0 - Introduction

Wednesday, August 26, 2015 6:52 AM

[Slides

- unstable $\pi_* S^2$
  - $\to$ stabilize $\quad \pi_*^s = \varinjlim_k \pi_{*+k} S^k$
  - $\to$ localize $\quad (-)_{(p)}$

- stable $(\pi_*^s)_{(p)}$]

J homomorphism:

$$SO(n) \longrightarrow \Omega^n S^n$$

$$A \longmapsto \left(A^+ : S^n \longrightarrow S^n\right)$$

$$\pi_k SO(n) \longrightarrow \pi_k \Omega^n S^n = \pi_{k+n} S^n$$

$\lim_{n\to\infty}$

$$\pi_k SO \longrightarrow \pi_{k+n} S^n$$

known Bott periodicity

k	1	2	3	4	5	6	7	8	9					
	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0 $\mathbb{Z}$ ...

8-fold periodic.

$\Longrightarrow$ periodic patterns in π_*^s? yes.
[slides]

Adams

$$(\mathrm{Im}\, J)_{8s} = \mathbb{Z}/2$$

$$(\mathrm{Im}\, J)_{8s+1} = \mathbb{Z}/2$$

Mahowald .....

$(\mathrm{Im}\, J) \quad = \mathbb{Z}/$

$\cdot\ 8s+1$

$$\left(\operatorname{Im} J_{(2)}\right)_{4\cdot 2^k s - 1} \cong \mathbb{Z}/2^{k+3}$$ Mahowald .....

$2 \nmid s$

p odd $$\left(\operatorname{Im} J_{(p)}\right)_{2(p-1)p^k s - 1} \cong \mathbb{Z}/p^{k+1}$$

Global answer $$\left|(\operatorname{Im} J)_{4k-1}\right| = \operatorname{denom}\left(\frac{B_k}{2k}\right)$$

Method: K-thy.

bott periodicity: $BO \times \mathbb{Z} \simeq \Omega^\infty KO$

$\pi_* KO =$

$\cdots \mathbb{Z}\ 0\ 0\ 0\ \mathbb{Z}\ \mathbb{Z}/2\ \mathbb{Z}/2\ 0\ \mathbb{Z}\ 0\ 0\ 0\ \mathbb{Z}\ \mathbb{Z}/2\ \mathbb{Z}/2\ 0 \cdots$

$-4 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 8$

X cpt: $KO^0(X) = \mathbb{Z}\{\mathrm{Vect}_{\mathbb{R}}(X)\} \big/ [V]+[W] = [V\oplus W]$

[Recall: Sp = Spectra $\quad E = \{E_i\} \quad E_i \xrightarrow{\simeq} \Omega E_{i+1}$

$E_i \in \mathrm{Top}_*$

$\updownarrow$

Coh thy's $\quad E^i(X) \cong [X, E_i]$

$E_* := \pi_* E$

$\pi_i E = E^{-i}(pt) = \pi_{i+k}(E_k)$

(can be nonzero for i negative)]

Adams' technique

$J_p \longrightarrow KO_p^\wedge \xrightarrow{\psi^\ell - 1} KO_p^\wedge \qquad \ell$ = chosen "well" wrt p

$\uparrow$

S

$$(\mathrm{Im}\, J)^\wedge_p \hookrightarrow (\pi_*^s)^\wedge_p \longrightarrow \pi_* J_p \qquad \text{split injection.}$$

"K-thy detects Im J"

ANSS's $\qquad E \qquad \mathrm{Ext}_{E_*E}(E_*, E_*(X)) \Rightarrow \pi_* X_E$

X = spectrum (e.g. $\Sigma^\infty Y$) $\qquad X_E$ = E-localization of X

$\pi_* X_E$ = "part of $\pi_* X$ detected by E-homology and its (higher) cohomology operations"

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e.g. $\quad E = H\mathbb{F}_p \qquad \mathrm{Ext}_A(H^*(X), \mathbb{F}_p) \Rightarrow \pi_*(X)^\wedge_p$

$\pi_{<0} X = 0$

$E = K^\wedge_p \qquad p$ odd

$$X_{K^\wedge_p} = J_p$$

$E = MU$ ← complex cobordism

$$\mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \Rightarrow \pi_*^s$$

Quillen gives description of this.

comm. 1-dim'l

Quillen: gives description of this.

