10 - generalized homology of BU

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Lemma: E cx orientable $\quad E_* BU \cong E_*[b_1, b_2, \ldots]$

$E_* \mathbb{C}P^\infty = E_*\{b_0, b_1, \ldots\}$

$b_i = (x^i)^*$

$$b_{i_1} \cdots b_{i_n} \in E_* BU(n) \cong E_*(\mathbb{C}P^\infty)^n / \Sigma_n \cong b_{i_1} \otimes \cdots \otimes b_{i_n} / \Sigma_n$$

$$b_{i_1} \cdots b_{i_n} \mapsto b_{i_1} \cdots b_{i_n} b_0 \in E_* BU(n+1)$$

$\square$ [Exercise - fill in details of this argument]

Prop: $E_* MU \cong E_*[b_1, b_2, \ldots]$

$$E \wedge BU(n)^{V_{univ}} \longrightarrow E \wedge BU(n)_+$$

$$\wr$$

$$E \wedge MU \longrightarrow E \wedge BU_+$$

(map of ring spectra + Bo)

Note: $E \wedge MU$ has two natural cx orientations

$$\mathbb{C}P^\infty \xrightarrow{x_E} \Sigma^2 E \longrightarrow \Sigma^2 E \wedge MU$$

$$\mathbb{C}P^\infty \xrightarrow[x_{MU}]{} \Sigma^2 MU \longrightarrow \Sigma^2 E \wedge MU$$

Exercise: in $(E \wedge MU)^*(\mathbb{C}P^\infty)$, $x_{MU} = \sum_i b_i x_E^{i+1}$

$\mathrm{Ring}(MU, E) =$ cx orientations of E

Fix one cx orientation of E $\quad x_E$

Given $MU \xrightarrow[x_E']{} E$, get $E_* MU \xrightarrow[\varphi]{} E_*$

$x'_E = \sum_i Q(b_i) x_E$

Thm: (Milnor) $\pi_* MU = \mathbb{Z}[x_1, x_2, \ldots]$ $|x_i| = 2i$

pf: use ASS $\mathrm{Ext}_{A_*}(\mathbb{F}_p, H_* MU) \Rightarrow (\pi_* MU)^\wedge_p$ for any p.

$H = H\mathbb{F}_p$
