3 - stable homotopy theory, cont

Monday, September 7, 2015 8:32 AM

Defn: $\left.\begin{array}{l} E \wedge X \\ E^X \end{array}\right\}$ ... $E \in Sp$, $X \in Top$.

Lemma: π_*^s is a homology theory (Homotopy excision)

$\Sigma^1 E = E \wedge S^1$

$\Omega E = E^{S^1}$.

Lemma: $\left.\begin{array}{l} E \to \Omega\Sigma^1 E \\ \Sigma^1 \Omega X \to E \end{array}\right\}$ - are equiv's

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ E \wedge I \arrow[r] & C(f) \end{tikzcd} \qquad \begin{tikzcd} F(f) \arrow[r] \arrow[d] & (E')^I \arrow[d] \\ E \arrow[r, "f"] & E' \end{tikzcd}$$

$\Sigma^{-i} E := \Omega^i E$

$S^i := \begin{cases} \Sigma^\infty S^i \\ \Omega^i \Sigma^\infty S^0 \end{cases}$

$S := \Sigma^\infty S^0$

$\pi_* E = [S^*, E]$

$X \to Y \to C(f) \to \Sigma^1 X \to \cdots$

$\cdots \to \Omega Y \to F(f) \to X \to Y$

$$\begin{tikzcd} \Sigma\Omega F(f) \arrow[r] \arrow[d] & \Sigma\Omega X \arrow[r] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma^1 F(f) \arrow[d] \\ F(f) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C(f) \arrow[d] \\ \Omega C(f) \arrow[r] & \Omega\Sigma^1 X \arrow[r] & \Omega\Sigma^1 Y \arrow[r] & \Omega\Sigma^1 F(f) \end{tikzcd}$$

$\Rightarrow$ LES on π_*

$Ho(Sp)$ is Δ'd

$E^{(-)}$, $E \wedge (-)$ preserves cofiber sequences.

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X \vee Y \arrow[r] \arrow[d] & Y \arrow[d, equal] \\ X \arrow[r] & X \times Y \arrow[r] & Y \end{tikzcd}$$

$\Rightarrow X \vee Y \simeq X \times Y$

$\Rightarrow [Z, X]$ is ab grp

E = CW-spectrum $\Rightarrow$ $(E^{[k]})_i = E_i^{[k+i]}$

$E \simeq \varinjlim_k E^{[k]}$

Exercise:

$E^{[k-1]} \to E^{[k]} \to \underset{\text{"\{k-cells of E\}"}}{\bigvee} S^k$

"skeletal filtration"

Defn: $\tilde{E}^* X = \pi_{-*} E^X = [\Sigma^\infty X, \Sigma^* E]$

$\tilde{E}_* X = \pi_* E \wedge X$

$\left(E = S \qquad \tilde{S}_* X = \pi_*^s X \right)$

$E = HA$

$\widetilde{HA}_*(X) = \tilde{H}_*(X; A)$

Smash Product + function spectra:

$K \longmapsto [X \wedge K, \Sigma^i Y]$ is a coh thy

$\Rightarrow \exists \; F(X,Y) \in Sp$

$$[X \wedge K, \Sigma^i Y] \cong [\Sigma^\infty K, F(X, \Sigma^i Y)]$$

$$[X \wedge K, Y] \cong [\Sigma^\infty K, F(X,Y)]$$

$$\left(\text{Note} \quad F(\Sigma^\infty K, Y) \simeq Y^K \quad \pi_* F(X,Y) \cong [\Sigma^* X, Y] \right)$$

Thm: $F(-,-)$ lifts to a functor $Ho(Sp^{op}) \times Ho(Sp) \to Ho(Sp)$

has left adjoint: $- \wedge - : Ho(Sp) \times Ho(Sp) \to Ho(Sp)$

$$[X \wedge Y, Z] \cong [X, F(Y,Z)]$$

Note: $Ho(Sp)$ symmetric monoidal $- \wedge -$

- $S = \Sigma^\infty S^0 \qquad X \wedge S = X$

- $X \wedge \Sigma^\infty K \simeq X \wedge K$

$- \wedge -$ & $F(-,-)$ preserve cofiber sequences in each variable.

$E, X \in Sp$

$$E^*(X) = \pi_{-*} F(X,E) = [X, \Sigma^* E]$$

$$E_*(X) = \pi_* E \wedge X$$

$K \in Top_*$

$\tilde{E}^*(K) = E^*(\Sigma^\infty K)$

$\tilde{E}_*(K) = E_*(\Sigma^\infty K)$

Ex construct these.

AHSS

$$H_*(X; E_*) \Rightarrow E_* X$$

$$H^*(X; E^*) \Rightarrow E^* X$$

(use skeletal filtration on X)