TOPOLOGICAL AUTOMORPHIC FORMS: A FANTASY

MARK BEHRENS

I jotted down some thoughts on bringing Shimura varieties into stable homotopy theory. I think this sort of idea is in many people's minds right now. Jack Morava convinced Paul Goerss to give a talk about Harris and Taylor's proof of the local langlands correspondence, hoping to start discussion on how to bring the "simple" Shimura varieties considered there into homotopy theory. Paul had suggested in his talk that Jacob Lurie's newly formed techniques might be applicable to these simple Shimura varieties, due to the presense of idempotents that give 1-dimensional summands of the formal groups involved.

These thoughts are an attempt to flesh out Jack's and Paul's suggestions in a simple case. They are really summaries of some discussions with Tyler Lawson, Mike Hopkins, Bob Kottwitz, Johan de Jong, and Jacob Lurie. To ease the passage to the "simple" Shimura varieties considered by Harris and Taylor in their proof of the local Langlands conjecture, in this document I will describe the set-up for some *very* simple Shimura varieties that were of relevance to the Langlands story for GL_2. At the end we say very little bit about the outlook for the Kottwitz-Harris-Taylor Shimura varieties. It should go without saying that this is a very non-rigorous document. The subject of Shimura varieties is so new to me that some things I say here might be outright wrong, so beware!

1. Topological Modular forms

For the sake of comparison, what is tmf, and how its constructed? We use the the modular curve $X(1)$, or more precisely the moduli stack $\mathcal{M}$ of generalized elliptic curves.

1.1. The Hopkins-Miller approach I: $K(2)$-local TMF.

If we only want to model the moduli stack $\mathcal{M}$ near a supersingular point in characteristic p, then Serre-Tate theory and the Hopkins-Miller theorem does all of the work. We have

$$TMF_{K(2)} = \left(\prod_{C \text{ supersingular}} E(\overline{\mathbb{F}}_p, \widehat{C})^{h\,\mathrm{Aut}(C)} \right)^{hGal(\overline{\mathbb{F}}_p/\mathbb{F}_p)}.$$

The product ranges over isomorphism classes of supersingular curves defied over $\overline{\mathbb{F}}_p$. $E(\overline{\mathbb{F}}_p, \widehat{C})$ is the spectrum that the Hopkins-Miller theorem associates to the formal group $\widehat{C}$.

1.2. The Hopkins-Miller approach II: elliptic spectra.

An elliptic spectrum (R, E, ϕ) consists of the following data.

- An even periodic ring-spectrum R, with associated formal group F_R.

Date: February 16, 2005.

- A generalized elliptic curve E defined over R_0, with associated formal group F_E.
- An isomorphism $\phi : F_R \to F_E$.

The spectrum tmf is the connected cover of the homotopy inverse limit

$$\overline{TMF} = \operatorname*{holim}_{\substack{(R,E,\phi) \\ E_\infty,\ \text{étale}}} R.$$

The homotopy inverse limit is over a suitable category of elliptic spectra (R, E, ϕ) so that R is E_∞ and the classifying map $C : \mathrm{spec}(R_0) \to \mathcal{M}$ is étale. Hopkins and Miller needed to show that there are "enough" elliptic spectra so that the homotopy inverse limit is interesting. They originally did this in the A_∞ context, and Goerss and Hopkins upgraded to E_∞.

Let $\mathcal{M}$ be the moduli stack of generalized elliptic curves. Define ω to be the line bundle over $\mathcal{M}$ whose fiber over a point corresponding to a generalized elliptic curve E is given by

$$\omega_E = t_E^*$$

where t_E is the tangent space at the identity of E. Then there is a spectral sequence

$$H^s(\mathcal{M}, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(\overline{TMF}).$$

