

# UNSTABLE HOMOTOPY GROUPS AND LIE ALGEBRAS

MARK BEHRENS AND CONNOR MALIN

ABSTRACT. We survey the role of Lie algebras in the study of unstable homotopy groups.

## CONTENTS

1. Introduction	1
2. Algebras and modules over operads	3
3. The Goodwillie spectral sequence	5
4. The mod $p$ lower central series and restricted Lie algebras	6
5. Lie algebra models of rational homotopy theory	9
6. Lie algebra models of unstable $v_n$ -periodic homotopy theory	10
References	14

## 1. INTRODUCTION

Let  $X$  be a pointed space. The Whitehead product

$$[-, -] : \pi_i(X) \otimes \pi_j(X) \rightarrow \pi_{i+j-1}(X)$$

gives  $\pi_*(X)$  the structure of a graded shifted Lie algebra. This structure is most easily conceptualized by its relationship to the Samelson product, which is given by the commutator on the loop space

$$\langle -, - \rangle : \pi_i(\Omega X) \otimes \pi_j(\Omega X) \rightarrow \pi_{i+j}(\Omega X).$$

Samelson showed that under the isomorphism  $\pi_{i+1}(X) \cong \pi_i(\Omega X)$ , the two products agree up to a sign [Sam53].

This Lie algebra structure is fundamentally unstable in nature — there is no vestige of it in the context of stable homotopy groups. It captures the difference between unstable and stable homotopy groups in a manner made precise by Curtis's lower

central series [Cur65], Rector’s mod  $p$  lower central series [Rec66] and its relationship to simplicial restricted Lie algebras, and Quillen’s differential graded Lie algebra model of unstable rational homotopy theory [Qui69].

In this paper we will review these now classical ideas, and their more recent development in the context of Goodwillie calculus [Goo03]. Specifically, we will discuss the results of Arone, Ching, Taggart and the second author on Koszul duality and its interaction with Goodwillie calculus [Chi05], [AC11], [Esp22], [MT24], Konovalov’s work on simplicial restricted Lie algebras [Kon23], and the generalization of Quillen’s Lie algebra model of rational homotopy theory to the unstable  $v_n$ -periodic context of Heuts, Rezk, and the first author [Heu21], [BR20a].

The recurring theme will be the following:

$$\text{Unstable homotopy groups} = \begin{pmatrix} \text{stable homotopy groups} \\ + \\ \text{Lie algebra information} \end{pmatrix}$$

**Conventions.** We will denote the following  $\infty$ -categories by

$$\begin{aligned} \text{sSet} &= \text{simplicial sets (a.k.a. spaces)} \\ \text{sSet}_* &= \text{pointed spaces} \\ \text{sSet}^{\geq n} &= (n-1)\text{-connected spaces} \\ \text{Sp} &= \text{spectra} \end{aligned}$$

We will use  $(-)^{\vee}$  to denote the Spanier-Whitehead dual. Given an  $\infty$ -category  $\mathcal{C}$ , and objects  $X, Y \in \mathcal{C}$ , we let

$$\mathcal{C}(X, Y)$$

denote the associated space of maps, and

$$[X, Y] = [X, Y]_{\mathcal{C}}$$

denote the corresponding set of homotopy classes of maps. If  $\mathcal{C}$  is a stable  $\infty$ -category, we will let

$$\underline{\mathcal{C}}(X, Y)$$

denote the mapping spectrum.

$p$  will always denote a prime number. For elements  $x_i$  of a Lie algebra  $L$ , we will let

$$[x_1, \dots, x_k] = [x_1, [x_2, \dots [x_{k-1}, x_k] \dots]]$$

denote the iterated Lie bracket.

**Acknowledgments.** The authors would like to thank Nikolay Konovalov, for his generous explanations of his work and related topics.

## 2. ALGEBRAS AND MODULES OVER OPERADS

**Symmetric sequences.** Fix a presentably symmetric monoidal stable  $\infty$ -category  $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ , and let

$$\text{Seq}_{\Sigma}(\mathcal{C})$$

denote the  $\infty$ -category of *symmetric sequences* in  $\mathcal{C}$ , whose objects are sequences

$$\{\mathcal{X}_i \in \mathcal{C}^{B\Sigma_i}\}_{i \geq 0}.$$

We will identify  $\mathcal{C}$  with the full subcategory of  $\text{Seq}_{\Sigma}(\mathcal{C})$  consisting of sequences concentrated in degree 0:

$$X := \{X, 0, 0, \dots\}.$$

The  $\infty$ -category  $\text{Seq}_{\Sigma}(\mathcal{C})$  has a monoidal structure  $\circ$  given by

$$(\mathcal{X} \circ \mathcal{Y})_i := \bigoplus_{i=i_1+\dots+i_k} \text{Ind}_{\Sigma_{i_1, \dots, i_k}}^{\Sigma_i} \mathcal{X}_{i_1} \otimes \mathcal{Y}_{i_2} \otimes \dots \otimes \mathcal{Y}_{i_k},$$

where  $\Sigma_{i_1, \dots, i_k} \leq \Sigma_i$  is the subgroup which preserves the partition  $i = i_1 + \dots + i_k$ . The unit of this monoidal structure is  $1_*$ , given by

$$1_* := \{0, 1_{\mathcal{C}}, 0, 0, \dots\} \in \text{Seq}_{\Sigma}(\mathcal{C}).$$

