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M20550 Calculus III Tutorial
Worksheet 3

1. Imagine a wheel of unit radius rolling from left to right along the x-axis in the xy-plane with a constant angular velocity of $\frac{1rad}{sec}$. Let p be the point on the wheel that has coordinates $(0, 0)$ at time $t = 0$. Find a vector valued function that describes the position of p at time t. What if the wheel had radius a? (The curve traced out by the motion of this point is called a cycloid.)

Solution: Notice that our function, $\mathbf{r}(t)$, may be written as a sum of three functions $\langle c(t), 0\rangle + \langle 0, 1\rangle + \boldsymbol{\alpha}(t)$, where $c(t)$ is the x-component of the center of the wheel at time t and $\boldsymbol{\alpha}(t) = -\boldsymbol{\phi}(t)$, where $\boldsymbol{\phi}$ was your solution to the last problem on the previous worksheet, i.e. $\boldsymbol{\alpha}(t) = \langle -\sin(t), -\cos(t)\rangle$. This can be seen geometrically by drawing a picture of the wheel at some time t, and drawing the three vectors in the sum above in such a way that the initial point of each one is the terminal point of the last, with the first one positioned so that its initial point is at the origin. The stipulation that the angular velocity of the wheel is 1 radian per second gives us that $c(t) = t$. So, $\mathbf{r}(t) = \langle t - \sin(t), 1 - \cos(t)\rangle$. If the radius of the wheel is a then by similar reasoning we get $\mathbf{r}(t) = \langle at - a\sin(t), a - a\cos(t)\rangle$. The following link will bring you to a page about Problem 1 on the site math-stackexchange that has a nice animation and some good solutions, please have a look:

https://math.stackexchange.com/questions/133604/
how-to-find-the-parametric-equation-of-a-cycloid

The solution by Robert Israel is particularly good.

2. Find an equation of the tangent line to the space curve $\mathbf{r}(t) = \langle 2t^3, 3t^2, 3t\rangle$ at the point $(-2, 3, -3)$.

Solution: First, we want to find t corresponds to the point $(-2, 3, -3)$. t corresponds to $(-2, 3, -3)$ must satisfy the equations

$$2t^3 = -2, \quad 3t^2 = 3, \quad 3t = -3.$$

From the third equation, we know $t = -1$.

Next, we want to find $\mathbf{r}'(-1)$, the tangent vector at $t = -1$. The derivative of $\mathbf{r}(t)$ is given by $\mathbf{r}'(t) = \langle 6t^2, 6t, 3\rangle$. So the tangent vector at $t = -1$ is $\mathbf{r}'(-1) = \langle 6, -6, 3\rangle$.

Then, the vector equation of the tangent line at $(-2, 3, -3)$ is

$$\langle x, y, z\rangle = \langle -2, 3, -3\rangle + t\,\langle 6, -6, 3\rangle.$$

3. Find the distance from the point $(0, 1, -1)$ to the space curve given by $\mathbf{r}(t) = \left\langle \sqrt{t}, 2t, -t \right\rangle$.

Solution: The distance from the point to the curve can be thought of as a function of t in that at each time we can compute the distance from the point to $\mathbf{r}(t)$. This we can write as $D(t) = \sqrt{(\sqrt{t})^2 + (2t-1)^2 + (-t-(-1))^2}$. We would like to minimize this quantity, which we can do by looking for critical points using its derivative. We also note that minimizing $D(t)$ also minimizes $D(t)^2$ and vice versa, so we compute

$$\frac{d}{dt}\left(D(t)^2\right) = \frac{d}{dt}(5t^2 - 5t + 2) = 10t - 5.$$

Since there is only one critical point, and the distance is certainly unbounded as t gets large, we reach the minimum distance at $t = \frac{1}{2}$ and see $D(\frac{1}{2}) = \sqrt{\frac{5}{4} - \frac{5}{2} + 2} = \frac{\sqrt{3}}{2}$ is the distance from the point to the curve.