1.3. **The Lurie approach: the derived moduli stack.** Jacob Lurie modifies the Hopkins-Miller approach with the introduction of "derived" schemes which are built out of E_∞-ring spectra instead of rings. In this way, he can replace the unnatural notion of an elliptic curve E defined over the ring R_0 with the more natural notion of a derived elliptic curve $\mathcal{E}$ defined over the E_∞-ring spectrum R. A derived elliptic curve $\mathcal{E}$ over an E_∞ ring spectrum R is a derived scheme

$$\mathcal{E} \to \mathrm{spec}\, R$$

whose underlying ordinary scheme is an elliptic curve over R_0. $\mathcal{E}$ must also be a very commutative abelian group object. That is to say, $\mathcal{E}$ must be endowed with the structure of a lift of the functor of points $\mathcal{E}(A) = \mathrm{Hom}_R(\mathrm{spec}(A), \mathcal{E})$ through abelian groups, as displayed below.

$$\begin{tikzcd} & \text{Topological abelian groups} \arrow[d] \\ \text{Commutative } R\text{-algebras} \arrow[ur, dashed] \arrow[r, "\mathcal{E}(-)"'] & \text{Spaces} \end{tikzcd}$$

Lurie defines an *enhanced elliptic spectrum* $(R, \mathcal{E}, \phi)$ to be the following data.

- An E_∞-ring spectrum R, with associated derived formal group $F_R = \mathrm{spf}\, R^{\mathbb{C}P^\infty_+}$.
- A derived elliptic curve $\mathcal{E}$ defined over R, with associated derived formal group $F_{\mathcal{E}}$.
- An isomorphism $\phi : F_R \to F_{\mathcal{E}}$.

Lurie proves a derived version of Artin's representablity theorem, and applies it to the functor

$$R \mapsto \{\text{enhanced elliptic spectra } (R, \mathcal{E}, \phi)\}$$

to produce a representing derived stack $\mathcal{M}_{der}$. The spectrum $\overline{TMF}$ is recovered as the global sections of the structure sheaf $\mathcal{O}_{ell}$ (a sheaf of E_∞-ring spectra over $\mathcal{M}$).

$$\overline{TMF} = \Gamma(\mathcal{M}, \mathcal{O}_{ell})$$

Hopkins and his collaborators had already produced the sheaf $\mathcal{O}_{ell}$, but their methods were more ad hoc.

2. From modular curves to Shimura varieties

Milne's 1979 paper "Points on Shimura varieties mod p" gives a beautiful exposition on the Shimura varieties we describe in this section. It's short, completely self-contained, and goes straight to positive characteristic - I heartily recommend it for the poor topologist trying to figure out what the heck a Shimura variety is. Let B be the rational indefinite quaternion algebra ramified at the primes

$$q_1, \dots, q_m$$

and fix a maximal order $\mathcal{O} \subset B$. Define the discriminant Δ to be the product of the q_i. Let $*$ be a positive involution on B. Define the affine group scheme $G = GL_1(\mathcal{O}^{op})$ to have R-points given by

$$G(R) = (\mathcal{O}^{op} \otimes R)^\times.$$

Fix an isomorphism

$$\mathcal{O} \otimes \mathbb{R} \cong M_2(\mathbb{R})$$

so that $*$ corresponds to the transpose. Under this isomorphism we may regard $GL_1(\mathbb{C})$ as being contained in $G(\mathbb{R})$. The adélic points of G is given by a restricted product

$$G(\mathbb{A}) = G(\mathbb{R}) \times \prod_p{}' G(\mathbb{Q}_p).$$

Define K to be the subgroup

$$GL_1(\mathbb{C}) \times \prod_p G(\mathbb{Z}_p).$$

You can the exchange the product above with more arbitrary compact open subgroups of the $G(\mathbb{A}_f)$, and this corresponds to "level structures". We choose this K for simplicity. Consider the adélic quotient

$$S(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K.$$

The resulting complex curve may be given the more familiar description as a quotient of the upper half-plane by a Fuchsian group.

$$S(\mathbb{C}) = \Gamma \backslash \mathcal{H}$$

Here Γ is the group $SL_1(\mathcal{O})$. It turns out that if B is ramified somewhere (i.e. is a division algebra) then $S(\mathbb{C})$ is a projective complex curve. There are no cusps!