The  $\infty$ -category  $\text{Seq}_{\Sigma}(\mathcal{C})$  also possesses a symmetric monoidal structure  $\otimes$  given by

$$(2.1) \quad (\mathcal{X} \otimes \mathcal{Y})_i := \bigoplus_{i=i_1+i_2} \text{Ind}_{\Sigma_{i_1} \times \Sigma_{i_2}}^{\Sigma_i} \mathcal{X}_{i_1} \otimes \mathcal{Y}_{i_2}.$$

**Operads.** An *operad* in  $\mathcal{C}$  is a monoid in  $(\text{Seq}_{\Sigma}(\mathcal{C}), \circ)$ .

- (2.2) We shall say that an operad  $\mathcal{O}$  is *reduced* if  $\mathcal{O}_0 = 0$  and  $\mathcal{O}_1 = 1$ .
- (2.3) Given an operad  $\mathcal{O}$  in  $\mathcal{C}$ , we will let  $\text{Mod}_{\mathcal{O}}^{rt}/\text{Mod}_{\mathcal{O}}^{lt}$  denote right/left modules over  $\mathcal{O}$ .
- (2.4) The  $\otimes$ -product of right  $\mathcal{O}$ -modules is again an  $\mathcal{O}$ -module, so  $\otimes$  endows  $\text{Mod}_{\mathcal{O}}^{rt}$  with a symmetric monoidal structure (see [Fre09]).
- (2.5) If an operad  $\mathcal{O}$  is reduced, then the canonical map

$$\mathcal{O} \rightarrow 1_*$$

is a map of operads, and in particular  $1_*$  is both a left and right  $\mathcal{O}$ -module.

- (2.6) An object  $X \in \mathcal{C}$  gives rise to a symmetric sequence  $X^{\otimes} \in \text{Seq}_{\Sigma}(\mathcal{C})$  with

$$(X^{\otimes})_i := X^{\otimes i}.$$

An  $\mathcal{O}$ -coalgebra structure on  $X$  induces a right  $\mathcal{O}$ -module structure on  $X^{\otimes}$ .

- (2.7) A left  $\mathcal{O}$ -module structure on  $X \in \mathcal{C}$  (regarded as a symmetric sequence concentrated in degree 0) is an  $\mathcal{O}$ -algebra structure on  $X$ .

**Coendomorphism operads.** For objects  $X, Y \in \mathcal{C}$ , define a symmetric sequence of spectra  $\mathcal{H}om_{\mathcal{C}}(X, Y)$  by

$$\mathcal{H}om_{\mathcal{C}}(X, Y)_i := \underline{\mathcal{C}}(X, Y^{\otimes i}).$$

- (2.8) The symmetric sequence  $\mathcal{H}om_{\mathcal{C}}(X, X)$  admits a canonical operad structure (sometimes referred to as the *coendomorphism operad*).
- (2.9) The symmetric sequence of spectra  $\mathcal{H}om_{\mathcal{C}}(X, Y)$  is canonically a right  $\mathcal{H}om_{\mathcal{C}}(Y, Y)$ -module and a left  $\mathcal{H}om_{\mathcal{C}}(X, X)$ -module.
- (2.10) The *n-sphere operad* is defined to be the coendomorphism operad in spectra

$$\mathcal{S}^n := \mathcal{H}om_{\text{Sp}}(\mathcal{S}^n, \mathcal{S}^n).$$

Using the fact that  $\mathcal{C}$  is tensored over spectra, we can define the *n-th suspension*  $\sigma^n \mathcal{O}$  of an operad  $\mathcal{O}$  to be the operad

$$\sigma^n \mathcal{O} := \mathcal{S}^n \otimes \mathcal{O}.$$

If  $A$  is an  $\sigma^n \mathcal{O}$ -algebra, then  $\Sigma^n A$  is an  $\mathcal{O}$ -algebra.

**Koszul duality.** Ching originally defined Koszul duality of operads/modules in spectra using bar constructions [Chi05]. Recently, Espic [Esp22] introduced a more conceptual homotopy invariant construction, which he showed was equivalent to Ching's.

Given a reduced operad of spectra  $\mathcal{O}$ , its *Koszul dual* is defined to be the coendomorphism operad

$$K(\mathcal{O}) := \mathcal{H}om_{\text{Mod}_{\mathcal{O}}^{\text{rt}}}(\mathbf{1}_*, \mathbf{1}_*).$$

Given  $\mathcal{M} \in \text{Mod}_{\mathcal{O}}^{\text{rt}}$ , its *Koszul dual* is defined to be the right  $K(\mathcal{O})$ -module

$$K_{\mathcal{O}}(\mathcal{M}) := \mathcal{H}om_{\text{Mod}_{\mathcal{O}}^{\text{rt}}}(\mathcal{M}, \mathbf{1}_*).$$

There are equivalences [Esp22], [MT24]

$$\begin{aligned} K(\mathcal{O}) &\simeq B(\mathbf{1}_*, \mathcal{O}, \mathbf{1}_*)^{\vee}, \\ K_{\mathcal{O}}(\mathcal{M}) &\simeq B(\mathcal{M}, \mathcal{O}, \mathbf{1}_*)^{\vee} \end{aligned}$$

where  $B(-, -, -)$  denotes the two-sided monoidal bar construction. Given an  $\mathcal{O}$ -coalgebra  $X$ , we define the spectrum of *primitives* by

$$\text{Prim}_{\mathcal{O}}(X) := \underline{\text{Mod}}_{\mathcal{O}}^{\text{rt}}(\mathbf{1}_*, X^{\otimes})$$

It follows from the definition of  $K(\mathcal{O})$  that  $\text{Prim}_{\mathcal{O}}(X)$  naturally has the structure of a  $K(\mathcal{O})$ -algebra.

