6 - Adams spectral sequence E2

Monday, September 21, 2015 6:53 AM

(*) Assume $E_*E$ flat$/E_*$

**Prop:** $E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X)$ (Ext of comodules)

$\shortparallel$

$R^s\mathrm{Hom}_{E_*E}(-,-)$

note: this is graded

$R\mathrm{Hom}^t_{E_*E}(M_*, N_*)$

$\shortparallel$

$f : M_* \to N_{*+t}$

**Lemma** If $M$ is flat$/E_*$

$N = E_*E \otimes_{E_*} N'$ as a comodule "cofree"

$\Rightarrow \mathrm{Ext}^s_{E_*E}(M, N) = 0, \quad s > 0.$

---

**Consequence:** $M$ flat

$N \to I^0 \to I^1 \to \cdots$ resolution of $N$ by cofree comodules $I_s = E_*E \otimes_{E_*} J_s$

$\Rightarrow \mathrm{Ext}^s_{E_*E}(M, N) \cong H^s(\mathrm{Hom}_{E_*E}(M, I^\bullet))$

---

$$\begin{tikzcd}
X \arrow[d] & \bar{E}\wedge X \arrow[l] \arrow[d] & \bar{E}^2\wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
E\wedge X & E\wedge\bar{E}\wedge X & E\wedge\bar{E}^2\wedge X & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \Sigma\bar{E}\wedge X \arrow[dr] & & & & \\
X \arrow[r] \arrow[d, equal] & E\wedge X \arrow[rr, "d_1"'] \arrow[ur] & & \Sigma E\wedge\bar{E}\wedge X \arrow[rr, "d_1"'] \arrow[dr] & & \Sigma^2 E\wedge\bar{E}^2\wedge X \arrow[r, "d_1"'] \arrow[ur] & \cdots \\
X \arrow[ur] & & & & \Sigma^2\bar{E}^2\wedge X \arrow[ur] & &
\end{tikzcd}$$

$E_1^{s,t} = \pi_t(\Sigma^s E\wedge\bar{E}^s\wedge X) =$

**Lemma 1** $0 \to E_*Y \to E_*(E\wedge Y) \to E_*(\Sigma\bar{E}\wedge Y) \to 0$

split exact.

**Lemma 2** $\quad E_*(E\wedge Y) \cong E_*E \otimes_{E_*} E_*Y$

**Lemma 3** $\quad \pi_t(E\wedge Y) = \mathrm{Hom}_{E_*}(E_*, E_*Y)_t = \mathrm{Hom}_{E_*E}(E_*, E_*E\otimes_{E_*} E_*Y)_t$

$$= \mathrm{Hom}_{E_*E}(E_*, E_*(E\wedge Y))_t$$

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & \\
 & & & E_*(\Sigma\bar{E}\wedge X) \arrow[ur] \arrow[dr] & & & & \\
 & E_*X \arrow[r] \arrow[d, equal] & E_*(E\wedge X) \arrow[ur] \arrow[rr] & & E_*(\Sigma E\wedge\bar{E}\wedge X) \arrow[rr] \arrow[dr] & & E_*(\Sigma^2 E\wedge\bar{E}^2\wedge X) & \cdots \\
 & E_*X \arrow[ur] & & & & E_*(\Sigma^2\bar{E}^2\wedge X) \arrow[ur] \arrow[dr] & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

((Cofree resolution of $E_*X$)

Apply $\mathrm{Hom}_{E_*E}(E_*, -)$

$$\begin{array}{ccccc}
\mathrm{Hom}_{E_*E}(E_*, E_*(E\wedge X))_t & \longrightarrow & \mathrm{Hom}_{E_*E}(E_*, E_*(\Sigma E\wedge\bar{E}\wedge X))_t & \longrightarrow & \cdots \\
\wr\Vert & & \wr\Vert & & \\
\pi_t(E\wedge X) & \xrightarrow{\quad d_1 \quad} & \pi_t(\Sigma E\wedge\bar{E}\wedge X) & \xrightarrow{\ d_1\ } & \cdots
\end{array}$$

$$\Longrightarrow \qquad E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X)$$

---

**Cobar complex** $\qquad (\Gamma, A) = $ Hopf algebroid

Cobar complex

(Computing)

$(\Gamma, A)$ = Hopf algebroid

$M$ = right comodule

$N$ = left comodule

$$M \otimes_A N \overset{\xi_M}{\underset{\xi_N}{\rightrightarrows}} M \otimes_A \Gamma \otimes_A N \overset{\xi_M}{\underset{\xi_N}{\overset{\psi}{\rightrightarrows}}} M \otimes_A \Gamma^{\otimes_A 2} \otimes_A N \rightrightarrows \cdots \quad \text{cosimplicial object}$$

$$\rightsquigarrow \quad C^\bullet(M, \Gamma, N) \quad \text{cochain cx}$$

$$M \otimes \Gamma^{\otimes s} \otimes N \longrightarrow M \otimes \Gamma^{\otimes s+1} \otimes N$$

$$m \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \otimes n$$

$$\Downarrow$$

$$m[\gamma_1 | \cdots | \gamma_s] n \longmapsto \sum m'[\gamma'' | \gamma_1 | \cdots | \gamma_s] n$$

$$- \sum m[\gamma_1' | \gamma_1'' | \cdots ] n$$

$$+ \sum m[\gamma_1 | \gamma_2' | \gamma_2'' | \cdots ] n$$

$$\vdots$$

$$\pm \sum m[\gamma_1 | \cdots | \gamma_s | \gamma'] n$$

$$\xi(m) = \sum m' \otimes \gamma''$$

$$\psi(\gamma_i) = \sum \gamma_i' \otimes \gamma_i''$$

$$\xi(n) = \sum \gamma' \otimes n''$$

$C^\bullet(\Gamma, \Gamma, N)$ (1) cofree

(2) $N \xrightarrow{\simeq} C^\bullet(\Gamma, \Gamma, N)$ (extra degeneracy)

$$\mathrm{Hom}_\Gamma(A, C^\bullet(\Gamma, \Gamma, N)) \cong C^\bullet(A, \Gamma, N)$$

$$H^*(C^\bullet(A, \Gamma, N)) \cong \mathrm{Ext}^*_\Gamma(A, N)$$

---