1 - Stable Homotopy Theory

Monday, August 31, 2015 6:29 AM

Goal: Sp, $Ho(Sp)$, ...

Preliminary: (co)Localization

$\mathcal{C}$ = Cat W = class of morphisms "equivalences"

Wish to form $\mathcal{C}[W^{-1}]$ (e.g. $Top/\simeq \; [w.e.^{-1}]$)

Localization: Def $Z \in \mathcal{C}$ is W-local if $\forall f: X \to Y$

$$f^*: \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

is an iso

(i.e. $\mathcal{C}(-, Z)$ takes equivalences to isos)

Lemma (Whitehead)

$f: Z \to Z'$ equivalence between two W-local objects

$\implies f$ iso (prove)

Def: A W-localization functor is a functor $(-)_W : \mathcal{C} \to \mathcal{C}$

and a natural transformation $\eta_X : X \longrightarrow X_W$

s.t. (1) X_W is W-local

(2) $\eta_X \in W$

Prop: Suppose $\exists$ W-localization functor.

$$\implies \mathcal{C}[W^{-1}] = \mathcal{C}^{W\text{-}loc}$$ (prove)

Co-localization Z = W-colocal $\iff$ $\mathcal{C}(Z, -)$ takes equivs

Co-localization $Z = W$-colocal $\iff$ $\mathcal{C}(Z,-)$ takes equiv's to isos
$\shortparallel$
localization in $\mathcal{C}^{op}$ Whitehead ✓ equiv between colocal $\Rightarrow$ equiv.

$$\mathcal{C}[W^{-1}] \simeq \mathcal{C}^{W\text{-colocal}}$$

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e.g. $\mathcal{C} = \mathrm{Top}/\simeq$ W = class of weak htpy equivalences.

- CW cx's are W-colocal
- functorial CW-replacement (co-localization functor)

$$\Rightarrow \mathrm{Ho}(\mathrm{Top}) = CW/\simeq \qquad [X,Y] := \mathrm{Top}[w.e.^{-1}](X,Y)$$
$$\shortparallel$$
$$\mathrm{Top}[w.e.^{-1}] \qquad [X,Y]_* := \mathrm{Top}_*[w.e.^{-1}](X,Y)$$

Spectra: Goal: represent cohomology

$\tilde{E}^*: \mathrm{Top}_* \longrightarrow \mathbb{Z}$-graded ab grps, $\sigma_k : \tilde{E}^k(X) \xrightarrow{\simeq} \tilde{E}^{k+1}(\Sigma X)$

is a reduced coh thy if

(1) $\tilde{E}^*(\text{w.h.e.}) = \text{iso}$

(2) $X \xrightarrow{f} Y \to Z$ cofiber sequence $Z = C(f)$

$\Rightarrow$ $\tilde{E}^*(Z) \leftarrow \tilde{E}^*(Y) \leftarrow \tilde{E}^*(X)$ is exact

(3) $\tilde{E}^*(\bigvee_\alpha X_\alpha) \xrightarrow{\simeq} \prod_\alpha \tilde{E}^*(X_\alpha)$

[LES cof]
[LES pair]

$$E^*(X) := \tilde{E}^*(X_+)$$
$$E^*(X,A) := \tilde{E}^*(C(A_+ \hookrightarrow X_+))$$

e.g. $\underline{E} \in \mathrm{Top}_*$ $\quad \tilde{E}(\ ) := [-, \underline{E}]_*$ satisfies (1)–(3)

so $\{\underline{E}_i\}$ $\quad \tilde{E}^i(-) := [-, \underline{E}_i]_*$

$$\sigma_i : \tilde{E}^i(-) \xrightarrow{\cong} \tilde{E}^{i+1}(\Sigma -)$$

$$\Updownarrow$$

$$[-, \underline{E}_i]_* \longrightarrow [\Sigma -, \underline{E}_{i+1}]_* \cong [-, \Omega \underline{E}_{i+1}]_*$$

$$\Updownarrow \text{ Yoneda.}$$

$$\sigma_0 : \underline{E}_i \xrightarrow{\cong} \Omega \underline{E}_{i+1}$$

So $\{\underline{E}_i\}_i$, $\sigma_i : \underline{E}_i \xrightarrow{\cong} \Omega \underline{E}_{i+1}$ gives coh. thy

"Ω-Spectrum"

Thm (Brown) Every coh thy is represented by an Ω-spectrum

$i \in \mathbb{Z}$ $\quad \pi_i E = E^{-i}(*) \cong \tilde{E}^k(S^{i+k}) = \pi_{i+k}(\underline{E}_k)$ $\quad \left(E_* := \pi_* E \text{ coefficients of } E \right)$

e.g. $A \in \mathrm{Ab}$ $\quad HA^*(*) = \begin{cases} A, & * = 0 \\ 0, & \text{o/w} \end{cases}$

regular cohom. coef in A

$\pi_j \underline{HA}_n = \begin{cases} A, & j = n \\ 0, & \text{o/w} \end{cases} \Rightarrow \underline{HA}_n = K(A, n)$

Prop: $\tilde{E}^*(-) \to \tilde{F}^*(-)$ map of coh thys (comp w/ susp isos)

$\pi_* E \to \pi_* F$ iso

$$\Rightarrow \tilde{E}^*(X) \to \tilde{F}^*(X) \text{ iso } \forall X.$$

[Ex]

Sp^{Ω} = category of Ω-spectra

objects: $\{E_i\}$, $\sigma_i : E_i \xrightarrow{\simeq} \Omega E_{i+1}$

Morph: $Sp^{\Omega}(E, E') = \{f_i : E_i \to E'_i \mid$

$$\begin{tikzcd} E_i \arrow[r, "f_i"] \arrow[d, "\simeq"'] & E'_i \arrow[d, "\simeq"] \\ \Omega E_{i+1} \arrow[r, "\Omega f_{i+1}"'] & \Omega E'_{i+1} \end{tikzcd}$$

$\}$

f stable equiv $\Leftrightarrow$ $\pi_* =$ iso

$$Ho(Sp^{\Omega}) = Sp[\text{stable equivalence}]$$

$f, f' : E \to E'$ homotopic if f_i^t, spectrum structure maps.

$t \in [0,1]$

$f_i^0 = f_i$, $f_i^1 = f_i'$

An Ω-spectrum is called "CW" if

(1) E_i pointed CW cx's

(2) $\Sigma E_i \to E_{i+1}$ inclusions of CW cx's

Lemma: CW-Ω-spectra are stably co-local. [Ex]

lemma: any Ω-spectrum is stably equiv to a CW-Ω-spectrum

Cor: $Ho(Sp^{\Omega}) = Sp^{CW\Omega}/\simeq$