**The Lie operad.** Let  $\text{Comm}$  be the reduced commutative operad in spectra, given by the symmetric sequence

$$\{0, S, S, S, \dots\}.$$

Define the spectral Lie operad to be the shift of the Koszul dual

$$\mathcal{L}ie := \sigma K(\text{Comm}).$$

It is shown in [Chi05] that there is an isomorphism of operads

$$H\mathbb{Z}_* \mathcal{L}ie \cong \mathcal{L}ie^{\mathbb{Z}},$$

where  $\mathcal{L}ie^{\mathbb{Z}}$  denotes the Lie operad in abelian groups. For a commutative ring  $k$ , algebras over

$$\mathcal{L}ie^k := k \otimes \mathcal{L}ie^{\mathbb{Z}}$$

in  $\text{Mod}_k$  are Lie algebras over  $k$ .

### 3. THE GOODWILLIE SPECTRAL SEQUENCE

**The Goodwillie tower.** Goodwillie calculus [Goo03], [Lur17, Ch.6] associates to a reduced functor between presentable pointed  $\infty$ -categories a *Taylor tower* of degree  $n$  polynomial approximations

$$F \rightarrow \cdots \rightarrow P_n(F) \rightarrow \cdots \rightarrow P_1(F).$$

In the context where  $F$  is the identity functor

$$\text{Id} : \text{sSet}_* \rightarrow \text{sSet}_*,$$

the fibers take the form [Joh95], [Chi05]

$$D_n(\text{Id})(X) \simeq \Omega^\infty \sigma^{-1} \mathcal{L}ie_n \otimes_{h\Sigma_n} \Sigma^\infty X^{\otimes n}$$

and for  $X$  connected and  $\mathbb{Z}$ -complete the map

$$X \rightarrow P_\infty(\text{Id})(X) := \varprojlim_n P_n(\text{Id})(X)$$

is an equivalence [AK98]. It follows that for a connected  $\mathbb{Z}$ -complete space there is a *Goodwillie spectral sequence*

$$(3.1) \quad {}^{gss}E_1^{t,*}(X) = \pi_t \mathcal{L}ie(\Sigma^{-1} \Sigma^\infty X) \Rightarrow \pi_{t+1} X,$$

where

$$\mathcal{L}ie(Y) \simeq \bigoplus_n \mathcal{L}ie_n \otimes_{h\Sigma_n} Y^{\otimes n}$$

is the free spectral Lie algebra on a spectrum  $Y$ . Note that the  $E_1$ -term is a Lie algebra, and the GSS converges to a Lie algebra, *but it has not been proven that this spectral sequence is a spectral sequence of Lie algebras.*

**The homotopy and homology of free spectral Lie algebras.** We are led to compute the homotopy groups of  $\mathcal{L}ie(Y)$  for  $Y \in \text{Sp}$ . We shall do this for the  $p$ -completions for every prime  $p$ .

The homotopy groups of any bounded below  $p$ -complete spectrum  $Z$  can be studied using the *mod  $p$  Adams spectral sequence*

$$\text{Ext}_{\mathcal{A}^{op}}^{s,t}(\mathbb{F}_p, (H\mathbb{F}_p)_* Z) \Rightarrow \pi_{t-s} Z.$$

Here  $\mathcal{A}$  is the mod  $p$  Steenrod algebra, whose dual action gives  $(H\mathbb{F}_p)_* Z$  an  $\mathcal{A}^{op}$ -module structure. Thus the input needed to study the homotopy groups of the  $p$ -completion of  $\sigma^{-1} \mathcal{L}ie(Y)$  is the homology  $(H\mathbb{F}_p)_* \sigma^{-1} \mathcal{L}ie(Y)$ .

We first consider the case of  $p = 2$ . Suppose that  $L$  is a 2-complete  $\mathcal{L}ie$ -algebra. Since

$$(3.2) \quad \mathcal{L}ie_2 \simeq S^{\sigma^{-1}}$$

where  $\sigma$  is the sign representation, the  $\mathcal{L}ie$ -algebra structure gives a map

$$(H\mathbb{F}_2)_*(\Sigma^{\sigma-1}L^{\otimes 2})_{h\Sigma_2} \rightarrow (H\mathbb{F}_2)_*L.$$

In addition to endowing  $(H\mathbb{F}_2)_*L$  with the structure of a graded Lie algebra, it also gives rise to *Lie-Dyer-Lashof operations* [Beh12]

$$(3.3) \quad \bar{Q}^i : (H\mathbb{F}_p)_t L \rightarrow (H\mathbb{F}_p)_{t+i} L$$

which satisfy the *allowability conditions* [AC20]

- $\bar{Q}^i x = 0$  if  $i < |x|$ .
- $\bar{Q}^i x = [x, x]$  if  $i = |x|$ .
- $[x, \bar{Q}^i y] = 0$ .