If B is split everywhere ($B = M_2(\mathbb{Q})$, $\mathcal{O} = M_2(\mathbb{Z})$) then we have $\Gamma = SL_2(\mathbb{Z})$, so we have

$$S(\mathbb{C}) = SL_1(\mathbb{Z}) \backslash \mathcal{H} = X(1) - \text{cusps}.$$

Therefore $S(\mathbb{C})$ parameterizes elliptic curves over $\mathbb{C}$.

What does $S(\mathbb{C})$ parameterize for more general B? A point of $S(\mathbb{C})$ corresponds to the data $(A, [\lambda], i)$, where

- A is a complex 2-dimensional abelian variety.

- $[\lambda]$ is a $\mathbb{Q}$-equivalence class of polarization $\lambda : A \to A^\vee$.
- i is an inclusion of rings $i : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ such that the involution $*$ on B is compatible with the Rosati involution on $\mathrm{End}^0(A)$.

Remark 2.1. Milne seems to indicate that the weak polarization $[\lambda]$ is determined by i. Therefore, it is redundant data.

We briefly explain this terminology. The dual abelian variety $A^\vee$ is the abelian variety $Pic^0(A)$. An invertible sheaf $\mathcal{L}$ on A gives rise to an isogeny

$$\lambda_{\mathcal{L}} : A \to A^\vee.$$

This isogeny is described as follows. Given $x \in A$, the invertible sheaf $\lambda_{\mathcal{L}}(x) \in A^\vee$ is the pullback of $\mathcal{L}$ under the composite

$$A = \{x\} \times A \hookrightarrow A \times A \xrightarrow{m} A$$

where m is the multiplication map. A polarization is an isogeny $\lambda : A \to A^\vee$ of the form $\lambda_{\mathcal{L}}$ for some $\mathcal{L}$.

Two polarizations λ and λ' are $\mathbb{Q}$-equivalent if, when viewed as being contained in the ring of quasi-isogenies

$$\mathrm{Hom}^0(A, A^\vee) = \mathrm{Hom}(A, A^\vee) \otimes \mathbb{Q}$$

there exists a $c \in \mathbb{Q}^\times$ so that $\lambda' = c \cdot \lambda$. A $\mathbb{Q}$-equivalence class of polarizations is called a weak polarization.

A polarization λ gives rise to an involution $*$ on $\mathrm{End}^0(A)$ called the Rosati involution. Given a quasi-endomorphism α, α^* is the unique quasi-endomorphism making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "\alpha^*"] \arrow[d, "\lambda"'] & A \arrow[d, "\lambda"] \\ A^\vee \arrow[r, "\alpha^\vee"'] & A^\vee \end{tikzcd}$$

The Rosati involution only depends on the $\mathbb{Q}$-equivalence class $[\lambda]$.

It turns out that the functor

$$R \mapsto \left\{ \begin{matrix} \text{weakly polarized abelian varieties over } R \text{ with } \mathcal{O} \text{ multiplication} \\ \text{compatible with the Rosati involution} \end{matrix} \right\}$$

satisfies Artin's representablity theorem if it is restricted to $\mathbb{Z}[\Delta^{-1}]$ algebras R. Thus there is a moduli stack $\mathcal{S}_{(G,K)}$ of this data defined over $\mathbb{Z}[\Delta^{-1}]$ such that the complex points of the associated course moduli space are given by $S(\mathbb{C})$.

There are analogs of the modular curves $X_0(N)$ where we place level structures on the abelian varieties. Enough level makes these moduli stacks representable by schemes.

Note that when B is split and $\mathcal{O} = M_2(\mathbb{Z})$ then there are commuting idempotents e_1, e_2 in $\mathcal{O}$.

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The pairs (A, i) are 2-dimensional abelian varieties with $\mathcal{O}$-multiplication. The idempotents e_i canonically split A as the product of two identical elliptic curves

$$A \cong E \times E.$$

Thus, as expected, $\mathcal{S}_{(G,K)}$ in this case is nothing more than the moduli space of elliptic curves.

