2 - Stable homotopy theory, cont

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Unfortunately, many spectra are not presented to us as Ω-spectra...

Def: A spectrum is a $\{E_i\}$, $\sigma_i : E_i \xrightarrow{\text{just a map}} \Omega E_{i+1}$

or equiv.

$\tilde{\sigma}_i : \Sigma E_i \to E_{i+1}$

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e.g. suspension spectrum

$X \in Top_*$ $\quad \Sigma^\infty X \in Sp$ $\quad \underline{\Sigma^\infty X}_i = \Sigma^i X$

(agrees w/ π_* for Ω-spectra) $\quad \pi_i E = \varinjlim \pi_{i+n} E_n$ $\quad$ (e.g. $\pi_* \Sigma^\infty X = \pi_*^s X$

stable equivalence $\Longleftrightarrow$ π_*-iso

Prop: Every spectrum is stably equiv to an Ω-spectrum.

Lemma: X CW spectrum

$\underline{wX}_i := \varinjlim_n \Omega^n X_{i+n}$

$X \to wX$ (Ω-spectrum; stable equiv. check.)

Lemma: every spectrum is levelwise equiv to CW spectrum.

$SHC = Ho(Sp)$ $\quad$ maps in SHC will be denoted $[X, Y]$

$\simeq$

$Sp[\text{stable equiv}^{-1}]$

$\simeq$

$\left(Sp[\text{levelwise equiv}^{-1}]\right)[\text{stable equiv}^{-1}]$

$= \left(Sp^{CW}/\simeq\right)[\text{stable equiv}^{-1}]$

$= Sp^{CW\ell}/\simeq$

$\Sigma^\infty : Top_* \rightleftarrows Sp : \Omega^\infty$

$\Sigma^\infty : Ho(Top_*) \rightleftarrows Ho(Sp) : \Omega^\infty$

Aside on derived functors....

Derived functors

$\mathcal{C} \supset W$ subcat of "weak equivalences"

$Ho(\mathcal{C}) := \mathcal{C}[W^{-1}]$ (if it exists)

$F : \mathcal{C} \to \mathcal{D}$ ⟿ ($\mathcal{D}$ is typically the homotopy category of some other category)

Right derived functor

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\Rrightarrow"] & \mathcal{D} \\ Ho(\mathcal{C}) \arrow[ur, dashed, "RF"'] & \end{tikzcd}$$

$F \xrightarrow[\eta]{} RF \circ \iota$

Given any other G, $\mu : F \to G \circ \iota$

$\exists!\ \delta : RF \to G$

s.t. F

$\eta \swarrow \quad \searrow \mu$

Typical implementation:

$$F(X) \xrightarrow{\eta} RF(X) = F(\rho(X))$$

Find subcat $\mathcal{C}'$ of $\mathcal{C}$

$\mathcal{C} \xrightarrow{\rho} \mathcal{C}'$

$\mathrm{Id} \to \rho$ natural w.e.

$F(\rho(f))$ is an iso $\forall f \in W$

(e.g. if $F|_{\mathcal{C}'}$ sends w.e. to isos)

s.t.

$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\mu"] & \\ RF \circ L \arrow[rr, "\delta \circ l"'] & & G \circ L \end{tikzcd}$$

e.g. $\underleftarrow{\mathrm{holim}}$

e.g.: $R\Omega^\infty E = E_0$ for Ω-spectra