The algebra of all such operations  $\bar{\mathcal{R}}$  is subject to Lie-Adem relations [Beh12, Sec. 1.4] which give rise to a basis of admissible monomials

$$\bar{Q}^{i_1} \dots \bar{Q}^{i_\ell}$$

with  $i_m > 2i_{m+1}$ .

Antolín Camarena [AC20] showed that  $(H\mathbb{F}_2)_*\mathcal{L}ie(Y)$  is the free allowable  $\bar{\mathcal{R}}$ -Lie algebra on  $(H\mathbb{F}_2)_*Y$ . Specifically, if  $\{x_j\}$  is a basis of  $(H\mathbb{F}_2)_*Y$ , then  $(H\mathbb{F}_2)_*\mathcal{L}ie(Y)$  has a basis

$$(3.4) \quad \bar{Q}^{i_1} \dots \bar{Q}^{i_\ell} [x_{j_1}, \dots, x_{j_k}]$$

where the brackets range over a basis of the free graded Lie algebra over  $\mathbb{F}_2$  on the generators  $\{x_j\}$ ,  $i_m > 2i_{m+1}$ , and  $i_\ell > |x_{j_1}| + \dots + |x_{j_k}|$ .

For  $p$  odd, Kjaer [Kja18] constructed the odd primary analog of the Lie-Dyer-Lashof operations (3.3), and he showed that  $(H\mathbb{F}_p)_*\mathcal{L}ie(Y)$  admits a basis analogous to (3.4). However he was unable to determine the odd primary Lie-Adem relations. In the case of the prime 2, they were deduced in [Beh12] from a classical computation of the transfer

$$(H\mathbb{F}_2)_*B\Sigma_4 \rightarrow (H\mathbb{F}_2)_*B\Sigma_2 \wr \Sigma_2$$

due to Kahn and Priddy [Pri73]. Surprisingly, the formula for the odd primary analog of this transfer was unknown. One interesting corollary of the work of [Kon23] (which we will discuss in Section 4) is that he is able to determine these odd primary Lie-Adem relations.

#### 4. THE MOD $p$ LOWER CENTRAL SERIES AND RESTRICTED LIE ALGEBRAS

**The Rector spectral sequence.** Adapting the work of Curtis [Cur65] to the  $p$ -primary setting, Rector [Rec66] studied the spectral sequence associated to the the mod  $p$ -lower central series

$$\Gamma_s^p G = \langle [g_1, \dots, g_i]^{p^j} : ip^j \geq s \rangle \leq G$$

of a simplicial group  $G$ , whose associated graded is a *simplicial graded restricted Lie algebra over  $\mathbb{F}_p$* . A graded restricted Lie algebra  $L_*$  (over a field of characteristic  $p$ ) is a graded Lie algebra which possesses an additional operation

$$\xi : L_t \rightarrow L_{pt}$$

which satisfies certain axioms (see [MM65]).

Consider the following diagram of  $\infty$ -categories and functors.

$$(4.1) \quad \begin{array}{ccccccc} \mathbf{sSet}_* & \xrightarrow{\Omega} & \mathbf{sGp} & \xrightarrow{\Gamma^\bullet} & \mathbf{sGp}^{fil} & \xrightarrow{gr_\bullet} & \mathbf{sLie}_{\mathbb{F}_p}^{gr.res} \\ \tilde{\mathbb{F}}_p \downarrow & & \mathbb{F}_p \downarrow & & & & \downarrow V \\ \mathbf{sCoAlg}_{\mathbb{F}_p}^{Comm} & \xrightarrow{C} & \mathbf{sHopf}_{\mathbb{F}_p} & \xrightarrow{I^\bullet} & \mathbf{sHopf}_{\mathbb{F}_p}^{fil} & \xrightarrow{gr_\bullet} & \mathbf{sHopf}_{\mathbb{F}_p}^{gr.prim} \end{array}$$

where

- $\mathbf{sGp}$  = simplicial groups
- $\mathbf{sLie}_{\mathbb{F}_p}^{gr.res}$  = simplicial graded restricted Lie algebras over  $\mathbb{F}_p$
- $\mathbf{sCoAlg}_{\mathbb{F}_p}^{Comm}$  = (non-counital) simplicial cocommutative coalgebras over  $\mathbb{F}_p$
- $\mathbf{sHopf}_{\mathbb{F}_p}$  = simplicial cocommutative Hopf algebras over  $\mathbb{F}_p$
- $\mathbf{sHopf}_{\mathbb{F}_p}^{gr.prim}$  = simplicial cocommutative primitively generated graded Hopf algebras

and

- $\Omega X$  = the Kan loop group of a simplicial set  $X$
- $\mathbb{F}_p X$  = the free simplicial  $\mathbb{F}_p$ -module of a simplicial set  $X$
- $\tilde{\mathbb{F}}_p X$  = the free reduced simplicial  $\mathbb{F}_p$ -module of a pointed simplicial set  $X$
- $V(L)$  = the universal enveloping algebra (of a graded restricted Lie algebra  $L$ )
- $C(A) = C(\mathbb{F}_p, \mathbb{F}_p \oplus A, \mathbb{F}_p)$ , the cobar construction on a non-counital coalgebra  $A$   
(where  $\mathbb{F}_p \oplus A$  denotes the coaugmented counital coalgebra associated to  $A$ )
- $I^\bullet A$  = the filtration of  $A$  given by powers of the augmentation ideal
- $\mathcal{C}^{fil}$  = filtered objects of  $\mathcal{C}$
- $gr_\bullet$  = the associated graded of a filtered object

