12 - BP

Wednesday, October 28, 2015 7:52 AM

F - formal gp$/R$ $\qquad R = \mathbb{Z}_{(p)}$-alg (torsion-free)

F is <u>p-typical</u> if $\log_F(x) = \sum' \ell_i x^{p^i}$

<u>Thm</u>

$$\begin{tikzcd} F \arrow[rr, "f"] \arrow[dr, "\sum m_i x^{i+1}"'] & & F_{p\text{-typ}} \arrow[dl, "\sum \ell_i x^{p^i}"] \\ & F_{add} & \end{tikzcd}$$

$$\log_{F_{p\text{-typ}}}(f(x)) = \log_F(x)$$

or $$f(x) = \exp_{F_{p\text{-typ}}}(\log_F(x))$$

$$= \sum_i{}^{F} \gamma_i \left(\sum m_j x^{j+1}\right)^{p^i}$$

<u>Consider:</u> $\quad f_\ell : F \longrightarrow F_\ell \qquad \ell \neq p$

$$f_\ell^{-1}(x) = x -_F \left[\tfrac{1}{\ell}\right]_F \left(\sum_{i=1}^{\ell}{}^{F} \zeta^i x\right)$$

$$F \xrightarrow{f} F_\ell$$

$$\log \searrow \quad \swarrow \log_\ell$$
$$F_{add}$$

$$\log_\ell(x) = \log_F\left(f_\ell^{-1}(x)\right) = \log_F(x) - \frac{1}{\ell}\left(\sum_{i=1}^{\ell} \log_F(\zeta^i x)\right)$$

$$= \sum_{i,j} m_j \frac{1}{\ell}\left(1 - \zeta^{(j+1)i}\right) x^{j+1}$$

$$-1 \begin{pmatrix} 0 \\ \omega \end{pmatrix} . \qquad = \sum_j \begin{cases} 0, & \ell \mid j+1 \\ m_j, & \ell \nmid j+1 \end{cases} x^{j+1}$$

Do this for all $\ell \neq p$

Get $f_{p\text{-typ}} : F \longrightarrow F_{p\text{-typ}}$

$$\log_{F_{p\text{-typ}}}(x) = \sum_i \lambda_i x^{p^i}$$

So every formal gp law can be uniquely "p-typicalized" $(F_{p\text{-typ}})_{p\text{-typ}} = F_{p\text{-typ}}$

<u>Universal case:</u>

$$MU_{(p)} \xrightarrow[\varepsilon]{} MU_{(p)} \qquad \varepsilon^2 = \varepsilon$$

$$BP \hookrightarrow MU_{(p)}$$

lemma:

$f \in R[[x]]$

$f : F_1 \longrightarrow F_2$

F_1 p-typical

F_2 is p-typical (iff if f is an iso)

$$\Rightarrow \quad f(x) = \sum^{F_2} \overline{t}_i \, x^{p^i}$$

Cor:

$$[p]_F(x) = \sum^F v_i \, x^{p^i}$$

$$p \log(x) = \sum \lambda_i v_j^{p^i} x^{p^{i+j}}$$

e.g. $m_1 p^p + v_1 = p m_1$

$\Rightarrow \quad m_1 = \dfrac{v_1}{p - p^p}$

$$p \sum \lambda_k x^{p^k} \quad \Rightarrow \quad p \lambda_k = \sum_{i+j=k} \lambda_i v_j^{p^i}$$

get

classifying map for p-typ.

$$\mathbb{Z}_{(p)}[v_1, v_2, \dots] \longrightarrow V \hookrightarrow \overline{V}$$

$$
\begin{tikzcd}
\mathbb{Z}_{(p)}[v_1, v_2, \dots] \arrow[r] & V \arrow[r, hook] & \overline{V} \arrow[d, phantom, "\shortparallel"] \\
 & & \mathbb{Z}_{(p)}[\lambda_1, \lambda_2, \dots] \\
 & L \arrow[uu] \arrow[r] & \overline{L} \arrow[u]
\end{tikzcd}
$$