3. Characteristic p

3.1. **Mod p points.** Suppose that p does not divide Δ. Consider a mod p point of $\mathcal{S}_{(G,K)}$. That is, we are given a triple $(A, [\lambda], i)$ where A is defined over $\overline{\mathbb{F}}_p$ and possesses $\mathcal{O}$-multiplication. In what follows, we will drop the weak polarization from the notation.

Let t_A be the tangent space of A at the identity, and let $A(p)$ be the p-divisible group of A. Then there is a short exact sequence of p-divisible groups

$$0 \to A(p)_f \to A(p) \to A(p)_{et} \to 0$$

where $A(p)_f$ is the 2-dimensional formal group given by completing A at the identity, and $A(p)_{et}$ is an étale group scheme given by the p-torsion points of A.

Since p does not divide Δ, B is split at p, and there is an isomorphism

$$\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p \cong M_2(\mathbb{Z}_p).$$

There are thus commuting idempotents e_1 and e_2 of $\mathcal{O}_p$ which gives decompositions

$$t_A \cong (t_A)_0 \oplus (t_A)_0$$
$$A(p) \cong A(p)_0 \oplus A(p)_0$$

where $(t_A)_0$ and $A(p)_0$ are 1-dimensional! Thus the structure of $\mathcal{O}$ multiplication on A gives rise to a 1-dimensional formal group $A(p)_{f,0}$.

Milne shows that A is either isogenous to a product of supersingular elliptic curves or a product of ordinary elliptic curves. There is one supersingular isogeny class, and there is one "ordinary" isogeny class for each totally imaginary quadratic extension $E/\mathbb{Q}$ in which p splits and which splits B. If A is in the supersingular isogeny class, then the height of $A(p)_{f,0}$ is 2 and $A(p)_{et,0} = 0$. Otherwise, the height of $A(p)_{f,0}$ is 1, and $A(p)_{et,0} \cong \mathbb{Z}/p$.

It is natural to look at the endomorphisms of A preserving i

$$\mathcal{O}' = \mathrm{End}_{\mathcal{O}}(A).$$

If A is supersingular, then $\mathcal{O}'$ is a maximal order of the definite rational quaternion algebra B' ramified at

$$p, q_1, \ldots, q_m, \infty.$$

If A is ordinary, $\mathcal{O}'$ is an order of the CM field E corresponding to the isogeny class of A.

3.2. **Deformation theory.** Let (A, i) be a mod p point of $\mathcal{S}_{(G,K)}$ as in the last section. Serre-Tate theory says that the category of deformations of A over a complete local ring with residue field $\overline{\mathbb{F}}_p$ is equivalent to the category deformations of the p-divisible group $A(p)$. Presumably, the deformations of A which extend the $\mathcal{O}$ multiplication are the same as deformations of the p-divisible $\mathcal{O}$-module $A(p)$. Since $\mathcal{O}$ is split at p, it would seem that this is the same thing as deformations of the 1-dimensional p-divisible group $A(p)_0$.

Assume that A supersingular. Then $A(p)_0$ is entirely formal and of height 2. Let $\widetilde{A(p)}_0$ be the Lubin-Tate universal deformation of $A(p)_0$ over $\mathbb{W}(\overline{\mathbb{F}}_p)[[u_1]]$. Serre-Tate theory gives a deformation $\widetilde{A}$ of A corresponding to the p-divisible group

$\widetilde{A(p)}_0 \oplus \widetilde{A(p)}_0$. Note that both the $\mathcal{O}$-multiplication on A (and the weak polarization?) may be extended to $\widetilde{A}$ because (1) the Lubin-Tate deformation is functorial and (2) Serre-Tate theory gives an equivalence of categories.

Let $\mathrm{Aut}(A, i)$ be the (finite) group of endomorphisms of A which preserve i. It is the group of units of the maximal order $\mathcal{O}'$. It is also a finite subgroup of the Morava stabilizer group $\mathbb{S}_2 = (\mathcal{O}'_p)^\times$.