The left-hand square of (4.1) commutes when restricted to  $\mathbf{sSet}_*^{\geq 2}$  by the convergence of the Eilenberg-Moore spectral sequence. The right-hand rectangle of (4.1) is shown to commute in [BC70]. The universal enveloping algebra functor  $V$  in (4.1) is an equivalence by [MM65], where the inverse functor is given by taking primitives

$$\text{Prim} : \mathbf{sHopf}_{\mathbb{F}_p}^{gr.prim} \rightarrow \mathbf{sLie}_{\mathbb{F}_p}^{gr.res}.$$

Because the Kan loop group is level-wise free, the image of  $X$  under the various functors of (4.1) is given by

$$\begin{array}{ccccccc} X & \longmapsto & \Omega X & \longmapsto & \Gamma^\bullet \Omega X & \longmapsto & \mathcal{L}ie^r(\Sigma^{-1} \tilde{\mathbb{F}}_p X) \\ & & \downarrow & & & & \downarrow \\ & & \mathbb{F}_p \Omega X & \longmapsto & I^\bullet \mathbb{F}_p \Omega X & \longmapsto & T(\Sigma^{-1} \tilde{\mathbb{F}}_p X) \end{array}$$

where  $\mathcal{L}ie^r$  denotes the free restricted Lie algebra and  $T$  denotes the tensor algebra. Note that each of these carries a natural grading with  $\Sigma^{-1} \tilde{\mathbb{F}}_p X$  in degree 1.

The filtration  $\Gamma_{\bullet}^p \Omega X$  gives rise to the Rector spectral sequence

$${}^{r,ss}E_1^{t,*}(X) = \pi_t \mathcal{L}ie^r(\Sigma^{-1} \widetilde{\mathbb{F}}_p X) \Rightarrow \pi_{t+1} X$$

which converges for  $X$  simply connected [Rec66].

**Remark 4.2.** The spectral sequence associated to the filtration  $I^{\bullet} \mathbb{F}_p \Omega X$  is the Eilenberg-Moore spectral sequence. Thus, by (4.1), the Hurewicz homomorphism induces a map from the Rector spectral sequence to the Eilenberg-Moore spectral sequence.

**The homotopy of free simplicial restricted Lie algebras.** In order to compute the  $E_1$ -term of the Rector spectral sequence, we observe that the homotopy groups of a simplicial restricted Lie algebra  $L$  over  $\mathbb{F}_p$  have algebraic structure [BC70]. The *restricted* structure arises from a factorization of the Lie algebra structure maps through maps

$$\mathcal{L}ie_n^{\mathbb{F}_p} \otimes^{h\Sigma_n} L^{\otimes n} \rightarrow L.$$

This endows  $\pi_* L$  with the structure of a graded restricted Lie algebra. Furthermore we get  $\lambda$ -operations coming from the mod  $p$  cohomology of  $\Sigma_p$ . For simplicity, we restrict attention to the case where  $p = 2$ . In this case, it follows from (3.2) that the map

$$\mathcal{L}ie_2^{\mathbb{F}_2} \otimes^{h\Sigma_2} L^{\otimes 2} \rightarrow L$$

induces operations

$$\lambda_i : \pi_t L \rightarrow \pi_{t+i} L$$

for  $i \geq 0$ , which satisfy the *instability conditions*

- $x\lambda_i = 0$  if  $i > |x|$ .
- $x\lambda_i = \xi(x)$  if  $i = |x|$ .
- $[x, y\lambda_i] = 0$  if  $i < |y|$ .

The algebra of all such operations  $\Lambda$  is subject to Adem relations [BCK<sup>+</sup>66], which give rise to a basis of admissible monomials

$$\lambda_{i_1} \cdots \lambda_{i_\ell}$$

with  $2i_m \geq i_{m+1}$ . The  $\Lambda$ -algebra is Koszul dual to the Steenrod algebra  $\mathcal{A}$  [Pri73] and as such possesses a differential  $d$  such that

$$H^*(\Lambda) = \text{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

Bousfield and Curtis [BC70] showed that for a simplicial  $\mathbb{F}_2$ -module  $Y$ , the homotopy groups  $\pi_* \mathcal{L}ie^r(Y)$  form the free unstable  $\Lambda$ -Lie algebra on  $\pi_* Y$ . Specifically, if  $\{x_j\}$  is a basis of  $\pi_* Y$ , then  $\pi_* \mathcal{L}ie^r(Y)$  has a basis

$$[x_{j_1}, \dots, x_{j_k}] \lambda_{i_1} \cdots \lambda_{i_\ell}$$

where the brackets range over a basis of the free graded Lie algebra over  $\mathbb{F}_2$  on the generators  $\{x_j\}$ ,  $2i_m \geq i_{m+1}$ , and  $i_1 < |x_{j_1}| + \cdots + |x_{j_k}|$ .

