# 11 - proof of quillen's theorem

Monday, October 12, 2015 7:12 AM

<u>Steenrod alg:</u>

$$\overset{H_*H}{A_*} = \begin{cases} \mathbb{F}_2[\xi_1, \xi_2, \dots], \quad |\xi_i| = 2^i - 1, & p = 2 \\ \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \tau_2, \dots], & p > 2 \end{cases}$$

$|\xi_i| = 2(p^i - 1)$

$|\tau_i| = 2p^i - 1$

<u>Defining property:</u> $H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[t]$ $\quad H_*(\mathbb{RP}^\infty) = \mathbb{F}_2\{\beta_0, \beta_1, \dots\}$

$$\theta x = \sum_k \langle \theta, \xi_k \rangle x^{2^k} \qquad \beta_i = (t^i)^*$$

(Something similar for $p$ odd)

$A$ contains a subalgebra $E = E[Q_0, Q_1, Q_2, \dots]$

dual to the quotient algebra:

$$P_* = \begin{cases} \mathbb{F}_2[\xi_1^2, \xi_2^2, \dots], \\ \\ \end{cases} \qquad \begin{cases} A_*/(\xi_1^2, \xi_2^2, \dots) \cong E[\xi_1, \xi_2, \dots], & p = 2 \\ A_*/(\xi_1, \xi_2, \dots) \cong E[\tau_0, \tau_1, \dots], & p > 2. \end{cases}$$

$$P_* = (A /\!/ E)_*$$

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<u>Lemma:</u> $H = H\mathbb{F}_p$

$$H_*MU \longrightarrow H_*H$$

$$b_i \longmapsto \begin{cases} \begin{Bmatrix} \xi_k^2 & p = 2 \\ \xi_k & p > 2 \end{Bmatrix} & i = p^k - 1 \\ 0, & \text{o/w.} \end{cases}$$

($MU \to H$ classical co orientation.)

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$$\left( \begin{array}{c} MU \to H \\ \text{usual} \\ \mathbb{C}^\infty \text{ orientation.} \end{array} \right) \qquad b_i \longmapsto \begin{cases} \left.\begin{array}{ll} \xi_k^2 & p = 2 \\ \xi_k & p > 2 \end{array}\right\} & i = p^k - 1 \\ 0 & \text{o/w.} \end{cases}$$

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idea:

$$\begin{array}{ccccc} x & H^*(\mathbb{C}P^\infty) & & x & \\ \downarrow & \downarrow & \searrow & \searrow & \\ \sum_i b_i x^{i+1} & (H \wedge MU)^*(\mathbb{C}P^\infty) & \longrightarrow & (H \wedge H)^* \mathbb{C}P^\infty & \sum_k \xi_k^2 x^{2^k} \\ & \cong & & \cong & \\ & \mathbb{F}_2[b_1, b_2, \ldots][[x]] & & A_*[[x]] & \end{array}$$

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Cor: $H_* MU \cong P_*$ as an $A_*$-comodule.

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Use ASS

$$\operatorname{Ext}_{A_*}(\mathbb{F}_p, H_* MU) \Rightarrow \pi_* MU_p^\wedge \cong \mathbb{Z}_p[x_1, x_2, \ldots]$$

$$\cong$$

$$\operatorname{Ext}_A(H^* MU, \mathbb{F}_p)$$

$$\cong$$

$$\operatorname{Ext}_A(A/\!/E, \mathbb{F}_p) \otimes N$$

$$\cong$$

$$\operatorname{Ext}_E(\mathbb{F}_p, \mathbb{F}_p) \otimes N$$

$$=$$

$$\mathbb{F}_p[v_0, v_1, v_2, \ldots] \otimes \mathbb{F}_p[b_i \mid i \neq p^k - 1] \qquad \begin{array}{l} v_0 \Longrightarrow p \\ v_k \Longrightarrow x_{p^k - 1} \end{array}$$

$|v_k| = (1, 2p^k - 1)$

$|b_i| = (0, 2i)$

$b_i \Rightarrow x_i$

$$\begin{array}{ccc} MU_* & \longrightarrow & \prod (MU_*)^\wedge_p \\ \downarrow & & \downarrow \\ (MU_*)_{\mathbb{Q}} & \xrightarrow{\cong} & (\prod (MU_*)^\wedge_p)_{\mathbb{Q}} \\ \downarrow & & \\ H\mathbb{Q}_* MU & & \end{array}$$

