

2 - Stable homotopy theory, cont

Monday, August 31, 2015 1:13 PM

Unfortunately, many spectra are not presented to us as Ω -spectra...

Def! A spectrum is a $\{E_i\}$, $\sigma_i: E_i \rightarrow \Omega E_{i+1}$ ^{just a map}
 or equiv.
 $\tilde{\sigma}_i: \Sigma E_i \rightarrow E_{i+1}$

e.g. suspension spectrum

$$X \in \text{Top}_* \quad \Sigma^\infty X \in \text{Sp} \quad \underline{\Sigma^i X} = \Sigma^i X$$

(agrees w/ τ_0 for Ω -spectra)

$$\tau_0 E = \varinjlim \tau_{i+n} E_n$$

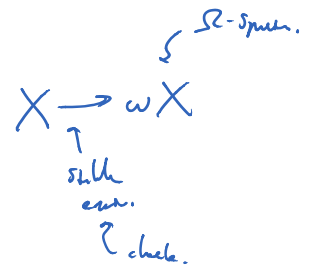
(e.g. $\tau_0 \Sigma^\infty X = \tau_0^S X$)

stable equiv. $\Leftrightarrow \tau_0$ -iso

Prop! Every spectrum is stably equiv to an Ω -spectrum.

Lemma! X CW spect

$$\omega X_i := \varinjlim_n \Omega^n X_{i+n}$$



Lemma! every spectrum is levelwise equiv to CW spectrum.

$$\text{SHC} = \text{Ho}(\text{Sp})$$

maps in SHC will be denoted $[X, Y]$

$$\begin{aligned}
& S_p \text{ (stable equiv)} \\
& \left(S_p \text{ [localise equiv]} \right) \text{ [stable equiv]} \\
& = \left(S_p^{CW} / \simeq \right) \text{ [stable equiv]} \\
& = S_p^{CW} / \simeq
\end{aligned}$$

$$\begin{aligned}
\Sigma^{\infty} : \mathcal{T}op_* &\Rightarrow Sp : \Omega^{\infty} \\
\Sigma^{\infty} : Ho(\mathcal{T}op_*) &\Rightarrow Ho(Sp) : \Omega^{\infty}
\end{aligned}$$

Aside
derived functors ----

Derived functors

$\mathcal{C} \supset \mathcal{W}$ subcat of "weak equivalences"

$$Ho(\mathcal{C}) := \mathcal{C}[\mathcal{W}^{-1}] \quad (\text{if it exists})$$

$F: \mathcal{C} \rightarrow \mathcal{D}$ is typically the homology category of \mathcal{C} (same other category)

Right derived functor

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow \iota & \searrow \dashrightarrow & \uparrow \\
Ho(\mathcal{C}) & & RF
\end{array}$$

$$F \xrightarrow{\eta} RF \hookrightarrow \mathcal{C}$$

Given any other G , $\mu: F \rightarrow G \hookrightarrow \mathcal{C}$

$$\exists! \delta: RF \rightarrow G$$

$$\text{s.t.} \quad \begin{array}{ccc} & F & \\ \eta \downarrow & & \downarrow \mu \end{array}$$

typical implementation:

$$\begin{array}{c}
 F(x) \\
 \downarrow \alpha \\
 RF(x) = F(\rho(x))
 \end{array}$$

eg. $\text{holim} \leftarrow$

Find subset e' of e

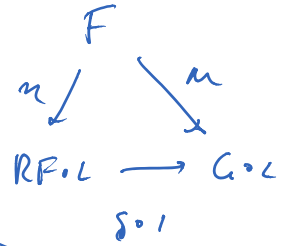
$$e \xrightarrow{\rho} e'$$

$$\begin{array}{c}
 \text{Id} \rightarrow \rho \\
 \uparrow \text{homotopy} \\
 \text{m.e.}
 \end{array}$$

$F(\rho(f))$ is an iso $\forall f \in W$

(e.g. if $F|_{e'}$ puts m.o. to iso)

s.t.



eg: $\text{R}\Omega^{\infty} E = E_0$ for Ω -spectra