The Rector spectral sequence is a spectral sequence of Lie algebras. The first potentially non-trivial differential on Lie algebra generators is given by the formula



[BC70]

$$(4.3) \quad d_{2\ell}^{r_{ss}}(\sigma^{-1}x \cdot \lambda_I) = \sum [\sigma^{-1}x'_i, \sigma^{-1}x_i] \lambda_I + \sigma^{-1}x \cdot d\lambda_I + \sum_j \sigma^{-1}x \text{Sq}_*^j \cdot \lambda_{j-1} \lambda_I$$

for  $x \in (\widetilde{H\mathbb{F}_2})_* X$  with  $\Delta(x) = \sum_i x'_i \otimes x''_i$ . It is shown in [BC70] that for sufficiently nice spaces, the Rector spectral sequence is isomorphic to the unstable Adams spectral sequence after re-indexing.

**The algebraic Goodwillie spectral sequence.** Konovalov [Kon23] related the Rector spectral sequence to the Goodwillie spectral sequence. Specifically, he showed that the *algebraic Goodwillie spectral sequence* associated to the Goodwillie tower of the functor<sup>1</sup>

$$\mathcal{L}ie^r(\Sigma^{-1}\widetilde{\mathbb{F}_p}(-)) : \text{sSet}_* \rightarrow \text{sLie}_{\mathbb{F}_p}^{res}$$

takes the form

$$(4.4) \quad {}^{agss}E_1 = (H\mathbb{F}_p)_* \mathcal{L}ie(\Sigma^{-1}\Sigma^\infty X) \otimes \Lambda \Rightarrow \pi_* \mathcal{L}ie^r(\Sigma^{-1}\widetilde{\mathbb{F}_p}X).$$

The spectral sequence (4.4) is a spectral sequence of Lie algebras, and Konovalov [Kon23, Rmk. 8.3.7] showed that the spectral sequence is *completely determined* by explicit differentials on Lie algebra generators given by formulas discovered by Lin [Lin81] in his proof of the algebraic Kahn-Priddy theorem.

The conjecture is that the algebraic Goodwillie spectral sequence fits into a “commuting square” of spectral sequences<sup>2</sup>

$$\begin{array}{ccc} (H\mathbb{F}_p)_* \mathcal{L}ie(\Sigma^{-1}\Sigma^\infty X) \otimes \Lambda & \xrightarrow{agss} & \pi_* \mathcal{L}ie^r(\Sigma^{-1}\widetilde{\mathbb{F}_p}X) \\ \text{ass} \downarrow & & \downarrow r_{ss} \\ \pi_* \mathcal{L}ie(\Sigma^{-1}\Sigma^\infty X) & \xrightarrow{gss} & \pi_{*+1} X \end{array}$$

Such a commuting square would allow for the lifting of AGSS differentials to GSS differentials.

## 5. LIE ALGEBRA MODELS OF RATIONAL HOMOTOPY THEORY

**Rational homotopy theory.** Quillen famously showed that simply connected rational homotopy theory can be modeled by simplicial Lie algebras over  $\mathbb{Q}$ . He

<sup>1</sup>Technically, Konovalov studied the case of restricted Lie algebras over the algebraic closure  $\overline{\mathbb{F}_p}$ . This was so he could use the action of the units of  $\overline{\mathbb{F}_p}$  to prove degeneration results — his results then carry over to  $\mathbb{F}_p$ .

<sup>2</sup>By “commuting square” we mean that the square of spectral sequences arises from a bifiltered object.

accomplished this by observing that diagram (4.1) simplifies to a diagram of equivalences of  $\infty$ -categories (the equivalence  $(*)$  was also studied by Sullivan [Sul77]).

$$(5.1) \quad \begin{array}{ccc} (\mathbf{sSet}_*)_{\mathbb{Q}}^{\geq 2} & \xrightarrow[\simeq]{\Omega} & \mathbf{sGp}_{\mathbb{Q}}^{\geq 1} \\ \downarrow \bar{\mathbb{Q}} \simeq & & \downarrow \mathbb{Q} \simeq \\ (\mathbf{sCoAlg}_{\mathbb{Q}}^{\mathcal{C}omm})^{\geq 2} & \xrightarrow[\mathcal{C}]{\simeq} & \mathbf{sHopf}_{\mathbb{Q}}^{\mathit{conn}} \\ \downarrow \text{Prim}_{\mathcal{C}omm} \simeq & & \uparrow U \simeq \downarrow \text{Prim} \\ \mathbf{Alg}_{\sigma^{-1}\mathcal{L}ie}(\mathbf{Sp}_{\mathbb{Q}})^{\geq 2} & \xrightarrow[\Sigma^{-1}]{\simeq} & \mathbf{sLie}_{\mathbb{Q}}^{\geq 1} \end{array}$$

Here,  $U$  refers to the universal enveloping algebra, and  $\text{Prim}_{\mathcal{C}omm}$  is the derived primitives construction described in Section 2.

**The rational Goodwillie tower.** In the rational case, the Goodwillie spectral sequence is the spectral sequence obtained from the bracket-length filtration on  $\text{Prim}\mathbb{Q}\Omega X$ , and takes the form [Wal06]

$$\mathcal{L}ie^{\mathbb{Q}}(\Sigma^{-1}(\widetilde{H\mathbb{Q}})_*X) \Rightarrow \pi_{*+1}X_{\mathbb{Q}}.$$

In this case the spectral sequence is known to be a spectral sequence of Lie algebras. The  $d_1$ -differential is determined by its effect on Lie algebra generators: for  $x \in (\widetilde{H\mathbb{Q}})_*X$  with  $\Delta(x) = \sum x'_i \otimes x''_i$  this differential is given by [Qui69, Apx B]

$$d_1^{gss}(\sigma^{-1}x) = \frac{1}{2} \sum_i (-1)^{|x'_i|} [\sigma^{-1}x'_i, \sigma^{-1}x''_i].$$

Thus if  $X$  is of finite type, the  $E_1$ -page is the Harrison complex associated to the ring  $(H\mathbb{Q})^*X$ , and the  $E_2$ -page is its Andre-Quillen cohomology.