(1) $H\mathbb{F}_{p*}BP = \mathbb{F}_{p*}[\lambda_1, \lambda_2, \dots]$

$$
\begin{tikzcd}
H\mathbb{F}_{p*}MU = \mathbb{F}_{p*}[b_1, b_2, \dots] = E_*[m_1, m_2, \dots] \arrow[u, mapsto]
\end{tikzcd}
$$

$$x_{MU} = \sum b_i x_E^{i+1}$$

$$x_E = \sum m_i x^{i+1}$$

(2) $\pi_* BP = \mathbb{Z}_{(p)}[\tilde{v}_1, \tilde{v}_2, \dots]$

(3)
$$
\begin{tikzcd}
\mathbb{Z}_{(p)}[\tilde{v}_1, \dots] \arrow[r, "\sim"] & \pi_* BP \arrow[r] \arrow[d, phantom, "\shortparallel"] & \overline{V} \\
 & V &
\end{tikzcd}
$$

natural iso

$\Rightarrow$ inj

hits indecomposables = surj

$$[p]_{F_R}(x) = \sum^{F_R} \eta_R(v_i) x^{p^i}$$

$$\begin{tikzcd} F_L & F_R \arrow[l, "f^{-1}"'] \arrow[d, "\log_{F_R}"] \\ & F_{add} \arrow[from=1-1, "\log_{F_L}"'] \end{tikzcd}$$

$$\log_{F_R}(x) = \sum_i \eta_R(m_i)\, x^{p^i}$$

$$\log_{F_L}(f^{-1}(x)) = \log_{F_R}(x) = \sum_i \eta_R(m_i)\, x^{p^i}$$

$$\shortparallel$$

$$\log_{F_L}\left(\sum^{F_L} t_i x^{p^i}\right)$$

$$\shortparallel$$

$$\sum m_j t_i^{p^j} x^{p^{i+j}}$$

So $\quad \eta_R(m_i) = \sum_{i_1+i_2=i} m_{i_1} t_{i_2}^{p^{i_1}}$

e.g. $\quad \eta_R(m_1) = t_1 + m_1$

$$\Longrightarrow \quad \eta_R(v_1) = \eta_R\big((p-p^p)m_1\big) = (p-p^p)\,t_1 + v_1$$

$$\equiv v_1 + p t_1 \mod p^2$$

Ψ: $\quad BP_*[t_1, t_2, \ldots) \xrightarrow{\psi} BP_*[t_1, \ldots, s_1, \ldots]$

$$F_0 \xrightarrow{f_1} F_1 \xrightarrow{f_2} F_2$$

$$f_2^{-1} = \sum_i^{F_1} s_i x^{p^i}$$

$$f_1^{-1} = \sum_i^{F_0} t_i x^{p^i}$$

$$f_1^{-1}(f_2^{-1}(x)) = \sum_i^{F_0} \psi(t_i)\, x^{p^i}$$

$$\shortparallel$$

$$f^{-1}\left(\sum^{F_1} s_i x^{p^i}\right)$$

$$\| $$

$$f^{-1}\left(\sum\nolimits^{F_1} s_i x^{p^i}\right)$$

$$\| $$

$$\sum\nolimits^{F_0} f_1^{-1}(s_i x^{p^i})$$

$$\| $$

$$\sum\nolimits^{F_0} t_i s_j^{p^i} x^{p^{i+j}}$$

Apply $\log_{F_0}$

$$\sum m_i t_j^{p^i} s_k^{p^{i+j}} x^{p^{i+j+k}} = \sum m_j \psi(t_j)^{p^i} x^{p^{i+j}}$$

e.g.

$$m_1 + t_1 + s_1 = m_1 + \psi(t_1)$$

$$\Longrightarrow \psi(t_1) = t_1 \otimes 1 + 1 \otimes t_1$$

$$\psi(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1^p$$

$$+ \frac{v_1}{(p^{p-1}-1)} \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} t_1^i \otimes t_1^{p-i}$$
