Def: E = comm ring spectrum

A vector bundle $\begin{matrix} V \\ \downarrow \\ B \end{matrix}$ $(\dim_{\mathbb{R}} = d)$ is E-orientable

if $\exists$ $[V] \in E^k(D(V), S(V))$ "Thom class"

s.t. $\forall x \in X$, $[V]_x \in E^k(D(V)_x, S(V)_x) \cong \underset{\pi_{d-k} E}{\overset{\wr\wr}{E^{k-d}(pt)}}$ is a unit. (typically $k = d$)

Thom Isomorphism

$$D(V)/S(V) = X^V \qquad \text{Thom cx.} \qquad [V] \in E^k(X^V)$$

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V \times 0 \arrow[d] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd} \rightsquigarrow X^V \xrightarrow[\Delta]{} (X \times X)^V = X_+ \wedge X^V \qquad \text{"Thom diagonal"}$$

$$E \wedge X^V \xrightarrow[1 \wedge \Delta]{} E \wedge X_+ \wedge X^V \xrightarrow[1 \wedge 1 \wedge [V]]{} E \wedge X_+ \wedge \Sigma^k E \xrightarrow[\mu]{} \Sigma^k E \wedge X_+$$

Thm (Thom Iso)
(Ex)

$$E \wedge X^V \xrightarrow{\sim} \Sigma^k E \wedge X_+$$

$$\left(\tilde{E}^{*+k}(X^V) \cong E^*(X) \right)$$

$$[V]x \longleftarrow x$$

Def: E is cx orientable

if $\begin{matrix} \xi \\ \downarrow \\ \mathbb{C}P^\infty \end{matrix}$ is E-orientable.

(E is <u>cx oriented</u> if we fix: $[L_{univ}] \in \tilde{E}^k((\mathbb{C}P^\infty)^{L_{univ}})$ cx orientation)

E cx oriented $\Rightarrow$ $\forall$ $\begin{matrix} L \\ \downarrow \end{matrix}$ cx line bundle, $\exists$ preferred

E cx-oriented $\Rightarrow$ $\forall$ $\begin{array}{c} L \\ \downarrow \\ B \end{array}$ cx line bundle, $\exists$ preferred orientation $[L] \in \tilde{E}^k(B^L)$

(Easy)

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L_{univ} \arrow[d] \\ B \arrow[r] & \mathbb{C}P^\infty \end{tikzcd}$$

Thm E cx oriented

$\Rightarrow$ every cx V.B. inherits an E-orientation, $[V] \in \tilde{E}^{dk}(B^V)$

$d = \dim_{\mathbb{C}} V$

Prop: E cx orientable $\Rightarrow$ $E^*(\mathbb{C}P^\infty) \cong E^*[[x]]$

Prop: $E^*(BU(d)) \xrightarrow{\cong} E^*((\mathbb{C}P^\infty)^d)^{\Sigma_d} \cong E^*[[c_1^E, \dots, c_n^E]]$

$c_i^E = e_i(c_1^E(L_1), \dots, c_1^E(L_d))$

Use map of AHSS's

$$\begin{tikzcd} X_+ \arrow[r] \arrow[d] & X_+ \wedge X_+ \arrow[d] \\ X^V \arrow[r] & X^V \wedge X_+ \end{tikzcd}$$

[Exercise: make above argument precise]

(pf of thm)

$BU(d-1)_+ \longrightarrow BU(d)_+ \longrightarrow (BU(d))^{V_d^{univ}}$

$0 \leftarrow E^*(BU(d-1)) \twoheadleftarrow E^*(BU(d)) \leftarrow (c_d) \leftarrow 0$

$0 \hookleftarrow c_d$

$\Rightarrow$ $(BU(d))^{V_d^{univ}} \cong (c_d)$ $E^*(BU(d))$-module

(in particular, free of rank 1)