## 6. LIE ALGEBRA MODELS OF UNSTABLE $v_n$ -PERIODIC HOMOTOPY THEORY

**The Bousfield-Kuhn functor.** Recall that a  $p$ -local finite complex  $X$  is called *type  $n$*  if it is  $K(n-1)$ -acyclic, and  $K(n) \otimes X \not\cong 0$ . The periodicity theorem of Hopkins-Smith [HS98] implies that a  $p$ -local finite complex  $V$  of type  $n$  admits an asymptotically unique  $v_n$ -self-map: a  $K(n)$ -equivalence

$$v : \Sigma^{t+N}V \rightarrow \Sigma^tV$$

for  $t \gg 0$ . The *unstable  $v_n$ -periodic homotopy groups (with coefficients in  $V$ )* of a pointed space  $X$  are defined to be

$$v_n^{-1}\pi_*(X; V) := v^{-1}[\Sigma^*V, X].$$

The corresponding stable  $v_n$ -periodic homotopy groups

$$v_n^{-1}\pi_*^s(X; V) := \varprojlim_k v_n^{-1}\pi_{*+k}(\Sigma^k X; V)$$

are the homotopy groups of the telescope

$$v^{-1}V^{\vee} \otimes \Sigma^{\infty}X.$$

Thus the stable  $v_n$ -periodic homotopy type of  $X$  is encoded in the Bousfield localization

$$(\Sigma^\infty X)_{T(n)} \in \mathrm{Sp}_{T(n)}$$

where  $T(n) := v^{-1}V^\vee$  (this localization is independent of the choice of  $V$  and  $v$ -self map).

The *Bousfield-Kuhn functor*

$$\Phi_n : \mathrm{sSet}_* \rightarrow \mathrm{Sp}_{T(n)}$$

encodes these unstable  $v_n$ -periodic homotopy groups, in the sense that there are natural isomorphisms

$$\pi_* \Phi_n(X) \otimes V^\vee \cong v_n^{-1} \pi_*(X; V).$$

The *completed unstable  $v_n$ -periodic homotopy groups* are defined to be

$$v_n^{-1} \pi_*^\wedge(X) := \pi_* \Phi_n(X).$$

**A generalization of rational homotopy theory.** Let  $v_n^{-1} \mathrm{sSet}_*$  denote the  $\infty$ -category obtained by inverting the  $v_n^{-1} \pi_*^\wedge$ -isomorphisms. Heuts [Heu21] showed that  $\Phi_n(X)$  canonically admits the structure of a  $\sigma^{-1} \mathcal{L}ie$  algebra which is compatible with the Whitehead product on homotopy groups, and proved that the induced functor

$$\Phi_n : v_n^{-1} \mathrm{sSet}_* \xrightarrow{\simeq} \mathrm{Alg}_{\sigma^{-1} \mathcal{L}ie}(\mathrm{Sp}_{T(n)})$$

is an equivalence of  $\infty$ -categories.

While this result gives a fantastic generalization of Quillen's simplicial Lie model of rational homotopy theory, it tells us nothing about the homotopy type of  $\Phi_n(X)$ . To that end, one can imitate Quillen's approach to the rational case. It is shown in [Heu21] (see also [BR20a]) that the diagram

$$\begin{array}{ccc} v_n^{-1} \mathrm{sSet}_* & \xrightarrow{\Sigma_{T(n)}^\infty} & \mathrm{CoAlg}_{\mathcal{C}omm}(\mathrm{Sp}_{T(n)}) \\ \Phi_n \simeq \downarrow & & \swarrow \mathrm{Prim}_{\mathcal{C}omm} \\ \mathrm{Alg}_{\sigma^{-1} \mathcal{L}ie}(\mathrm{Sp}_{T(n)}) & & \end{array}$$

is lax commutative in the sense that there is a natural transformation called the *comparison map*

$$(6.1) \quad c_X : \Phi_n(X) \rightarrow \mathrm{Prim}_{\mathcal{C}omm} \Sigma_{T(n)}^\infty X.$$

We shall say the a space  $X$  is  $\Phi_n$ -good if the map

$$(6.2) \quad X \rightarrow P_\infty(\Phi_n)(X)$$

is an equivalence.

One of the main results of [Heu21] is

**Theorem 6.3** (Heuts). *The comparison map (6.1) is an equivalence for  $X$  which are  $\Phi_n$ -good.*

Theorem 6.3 improved upon the main result of [BR20a], which showed that if  $X$  is finite with (6.2) a  $K(n)$ -equivalence (i.e.  $X$  is  $\Phi_{K(n)}$ -good), then the comparison map (6.1) is a  $K(n)$ -equivalence. If  $n = 1$ , the validity of the telescope conjecture [Bou79, Prop. 4.2] implies  $\Phi_1(X)$  is  $K(1)$ -local. The telescope conjecture has been shown to be false for  $n > 1$  [BHLS23].