It would seem that if A is a supersingular point, then a formal neighborhood of the point (A, i) of $\mathcal{S}_{(G,K)}$ is modeled by the stacky quotient

$$\mathrm{spf}(\mathbb{W}(\overline{\mathbb{F}}_p)[[u_1]])// \mathrm{Aut}(A, i)$$

4. Automorphic forms

Automorphic forms are complex valued functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ that satisfy certain conditions. The group $G(\mathbb{A})$ acts on this space of functions by right translation. We don't to consider all automorphic forms, at least not yet, just the ones that are the appropriate generalizations of modular forms.

The "strong approximation theorem" states that the group $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the finite ádeles. It follows that there is a decomposition

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+G(\widehat{Z})$$

where $G(\mathbb{R})^+ \cong GL_2(\mathbb{R})^+$ consists of those matrices with positive determinant. We may therefore write every $G(\mathbb{Q})$-coset $G(\mathbb{Q})g$ in $G(\mathbb{Q})\backslash G(\mathbb{A})$ in the form

$$G(\mathbb{Q})g = G(\mathbb{Q})g_\infty u$$

for $g_\infty \in G(\mathbb{R})^+$ and $u \in G(\widehat{\mathbb{Z}})$.

Assume that B is split, so that $G = GL_2$. A modular form of weight f of weight k restricts away from the cusps to a section of a line bundle $\omega^{\otimes k}$ over

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K = \Gamma\backslash\mathcal{H}$$

where $\Gamma = G(\mathbb{Q})\cap G(\mathbb{R})^+G(\widehat{\mathbb{Z}}) = SL_2(\mathbb{Z})$. The fiber of ω over a point corresponding to an elliptic curve E is given by

$$\omega_E = t_E^*$$

where t_E is the tangent space at the identity of E.

Given a modular form $f(\tau)$ of weight k (defined on the upper half plane) we may define an automorphic form

$$\phi_f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}.$$

The function ϕ_f applied to a coset $G(\mathbb{Q})g = G(\mathbb{Q})g_\infty u$ given by

$$\phi_f(G(\mathbb{Q})g) = f(g_\infty(i))(\det(g_\infty)^{-1/2}(ci + d))^{-k}$$

where

$$g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The function ϕ_f is nearly K-invariant, but not completely. When the action of K is restricted to the subgroup $U(1) < K$, the 1 dimensional representation spanned by ϕ_f is of weight k.

We now move back to the case of a general indefinite quaternion algebra B. Consider the complex projective curve

$$S(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/K = \Gamma\backslash\mathcal{H}.$$

Our isomorphism $\mathcal{O} \otimes \mathbb{R} \cong M_2(\mathbb{R})$ gives commuting idempotents e_1 and e_2 on the tangent space t_A. These idempotents decompose t_A into two identical summands of complex dimension 1

$$t_A = (t_A)_0 \oplus (t_A)_0.$$

Let ω be the line bundle over $S(\mathbb{C})$ whose fiber over a point corresponding to $(A, [\lambda], i)$ is given by

$$\omega_{(A,[\lambda],i)} = (t_A)_0^*.$$

Then we are interested in the subspace $\mathcal{A}(G, K)_k$ of the space of automorphic forms on $G(\mathbb{A})$ associated to the global sections

$$H^0(S(\mathbb{C}), \omega^{\otimes k}).$$

I don't really have any idea what the topological analog of a general automorphic form on $G(\mathbb{A})$ is, just the ones that arise from in the manner described above.

4.1. **$K(2)$-local TAF.** Here the situation is easy given the Hopkins-Miller theorem, and our description of $\mathcal{S}_{(G,K)}$ near the supersingular locus. We might define $K(2)$-local $TAF(G, K)$ by

$$TAF(G, K)_{K(2)} = \left(\prod_{(A,\, i) \text{ supersingular}} E(\overline{\mathbb{F}}_p, A(p)_0)^{h\,\mathrm{Aut}(A,i)} \right)^{hGal(\overline{\mathbb{F}}_p/\mathbb{F}_p)}.$$

4.2. **p-complete TAF.** Assume that p does not divide the discriminant Δ. Let's say a *p-complete automorphic spectrum of type* (G, K) is a tuple $(R, A, [\lambda], i, \phi)$ consisting of the following data. Fix an isomorphism $\mathcal{O}_p \cong M_2(\mathbb{Z}_p)$ and let e_1 and e_2 be the corresponding commuting idempotents.