R = Ring

A 1-dim'l comm. formal gp F/R is a power series

$$x +_F y \in R[[x,y]] \qquad x +_F y = \sum_{i,j} a_{ij} x^i y^j \qquad a_{ij} \in R,$$

satisfying

$$x +_F 0 = 0 +_F x = x$$
$$(x +_F y) +_F z = x +_F (y +_F z)$$
$$x +_F y = y +_F x$$

e.g. add. $x +_F y = x + y$

mult. $x +_F y = x + y + xy$

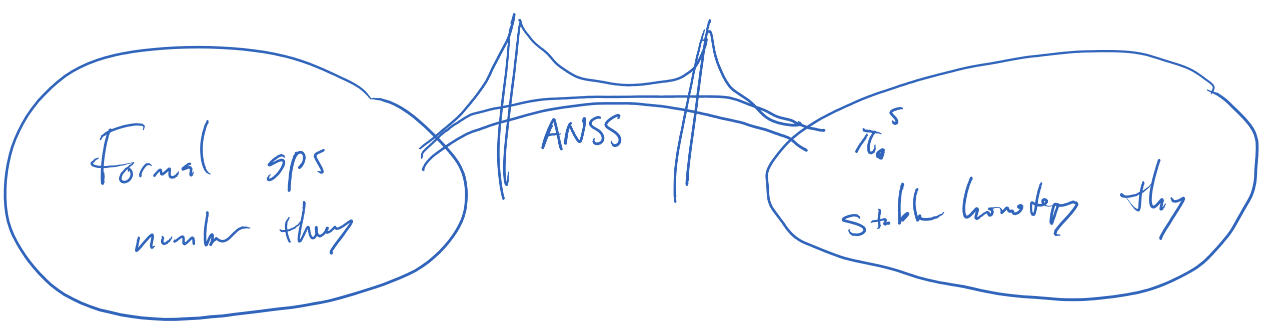
$\mathcal{M}_{FG}$ = "moduli space of formal gps" (algebro-geometric object)

Thm (Quillen)

$$\mathrm{Ext}^*_{MU_*MU}(MU_*, MU_*) \cong H^*(\mathcal{M}_{FG})$$

Morava's Idea Use arithmetic & geometry of formal groups to describe π_*^s (or more generally $\pi_*^s X$)

"Chromatic homotopy thy"



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key feature: k = fld char. p

F = formal gp$/k$

$$[p](x)_F := \underbrace{x +_F \cdots +_F x}_{p} = \underbrace{p}_{\text{zero}} x + \cdots$$

$$= u x^{p^h} + \cdots$$

$u \neq 0 \in K$

"F has height h"

$$\mathcal{M}_{FG} = \bigcup \mathcal{M}_{FG}^{ht \leq h}$$

$$\Longrightarrow \quad H^*(\mathcal{M}_{FG}) = \varprojlim H^*(\mathcal{M}_{FG}^{ht \leq h})$$

super complicated — less complicated.

Inductively get from $H^*(\mathcal{M}_{FG}^{ht \leq h-1})$ to $H^*(\mathcal{M}_{FG}^{ht \leq h})$

by $H^*(\mathcal{M}_{FG}^{ht = h})$

"computable" and <u>periodic</u>.

e.g. $\quad H^*(\mathcal{M}_{FG}^{ht=1}) = \mathrm{Ext}_{K_*K} \Longrightarrow \pi_* S_{K_p^\wedge} = \pi_* J_p$