4 - Adams spectral sequence

Wednesday, September 9, 2015 6:59 AM

E is a ring spectrum if

$$\begin{tikzcd} E\wedge E \arrow[r, "\mu"'] & E \\ S \arrow[r, "\eta"'] & E \end{tikzcd}$$

$$\begin{tikzcd} E\wedge E\wedge E \arrow[r, "1\wedge\mu"] \arrow[d, "\mu\wedge 1"'] & E\wedge E \arrow[d, "\mu"] \\ E\wedge E \arrow[r, "\mu"'] & E \end{tikzcd}$$

$$\begin{tikzcd} E = S\wedge E \arrow[r] \arrow[dr, equal] & E\wedge E \arrow[d] & E\wedge S = E \arrow[l] \arrow[dl, equal] \\ & E & \end{tikzcd}$$

(commutative

$$\begin{tikzcd} E\wedge E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E\wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

)

e.g. HR R = ring.

$\Sigma^\infty X_+$ X - H-space.

E = (commutative) ring spectrum $\Rightarrow$ $\pi_* E$ = graded commutative ring

E = (comm) ring spectrum

$X \in Top$ $\Rightarrow$ E^{X_+} = (comm) ring spectrum

$$S \longrightarrow E = E^{S^0} \longrightarrow E^{X_+}$$

$$E^{X_+}\wedge E^{X_+} \longrightarrow (E\wedge E)^{X_+\wedge X_+} = (E\wedge E)^{(X\times X)_+} \xrightarrow[\Delta^*]{} (E\wedge E)^{X_+} \xrightarrow[\mu_*]{} E^{X_+}$$

$\Rightarrow$ $E^*(X)$ is a (graded comm ring)

$\quad\; \parallel$

$\pi_* E^{X_+}$ (actually an E_*-alg)

Exercise: E = ring spectrum $E\wedge X$ flat$/E_*$

$X \in Sp$

(use skeletal filtration on X)

Exercise: E = ring spectrum, $X \in Sp$, E_*X flat$/E_*$

$\Rightarrow$ $E_*X \otimes_{E_*} E_*Y \cong E_*(X\wedge Y)$

(use skeletal filtration on X)

(Note: $\_\wedge\_ = \_\wedge^{L}\_$

$X\wedge^{L}Y = \tilde{X}\wedge^{c}\tilde{Y}$.)

Hurewicz homo

$$S \longrightarrow E$$

$\forall X \in Sp \Rightarrow \pi_*X \longrightarrow E_*X$

Adams Spectral sequence: A machine for computing

(a localization of) π_*X out of E_*X

(Topology) (alg analogy)

$$\begin{tikzcd} \bar{E} \arrow[r] & S \arrow[r] & E \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r] & R \arrow[r] & k \end{tikzcd}$$

"augmentation ideal"

$$\begin{tikzcd} \bar{E}\wedge X \arrow[r] & X \arrow[r] & E\wedge X \end{tikzcd} \qquad \begin{tikzcd} IM \arrow[r] & M \arrow[r] & M/IM \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d] & \bar{E}\wedge X \arrow[l] \arrow[d] & \bar{E}^{\wedge 2}\wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\ E\wedge X & E\wedge\bar{E}\wedge X & E\wedge\bar{E}^{2}\wedge X & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d] & IM \arrow[l] \arrow[d] & I^{2}M \arrow[l] & \cdots \\ M/I & IM/I^{2}M & & \end{tikzcd}$$

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unfinished

$$M^{\wedge}_{I} = \varprojlim_{s} M/I^{s}M$$

SS:

$$E_1^{s,t} = \underbrace{\pi_{t-s}\bar{E}^{\wedge s}\wedge X}_{\pi_t(\Sigma^{s}\bar{E}^{s}\wedge X)} \Rightarrow \pi_{t-s}\underbrace{\varprojlim_{s}\left(X/\bar{E}^{\wedge s}\wedge X\right)}_{X^{\wedge}_{E}}$$

$X^\wedge_E$ "E-completion of X"

(e.g. $E = H\mathbb{F}_p$
X = connective spectrum
$\pi_*(X^\wedge_{H\mathbb{F}_p}) = (\pi_* X)^\wedge_p$)

E-localization

Want: $f: X \to Y$ in Sp is an E-equiv. if $E \wedge f : E \wedge X \to E \wedge Y$ is an equiv.

$$Sp[E\text{-equiv}] = Ho(Sp)[E\text{-equiv}] = Ho(Sp)^{E\text{-local}}$$

E-local ⟺ local wrt E-equiv.

Thm (Bousfield) E-localization exists

$$\exists \quad (-)_E : Sp \longrightarrow Sp$$

$$X \longrightarrow X_E$$

E-equiv. E-local

Punchline: "often" $X_E = X^\wedge_E$

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e.g. $E = M\mathbb{Z}[\tfrac{1}{p}] = \underrightarrow{\operatorname{holim}}\left(S \xrightarrow[p]{} S \xrightarrow[p]{} S \longrightarrow \cdots\right)$

$\left(\text{c.f.} \quad \mathbb{Z}[1/p] = \varinjlim\left(\mathbb{Z} \xrightarrow[\cdot p]{} \mathbb{Z} \xrightarrow[\cdot p]{} \mathbb{Z} \longrightarrow \cdots\right)\right)$

$M\mathbb{Z}[p^{-1}]$ . . . my spectra

$M\mathbb{Z}_{(p)}$

$$\pi_* X \wedge M\mathbb{Z}[\wp^{-1}] = \pi_* X[\wp^{-1}]$$

Ex:

$$X \xrightarrow[M\mathbb{Z}[\wp^{-1}]\text{ - equiv}]{} \overset{M\mathbb{Z}[\wp^{-1}]\text{ - local.}}{X \wedge M\mathbb{Z}[\wp^{-1}]} = X_{M\mathbb{Z}[\wp^{-1}]} =: X[\wp^{-1}]$$

E.g. $\wp =$ all primes

$M\mathbb{Q} = H\mathbb{Q} \qquad X_{\mathbb{Q}} = H\mathbb{Q} \wedge X$

$E = M(p) \qquad S \xrightarrow[p]{} S \longrightarrow M(p)$

$$X_{M(p)} = X^{\wedge}_p = \varprojlim_i X \wedge M(p^i)$$

$$\pi_*(X^{\wedge}_p) = (\pi_* X)^{\wedge}_p$$

(If $\pi_* X$ is f.g. in each degree...)

In general, X_E is very mysterious

(e.g. $S_{K\wedge M(p)} \simeq J_p$)

There is always a map

$$X_E \to X^{\wedge}_E$$

Thm (Bousfield) Suppose E is a connective ring spectrum

$(\pi_{<0} E = 0)$

and X is bounded below

$(\pi_{<-N} X = 0)$

If

(1) $\pi_0 E = \mathbb{Z}[\wp^{-1}]$

then $X_E \simeq X^{\wedge}_E = X[\wp^{-1}]$

(2) $\pi_0 E = \mathbb{Z}/p^i$

then $X_E = X^{\wedge}_E = X^{\wedge}_p$

E.g. $E = H\mathbb{Z}$, $E = H\mathbb{F}_p$