4. Find $\mathbf{r}(t)$ if $\mathbf{r}''(t) = e^t\mathbf{i}$, $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, and $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Solution:

$$\mathbf{r}'(t) = \int \mathbf{r}''(t)\,dt = \int \left\langle e^t, 0, 0\right\rangle\,dt = \left\langle e^t, 0, 0\right\rangle + \mathbf{c}.$$

To find $\mathbf{c}$, we use the information $\mathbf{r}'(0) = \langle 1, 1, 1\rangle$. From the above, we have $\mathbf{r}'(0) = \langle e^0, 0, 0\rangle + \mathbf{c}$. So, $\langle e^0, 0, 0\rangle + \mathbf{c} = \langle 1, 1, 1\rangle \implies \mathbf{c} = \langle 1, 1, 1\rangle - \langle e^0, 0, 0\rangle = \langle 0, 1, 1\rangle$. Thus, we get

$$\mathbf{r}'(t) = \left\langle e^t, 0, 0\right\rangle + \langle 0, 1, 1\rangle \implies \mathbf{r}'(t) = \left\langle e^t, 1, 1\right\rangle.$$

Then

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \left\langle e^t, 1, 1\right\rangle dt = \left\langle e^t, t, t\right\rangle + \mathbf{d}.$$

To find $\mathbf{d}$, we use the information $\mathbf{r}(0) = \langle 2, 3, 2\rangle$. We have $\mathbf{r}(0) = \langle e^0, 0, 0\rangle + \mathbf{d} = \langle 2, 3, 2\rangle$. So, $\mathbf{d} = \langle 2, 3, 2\rangle - \langle e^0, 0, 0\rangle = \langle 1, 3, 2\rangle$.

Finally, we get

$$\mathbf{r}(t) = \left\langle e^t, t, t\right\rangle + \langle 1, 3, 2\rangle \implies \mathbf{r}(t) = \left\langle e^t + 1, t + 3, t + 2\right\rangle.$$

5. Find the unit tangent vector, the principal unit normal vector, and the unit binormal vectors to the curve $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, 3t^2\rangle$ at $t = \pi$.

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Solution: We have $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, 3t^2 \rangle$. So

$$\mathbf{r}'(t) = \langle 2\cos 2t, -2\sin 2t, 6t \rangle \implies \mathbf{r}'(\pi) = \langle 2, 0, 6\pi \rangle .$$

$$\mathbf{r}''(t) = \langle -4\sin 2t, -4\cos 2t, 6 \rangle \implies \mathbf{r}''(\pi) = \langle 0, -4, 6 \rangle .$$

Also,

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \langle 2, 0, 6\pi \rangle \times \langle 0, -4, 6 \rangle = \langle 24\pi, -12, -8 \rangle = 4 \langle 6\pi, -3, -2 \rangle .$$

Then,

$$\mathbf{T}(\pi) = \frac{\mathbf{r}'(\pi)}{|\mathbf{r}'(\pi)|} = \frac{\langle 2, 0, 6\pi \rangle}{|\langle 2, 0, 6\pi \rangle|} = \frac{1}{\sqrt{4 + 36\pi^2}} \langle 2, 0, 6\pi \rangle .$$

$$\mathbf{B}(\pi) = \frac{\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)}{|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)|} = \frac{4 \langle 6\pi, -3, -2 \rangle}{4 |\langle 6\pi, -3, -2 \rangle|} = \frac{1}{\sqrt{13 + 36\pi^2}} \langle 6\pi, -3, -2 \rangle .$$

$$\begin{aligned}
\mathbf{N}(\pi) = \mathbf{B}(\pi) \times \mathbf{T}(\pi) &= \frac{1}{\sqrt{13 + 36\pi^2}} \langle 6\pi, -3, -2 \rangle \times \frac{1}{\sqrt{4 + 36\pi^2}} \langle 2, 0, 6\pi \rangle \\
&= \frac{1}{\sqrt{13 + 36\pi^2}} \frac{1}{\sqrt{4 + 36\pi^2}} \langle 6\pi, -3, -2 \rangle \times \langle 2, 0, 6\pi \rangle \\
&= \frac{1}{\sqrt{13 + 36\pi^2}} \frac{1}{\sqrt{4 + 36\pi^2}} \left\langle -18\pi, -4 - 36\pi^2, 6 \right\rangle .
\end{aligned}$$

6. Find the equation for the normal and osculating planes to the curve $\mathbf{r}(t) = 2\cos(3t)\mathbf{i} + t\mathbf{j} + 2\sin(3t)\mathbf{k}$ at the point $(-2, \pi, 0)$.

