9 - complex cobordism

Monday, October 5, 2015 2:35 AM

$$\underline{MU}_n = \begin{cases} BU(k)^{V^k_{univ}}, & n = 2k \\ \Sigma\, \underline{MU}_{n-1}, & n = 2k+1 \end{cases}$$

Ring spectra:

$$BU(k)^{V^k_{univ}} \wedge BU(\ell)^{V^\ell_{univ}} \longrightarrow BU(k+\ell)^{V^{k+\ell}_{univ}}$$

$\underline{MU}_2$: $(\mathbb{CP}^\infty)^{L_{univ}} \longrightarrow \Sigma^2 MU$ (canonical orientation)

$$MU_k(X) = \{M^k \xrightarrow{f} X,\ M \text{ almost cx}\} \big/ \text{almost cx cobordism}$$

Q: what is $\pi_* MU$?

what is $x +_{MU} y$?

Formal group law = formal group up to equality

Formal group = formal gp up to isomorphism.

Lemma: $\exists$ ring L

so that $\mathrm{Ring}(L, R) \cong FGL(R)$

Pf) $L = \mathbb{Z}[a_{ij}] / \text{rels.}$

Let F_{univ} = universal formal gp$/L$

Thm: (Lazard)

$$L \cong \mathbb{Z}[x_1, x_2, \dots]$$

we will not prove this

Thm: (Quillen)

the map $L \longrightarrow MU$ classifying $x +_{MU} y$ is an iso.

(•) $MU_* = \mathbb{Z}[x_1, x_2, \ldots]$ $\quad |x_i| = 2i$

(•) $x +_{MU} y = x +_{F_{univ}} y$

(pf deferred)

Logarithms

Prop every FG over a $\mathbb{Q}$-alg R is uniquely strictly isomorphic to F_{add}

(pf) $\log_F : F \longrightarrow F_{add}$

$$\log_F(x) = \int \frac{dx}{F_y(x,0)}$$

[Exercise - check this works]

Write
$$\log_F(x) = \sum_i m_i x^{i+1}$$
$$\exp_F(x) := \log_F^{-1}(x) = \sum_i b_i x^{i+1}$$

$$x +_F y = \exp_F(\log_F(x) + \log_F(y))$$

Define
$$L = \mathbb{Z}[b_1, b_2, \ldots]$$
$$\exp_{\bar{F}}(x) := \sum b_i x^{i+1}$$
$$\log_{\bar{F}}(x) := \exp_{\bar{F}}^{-1}(x)$$
$$x +_{\bar{F}} y := \exp_{\bar{F}}(\log_{\bar{F}}(x) + \log_{\bar{F}}(y))$$

$\bar{L}$ classifies formal groups strictly isomorphic to F_{add}

Thus $\bar{L} \otimes \mathbb{Q} \cong L \otimes \mathbb{Q}$.

Lemma (Lazard)

Consider the map

$$L \xrightarrow[f]{} \bar{L} \quad \text{classifying } \bar{F}$$

Then

$$f(x_i) \equiv \begin{cases} b_i \text{ mod decomposables}, & i \neq p^k - 1, \ p \text{ prime}, \\ p b_i \text{ mod decomposables}, & i = p^k - 1, \ p \text{ prime}. \end{cases}$$