for the standard symplectic (=skew-symmetric + non-degenerate) form on $\mathbb{C}^{2n}$. Thus $Sp(n)/U(n) \subset G_n\mathbb{C}^{2n} \subset G_n\mathbb{C}^\infty$, and j classifies the universal example of a complex n-plane bundle with trivial symplectification.

Example 5. Contemplate the fibre sequences as above associated with $O(2n)/U(n)$ and $U(2n)/Sp(n)$.

Example 6. Let G be a discrete group, H any subgroup. Then up to homotopy, $BH \longrightarrow BG$ is the covering space of BG corresponding to the subgroup $H \subset G = \pi_1 BG$. The homotopy-fibre is the discrete space G/H.

Remark. Recall that for any pointed map $f : X \longrightarrow Y$, maps of a space W into the homotopy-fibre L_f correspond to maps $\phi : W \longrightarrow X$ together with a nullhomotopy of $f \circ \phi$. Thus homotopy classes of maps into G/H should correspond in some sense to principal H-bundles $P \longrightarrow W$ together with a trivialization of the induced principal G-bundle $P \times_H G \longrightarrow W$ (cf. the examples above). As an interesting thought experiment, contemplate how you might make this precise.

When the subgroup H is normal, it turns out that we can extend the fibre sequence in the other direction:

Theorem 11.4 *Let G be any topological group, H an admissible normal subgroup. Then there is a homotopy-fibre sequence* $BH \xrightarrow{Bi} BG \xrightarrow{B\rho} B(G/H)$, *where* $i : H \longrightarrow G$ *is the inclusion and* $\rho : G \longrightarrow G/H$ *is the quotient map.*

Proof: First note the following general fact: If $P \longrightarrow B$ is a principal G-bundle, and X is any G-space, then $P \times X \longrightarrow P \times_G X$ is again a principal G-bundle. In particular, when $P = EG$ and X is weakly contractible, we conclude that $EG \times_G X$ is a model for BG.

Here we will take $X = E(G/H)$, with G acting in the evident way. Then $EG \times_G E(G/H)$ is a model for BG, and one can easily check that the natural map

$$EG \times_G E(G/H) \longrightarrow [E(G/H)]/G = [E(G/H)]/(G/H) = B(G/H)$$

is a model for $B\rho$. Now since H is normal, we can take the orbits of any G-space Y by first taking the H-orbits and then taking the G/H-orbits: $Y/G = (Y/H)/(G/H)$. Furthermore EG/H is a model for BH, since H is admissible. Thus

$$EG \times_G E(G/H) = BH \times_{G/H} E(G/H).$$

This also show that the model for $B\rho$ constructed above can be identified with the fibre bundle

$$E(G/H) \times_{G/H} BH \longrightarrow B(G/H)$$

and hence $B\rho$ has homotopy-fibre BH, as desired. Finally, one can easily check that the resulting map $BH \longrightarrow BG$ is indeed a model for Bi.

Example 7. The exact sequences $1 \to SO(n) \to O(n) \to \mathbb{Z}/2 \to 1$ and $1 \to SU(n) \to U(n) \to S^1 \to 1$ give rise to fibre sequences

$$BSO(n) \longrightarrow BO(n) \longrightarrow \mathbb{R}P^\infty$$

and

$$BSU(n) \longrightarrow BU(n) \longrightarrow \mathbb{C}P^\infty.$$

Example 8. Consider the exact sequence of groups

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow Spin\,(n) \longrightarrow SO(n) \longrightarrow 1$$

This induces a homotopy-fibre sequence

$$\mathbb{R}P^\infty \longrightarrow BSpin(n) \longrightarrow BSO(n)$$

This leads to a cohomological criterion for the existence of a Spin-structure. Let ξ be an oriented n-plane bundle over B, $f : B \longrightarrow BSO(n)$ a classifying map for ξ. Let $\mathcal{O}_\xi$ denote the primary obstruction to producing a lift in the diagram

$$\begin{tikzcd} & BSpin(n) \arrow[d] \\ B \arrow[ur, dotted] \arrow[r, "f"'] & BSO(n) \end{tikzcd}$$

Since $\mathbb{R}P^\infty$ is an Eilenberg-Maclane space of type $(\mathbb{Z}/2, 1)$, this primary obstruction is the only obstruction. The universal obstruction class (obtained by taking $B = BSO(n)$ and f the identity) lies in $H^2(BSO(n); \mathbb{Z}/2)$, which we know is isomorphic to $\mathbb{Z}/2$ with generator $w_2 = w_2(\tilde{\gamma}^n)$. Hence it is either zero or w_2. If it is zero then $BSpin(n) \longrightarrow BSO(n)$ admits a section. But this is impossible since then $\mathbb{R}P^\infty \longrightarrow BSpin(n)$ would induce a monomorphism on homotopy groups, contradicting the fact that $BSpin(n)$ is simply-connected (to see this last point, use the corollary above). Thus if f classifies the oriented n-plane bundle ξ, we conclude that ξ admits a spin-structure if and only if $w_2\xi = 0$.

12 References

[Brocker-tom Dieck] *Representations of compact Lie groups.*

This beautifully written book has an excellent introductory chapter on basic Lie group facts.

[Dold] Partitions of unity in the theory of fibrations, Annals of Math. 78 (1963), 223-255.

In the usual approach to universal bundles, some restriction has to be made on the base spaces involved. In these notes we usually assume the base space is a CW-complex, although many of the results go through for a paracompact Hausdorff base. Dold's clever but simple idea is that we don't need partitions of unity for arbitrary covers; it is enough to have partitions of unity for at least one trivializing cover of the given bundle. Bundles with this property are called "numerable". The advantage of this approach is that by allowing only numerable bundles one can then work over an arbitrary base space. Furthermore, since every bundle over a paracompact Hausdorff space is numerable, all the classical results are recovered. The details are rather technical, but if you prefer partitions of unity to CW-complexes, Dold's elegant theory is the way to go.

[May] *A concise introduction to algebraic topology.*
Written by one of the world leaders in the field, this recently published book lives up to its title. While not really suitable as an introduction, it is an excellent source for anyone who has already had a first course in algebraic topology.

[Milnor] On the construction of universal bundles II, Annals of Math. 63 (1956), 430-436.
This elegant paper is independent of "Universal bundles I". Very readable.

[Steenrod] *The topology of fibre bundles.*
A classic by one of the ancient masters. Although much of the notation has changed since 1950, and many of the unsolved problems have been solved, this is still an excellent text.