$\Rightarrow MU_* = \mathbb{Z}[x_1, x_2, \ldots]$

pf of Quillen

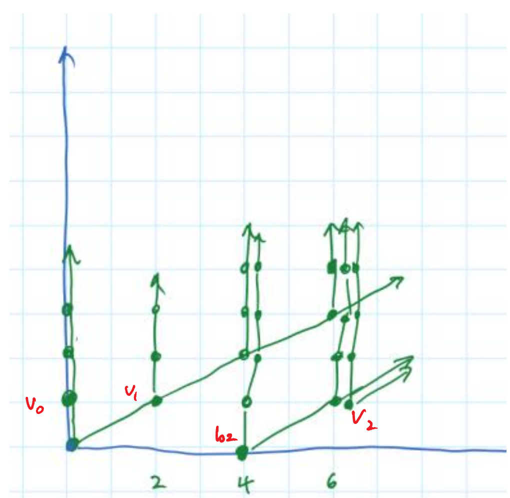
$$\begin{array}{ccccc} \mathbb{Z}[y_1, y_2, \ldots] = & L & \xrightarrow{F_{univ}} & MU_* & = \mathbb{Z}[x_1, x_2, \ldots] \\ & \downarrow & & \downarrow & \\ \mathbb{Z}[b_1, b_2, \ldots] = & \bar{L} & \xrightarrow[F_{univ} \cong F_{add}]{\cong} & H\mathbb{Z}_* MU & = \mathbb{Z}[b_1, b_2, \ldots] \end{array}$$

$$y_i \mapsto \begin{cases} p b_i + \text{decomp} & i = p^k - 1 \\ b_i + \text{decomp} & \text{o/w} \end{cases}$$

$$x_i \mapsto \begin{cases} p b_i + \text{decomp} & i = p^k - 1 \\ b_i + \text{decomp} & i \neq p^k - 1 \end{cases}$$

Map of ASS's

$$\begin{array}{ccc} \pi_* X & \longrightarrow & H\mathbb{Z}_* X \\ \Uparrow & & \Uparrow \\ \mathrm{Ext}_E \otimes N & \longrightarrow & \mathrm{Ext}_{E(0)} \otimes P_* \otimes N \end{array}$$




(no room for diff'ls)

$\Rightarrow \pi_* MU^\wedge_p \cong \mathbb{Z}^\wedge_p [x_1, x_2, \ldots]$

$v_0 \Rightarrow p$

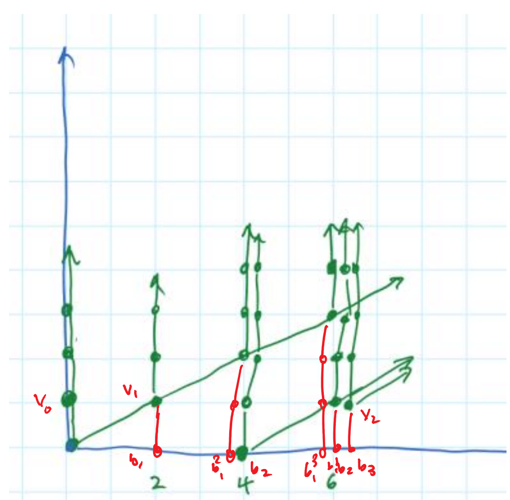
$v_i \Longrightarrow x_{p^i-1}$

$b_i \Longrightarrow x_i \qquad i \neq p^j - 1$

Thm (May-Milgram)

ASS for $H\mathbb{Z}\wedge X$ is the BSS.

ASS for $H\mathbb{Z}\wedge MU$




$v_i \longmapsto p\, b_{p^i-1}$ mod decomposables

(no room for diff'ls)

Note: (1) $H\mathbb{Z}_*X$ torsion-free $\Rightarrow$ ASS for $H\mathbb{Z}\wedge X$ collapses

$\Longrightarrow$ (2) $v_0^{-1}\,\mathrm{Ext}_{E[Q_0]} \otimes P_* \otimes N \Rightarrow (H\mathbb{Q}_p^\wedge)_* MU$ collapses.

$$v_0^{-1} \operatorname{Ext}_E \otimes N \Longrightarrow \pi_* MU_p^\wedge \otimes \mathbb{Q}$$

(3) deduce $v_i \longmapsto v_0 b_{p^i - 1}$ mod decomposables.

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Now, the proof follows from analysis of:

$$\begin{tikzcd} L \arrow[r] \arrow[d] & MU_* \arrow[d] \\ \bar{L} \arrow[r] & H\mathbb{Z}_* MU \end{tikzcd}$$

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**Thm** $\forall\ R$,

$$\Big(\operatorname{Spec}(MU_*)(R),\ \operatorname{Spec}(MU_*MU)(R)\Big)$$

is the groupoid:

objects: $FGL(R)$

Morph: strict isos.

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