8 - formal group laws

Wednesday, September 23, 2015 7:40 AM

A formal group F over a ring R is a power series

(commutative 1-dim'l)

$$x +_F y \in R[[x,y]]$$

satisfying

$$(1) \quad x +_F 0 = 0 +_F x = x$$

$$(2) \quad (x +_F y) +_F z = x +_F (y +_F z)$$

$$(3) \quad x +_F y = y +_F x$$

Note $\quad x +_F y = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j$

Formal group law $\quad E$ cx oriented of deg k

$$c_i^E(V_1 \oplus V_2) = \sum_{i_1+i_2=i} c_{i_1}^E(V_1)\, c_{i_2}^E(V_2)$$

$$c_1^E(L_1 \otimes L_2) = \sum a_{ij}\, c_1^E(L_1)^i\, c_1^E(L_2)^j = c_1^E(L_1) \, c_1^E(L_2)$$

universal example

$$\begin{array}{ccc} L_1 \otimes L_2 & \longrightarrow & L_{univ} \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

$$a_{ij} = E_{k(i+j-1)}$$

$$\begin{array}{ccc} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) & \longleftarrow & E^*(\mathbb{C}P^\infty) \\ \| & & \| \\ E^*[[x,y]] & & E^*[[x]] \\ x +_E y & \longleftarrow\!\shortmid & x \end{array}$$

Satisfies

$$x +_E 0 = 0 +_E x = x$$

$$x +_E (y +_E z) = (x +_E y) +_E z$$

$$x +_E y = y +_E x$$

Note $\quad x +_E y = x + y + \sum_{i,j>0} a_{ij} x^i y^j$

Examples $\quad E = HR \quad x +_{HR} y = x + y \quad$ additive.

K-thy $\quad K^0(X) = \mathbb{Z}\{\mathrm{Vect}_{\mathbb{C}}(X)\} \big/ [V] + [W] = [V \oplus W]$ (a ring)

$BU \times \mathbb{Z} \quad \Omega \quad BU \times \mathbb{Z} \quad \Omega$

(Fact: K is a comm ring spectrum)

$$\pi_i K = \pi_0 \underline{K}_i = \begin{cases} \mathbb{Z}, & i = \text{even} \\ 0, & i = \text{odd} \end{cases}$$

$\tilde{K}$ = v.b of v.d = 0

$$\underset{\substack{\Downarrow \\ \beta}}{\pi_2 K} = \tilde{K}^0(S^2) = \tilde{K}^0(\mathbb{C}P^1) = \mathbb{Z}\{[\xi] - 1\}$$

$$([\xi] - 1)^2 = 0$$

$\pi_* K = \mathbb{Z}[\beta, \beta^{-1}]$

(cf. $\tilde{H}^2(S^2) = \mathbb{Z}\{x\}$, $x^2 = 0$)

$$\tilde{K}^{i-2}(X) \xrightarrow[\cdot \beta]{\cong} \tilde{K}^i(X)$$

(cx orientation) $\longleftrightarrow$ choice of generator of $E^*((\mathbb{C}P^\infty)^{L_{\mathrm{univ}}})$ "choice of c_1^E"

$$\cong \tilde{E}^*(\mathbb{C}P^\infty)$$

$$\underset{\psi}{\tilde{E}^*(\mathbb{C}P^\infty)} \longrightarrow \tilde{E}^*(\mathbb{C}P^1) = \underset{\psi}{\pi_{-*+2} E}$$

$$\begin{array}{ccc} E^*(\mathbb{CP}^\infty) & \longrightarrow & E^*(\mathbb{CP}^1) = \pi_{-*+2}E \\ \ni & & \ni \\ x & \longmapsto & u \end{array}$$

unit.

$$\begin{array}{ccccc} \tilde{K}^0(\mathbb{CP}^\infty) & \longrightarrow & \tilde{K}^0(\mathbb{CP}^1) & \cong & K_2 \\ \ni & & \ni & & \ni \\ [L_{univ}] - 1 & \longmapsto & [\xi] - 1 & \longmapsto & \beta \end{array}$$

FGL: $\quad x +_K y = x + y + xy \quad$ Multiplicative

Def: a map of FGL's $/R$

$$f : F \to F'$$

is a power series $f(x) \in R[[x]]$

satisfying $\quad f(x +_F y) = f(x) +_{F'} f(y)$

Note, f is an isomorphism if

$$f(x) = ux + \sum_{i \geq 1} b_i x^{i+1} \qquad u \in R^\times$$

i.e. $f'(0) \in R^\times$

Def f is a strict iso if $f'(0) = 1$

i.e. $f(x) = x + \sum_{i \geq 1} b_i x^{i+1}$

Lemma: Let $x +^1_E y$ and $x +^2_E y$ be two formal gps over E_* arising from two different cx orientations of E

then $x +^1_E y \cong x +^2_E y$