- An p-complete even periodic ring-spectrum R, with associated formal group F_R.
- An abelian variety A defined over R_0, with weak polarization $[\lambda]$, and an inclusion $i : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ compatible with the Rosati involution. I am led to believe that we must also require that the summands $e_i(t_A)$ of the tangent space t_A are rank 1 locally free R_0-modules.
- An isomorphism $\phi : F_R \to A(p)_{f,0}$.

The spectrum $TAF(G, K)_p$ could be defined as the homotopy inverse limit

$$TAF(G, K)_p = \operatorname*{holim}_{\substack{(R,A,[\lambda],i,\phi) \\ p\text{-complete } E_\infty,\ \text{étale}}} R.$$

The homotopy inverse limit should be taken over a suitable category of automorphic spectra $(R, A, [\lambda], i, \phi)$ so that R is p-complete and E_∞, and such that the classifying map $(A, [\lambda], i) : \mathrm{spec}(R_0) \to \mathcal{S}_{(G,K)}$ is étale. One might expect this spectrum to have a nice connective cover $taf(G, K)_p$, since the Shimura variety has no cusps!

Define ω to be the line bundle over

$$\mathcal{S}_{(G,K),p} = \mathcal{S}_{(G,K)} \otimes_{\mathbb{Z}[\Delta^{-1}]} \mathbb{Z}_p$$

whose fiber over a point corresponding to a tuple $(A, [\lambda], i)$ is given by

$$\omega_{(A,[\lambda],i)} = e_1(t_A)^*.$$

If there are enough p-complete E_∞ étale automorphic spectra, then there will be a spectral sequence

$$H^s(\mathcal{S}_{(G,K),p}, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(TAF(G,K)_p).$$

4.3. TAF **over** $\mathbb{Z}[\Delta^{-1}]$**.** How do get the integral version? Well, let's invert the discriminant for now. We'll try to imitate the definition of $\mathcal{S}_{(G,K)}$ over $\mathbb{Z}[\Delta^{-1}]$.

Define an automorphic spectrum of type (G,K) to be a tuple $(R, A, [\lambda], i, \phi)$ consisting of the following data.

- An even periodic ring-spectrum R, such that Δ is invertible in R_0, with associated formal group F_R.
- An abelian variety A defined over R_0, with weak polarization $[\lambda]$, and an inclusion $i : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ compatible with the Rosati involution. I think we also need that if R' is an R_0-module which splits B, then the associated summands $e_i(t_A \otimes_{R_0} R')$ of the tangent space $t_A \otimes_{R_0} R'$ are rank 1 locally free R'-modules.
- For each R' as above, an isomorphism $\phi_{R'} : F_R \otimes_{R_0} R' \to e_1(\widehat{A} \otimes_{R_0} R')$.

The spectrum $TAF(G,K)$ then would be defined as the homotopy inverse limit

$$TAF(G,K) = \underset{\substack{(R,A,[\lambda],i,\phi) \\ E_\infty,\ \text{étale}}}{\mathrm{holim}} R.$$

We'd take $taf(G,K)$ to be the connective cover.

Is there a line bundle ω over $\mathcal{S}_{(G,K)}$ whose fiber over a point corresponding to a tuple $(A, [\lambda], i)$ (defined over a ring R' which splits B) is given by

$$\omega_{(A,[\lambda],i)} = e_1(t_A)^*?$$

Then we'd expect a spectral sequence

$$H^s(\mathcal{S}_{(G,K)}, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(TAF(G,K)).$$

4.4. **Derived Shimura varieties.** Of course now it seems natural to follow the approach of Lurie. A derived abelian variety $\mathcal{A}$ over an E_∞ ring spectrum R should be a derived scheme

$$\mathcal{A} \to \operatorname{spec} R$$

whose underlying ordinary scheme is an abelian variety over R_0. $\mathcal{A}$ must also be a very commutative abelian group object. Would it be enough to have a polarization on the underlying abelian variety, or would we have to develop some sort of derived version of polarization?