Solution: First, we note that t corresponds to the point $(-2, \pi, 0)$ is $t = \pi$ since $\mathbf{r}(t) = \langle 2\cos(3t), t, 2\sin(3t) \rangle = \langle -2, \pi, 0 \rangle$ implies $t = \pi$ by looking at the second component.

A normal vector of the normal plane at $t = \pi$ is $\mathbf{r}'(\pi)$. We have

$$\mathbf{r}'(t) = \langle -6\sin(3t), 1, 6\cos(3t) \rangle \implies \mathbf{r}'(\pi) = \langle 0, 1, -6 \rangle .$$

So, the normal plane at the point $(-2, \pi, 0)$ is given by

$$\langle 0, 1, -6 \rangle \cdot \langle x, y, z \rangle = \langle 0, 1, -6 \rangle \cdot \langle -2, \pi, 0 \rangle \implies y - 6z = \pi.$$

A normal vector of the osculating plane at $t = \pi$ is $\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)$. We have, $\mathbf{r}''(t) = \langle -18\cos(3t), 0, -18\sin(3t) \rangle$ and so $\mathbf{r}''(\pi) = \langle 18, 0, 0 \rangle$. Then,

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \langle 0, 1, -6 \rangle \times \langle 18, 0, 0 \rangle = 18 \langle 0, 1, -6 \rangle \times \langle 1, 0, 0 \rangle = 18 \langle 0, -6, -1 \rangle .$$

So, we can take $\langle 0,6,1\rangle$ to be a normal vector for this osculating plane. And the equation is

$$\langle 0,6,1\rangle \bullet \langle x,y,z\rangle = \langle 0,6,1\rangle \bullet \langle -2,\pi,0\rangle \implies 6y+z=6\pi.$$

7. Find the length of the curve $\mathbf{r}(t)=\left\langle 2t,t^2,\frac{1}{3}t^3\right\rangle$ from $(0,0,0)$ to $\left(2,1,\frac{1}{3}\right)$.

Solution:
First, we need the derivative:

$$\mathbf{r}'(t)=\left\langle 2,2t,t^2\right\rangle$$

and its magnitude

$$|\mathbf{r}'(t)|=\sqrt{4+4t^2+t^4}=\sqrt{(2+t^2)^2}=2+t^2 \qquad \text{since } 2+t^2>0.$$

And now, the point $(0,0,0)$ corresponds to $t=0$ and the point $\left(2,1,\frac{1}{3}\right)$ corresponds to $t=1$. Then, we have the length of $\mathbf{r}$ is

$$L=\int_{t=0}^{t=1}|\mathbf{r}'(t)|\,dt=\int_0^1(2+t^2)dt = 2t+\frac{1}{3}t^3\Big|_0^1=\frac{7}{3}-0=\frac{7}{3}.$$

8. A particle moves with position function $\mathbf{r}(t)=\langle \sin t,\cos t,\cos^2 t\rangle$. Find the tangential and normal components of acceleration when $t=\pi/4$.

Solution: We have

$$\mathbf{r}'(t)=\langle \cos t,-\sin t,-2\cos t\sin t\rangle=\langle\cos t,-\sin t,-\sin(2t)\rangle$$

$$\implies \mathbf{r}'(\pi/4)=\left\langle \frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-1\right\rangle,$$

$$\mathbf{r}''(t)=\langle -\sin t,-\cos t,-2\cos(2t)\rangle \implies \mathbf{r}''(\pi/4)=\left\langle -\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right\rangle.$$

And so

$$a_T=\frac{\mathbf{r}'\left(\pi/4\right)\bullet\mathbf{r}''\left(\pi/4\right)}{|\,\mathbf{r}'\left(\pi/4\right)|}=0.$$

We know $\mathbf{a}=a_T\mathbf{T}+a_N\mathbf{N}$. Since $a_T=0$, we get $\mathbf{a}=a_N\mathbf{N}$. So,

$$|\mathbf{a}|=a_N|\mathbf{N}|=a_N\cdot 1=a_N.$$

Thus,

$$
\begin{aligned}
a_N = |\mathbf{a}| &= |\,\mathbf{r}''\,(\pi/4)\,| \\
&= \left|\,\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\rangle\,\right| \\
&= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\
&= 1
\end{aligned}
$$