May as well define *enhanced automorphic spectra*. Perhaps Lurie's representablity theorem applies to the functor

$$R \mapsto \{\text{enhanced automorphic spectra of type } (G,K)\}$$

to produce a derived Shimura variety $\mathcal{S}_{(G,K),der}$. One would naïvely expect the spectrum $TAF(G,K)$ to be recovered as the global sections of the structure sheaf.

5. Looking toward the future

5.1. **The $K(2)$-local sphere.** The spectra $TAF(G,A)_{K(2)}$ should be relevant for a decomposition of the $K(2)$-local sphere of the nature studied by Goerss, Henn, Mahowald, and Rezk. The building associated to $GL_2(\mathbb{Q}_\ell)$ came into play with the ℓ-adic Tate modules of elliptic curves. The same deal should happen here, except that we will be looking at summands of the ℓ-adic Tate modules of the Abelian varieties. Here, ℓ would have to be coprime to $p\Delta$.

Tyler Lawson and I produced, using endomorphisms of elliptic curves, a dense subgroup of the Morava stabilizer group. Seems that that the same methods should extend to show that the group

$$\Gamma = (\mathcal{O}' \otimes \mathbb{Z}[1/\ell])^\times$$

is dense in the Morava stabilizer group if ℓ is a topological generator of $\mathbb{Z}_p^\times$. I haven't checked all of the details, but at least I know its closure contains the norm 1 p-Sylow subgroup for $p > 2$.

Mahowald always studies connective covers. In some sense TAF may be better suited for understanding Mahowald's computations - we don't need to add any cusps.

5.2. **The Shimura varieties of Kottwitz, Harris, and Taylor.** Harris and Taylor proved the local Langlands correspondence for GL_n by studying the "vanishing cycles" cohomology of the Drinfeld moduli of 1-dimensional height n formal $\mathcal{O}_F$-modules, following some conjectures of Carayol. Harris and Taylor studied the vanishing cycles cohomology group associated to some simple Shimura varieties originally studied by Kottwitz. These Shimura varieties are less simple then the ones I talked about, but roughly they correspond to B being an n^2 division algebra with center a CM field E over $\mathbb{Q}$ where p splits into w and w'. B is split at w, so the Barsotti-Tate modules at w split into 1-dimensional summands.

The locus of the associated Shimura variety S of abelian varieties whose Barsotti-Tate groups at w split into height n summands is zero dimensional. The completion of S at any of these points gives a universal deformation.

The associated spectra $TAF_{K(n)}$ would be approximations to the $K(n)$-local sphere. The whole $K(n)$-local sphere could be studied using a decomposition of the type Goerss-Henn-Mahowald-Rezk produced for $n = 2$. Here, the building for $GL_n(\mathbb{Q}_\ell)$ comes into play.

5.3. **Jacobians of curves.** Gorbounov, Hopkins, and Mahowald studied EO_{p-1} by considering certain families of curves X whose Jacobian varieties had formal summands of dimension 1 and height $n = p - 1$. Ravenel recently generalized this approach to all chromatic levels of the form $n = (p-1)f$ (these are precisely the n for which the the nth Morava stabilizer group has infinite cohomological dimension).

I don't know how the Jacobian approach is related to the approach of this document. I will remark that one advantage to the Jacobian approach is that you get polynomial equations that you can compute with explicitly. The Shimura varieties don't give you this.

The automorphisms of these families of curves act on the 1-dimensional summand and give rise to finite subgroups of the Morava stabilizer group. The problem is that at large chromatic levels only a fraction of the p-torsion of the Morava stabilizer group is accounted for. I'd like to think that the Shimura varieties have a better chance to capture all of the p-torsion, or, as I mentioned before, to produce dense subgroups of the Morava stabilizer group at all chromatic levels. So the Shimura variety approach may sacrifice computability for power.