Arone and Ching discovered yet another approach to Theorem 6.3 in the case where  $X$  is finite, assuming certain results about Koszul duality of right modules, which was described in [BR20b, Sec. 9]. These Koszul duality results have now been proven [MT24], and in the next two subsections we will proceed to give a concise recapitulation of the Arone-Ching approach to Theorem 6.3.

**Koszul duality and calculus.** For a functor

$$F : \mathbf{sSet}_* \rightarrow \mathbf{Sp}$$

define the *Koszul dual derivatives* to be the spectrum of natural transformations

$$\partial^k(F) := \mathbf{Nat}_X(F(X), \Sigma^\infty X^{\otimes k}).$$

The diagonal of  $X$  induces a right  $\mathcal{C}omm$ -module structure on  $\partial^*(F)$ . Using the Yoneda lemma, there is a natural transformation

$$F(X) \rightarrow \mathbf{Mod}_{\mathcal{C}omm}^{rt}(\partial^*(F), \Sigma^\infty X^{\otimes})$$

which gives an approximation of  $F(X)$ .

The derivatives  $\partial_*(F)$  were shown in [AC11] to possess a right  $\sigma^{-1}\mathcal{L}ie$ -module structure. The reason that we refer to  $\partial^*(F)$  as the Koszul dual derivatives of  $F$  is that if each  $\partial_i F$  is a finite spectrum, there is an equivalence [AC11, Example 17.28] of right  $\mathcal{C}omm = K(\sigma^{-1}\mathcal{L}ie)$ -modules

$$(6.4) \quad \partial^*(F) \simeq K_{\sigma^{-1}\mathcal{L}ie}(\partial_*(F)).$$

It follows from the results of [MT24] that if  $X$  and all of the derivatives  $\partial_k(F)$  are finite, then Koszul duality gives an equivalence

$$\mathbf{Mod}_{\mathcal{C}omm}^{rt}(\partial^*(F), \Sigma^\infty X^{\otimes}) \simeq \mathbf{Mod}_{\sigma^{-1}\mathcal{L}ie}^{rt}(\partial_*(\Sigma^\infty \mathbf{sSet}_*(X, -)), \partial_*(F)) =: \Psi(F)(X)$$

where

$$\Psi(F)(X) \simeq \varprojlim_k \Psi_k(F) = \varprojlim_k \mathbf{Mod}_{\sigma^{-1}\mathcal{L}ie}^{rt}(\partial_{\leq k}(\Sigma^\infty \mathbf{sSet}_*(X, -)), \partial_{\leq k}(F))$$

is the *fake Taylor tower* of [AC15].<sup>3</sup>

There is a map from the Taylor tower to the fake Taylor tower, giving a diagram of fiber sequences [AC11, Rmk. 4.2.27]

$$(6.5) \quad \begin{array}{ccccc} \partial_k(F) \otimes_{h\Sigma_k} \Sigma^\infty X^{\otimes k} & \longrightarrow & P_k(F)(X) & \longrightarrow & P_{k-1}(F)(X) \\ \downarrow N & & \downarrow & & \downarrow \\ \partial_k(F) \otimes_{h\Sigma_k} \Sigma^\infty X^{\otimes k} & \longrightarrow & \Psi_k(F)(X) & \longrightarrow & \Psi_{k-1}(F)(X) \end{array}$$

where  $N$  is the norm map.

<sup>3</sup>In [AC11],[AC15], the notation  $\Phi_k(F)$  is used, but we instead use  $\Psi_k$  to avoid conflict with the notation for the Bousfield-Kuhn functor.

**Proof of Theorem 6.3.** We may now explain how the theory of the previous subsection specializes in the case of  $F = \Phi_n$  to prove Theorem 6.3 in the case where  $X$  is finite.

Firstly, it follows from the general theory of [Goo03] that there is a natural equivalence

$$\Phi_n(P_k(\text{Id}))(X) \simeq P_k(\Phi_n)(X)$$

and therefore

$$\partial_*(\Phi_n) \simeq (\sigma^{-1}\mathcal{L}ie_*)_{T(n)}.$$

It follows from (6.4) that the dual derivatives are given by

$$\partial^*(\Phi_n) \simeq (1_*)_{T(n)}$$

Since  $\sigma^{-1}\mathcal{L}ie_k$  is level-wise finite, we may apply Koszul duality for right modules [MT24] to deduce that there is an equivalence

$$\Psi(\Phi_n)(X) \simeq \underline{\text{Mod}}_{\mathcal{C}omm}^{rt}(1_*, \Sigma_{T(n)}^\infty X^\otimes) = \text{Prim}_{\mathcal{C}omm}(\Sigma_{T(n)}^\infty X).$$

Finally, since Kuhn proved norm maps in  $\text{Sp}_{T(n)}$  are equivalences [Kuh04], it follows from (6.5) that there is a natural equivalence

$$P_\infty(\Phi_n)(X) \xrightarrow{\simeq} \Psi(\Phi_n)(X).$$

We deduce that the comparison map (6.1) may be identified with the composite

$$\Phi_n(X) \rightarrow P_\infty(\Phi_n)(X) \xrightarrow{\simeq} \text{Prim}_{\mathcal{C}omm}(\Sigma_{T(n)}^\infty X).$$

Theorem 6.3 follows.

**The  $v_n$ -periodic Goodwillie spectral sequence.** The Taylor tower for  $\Phi_n$  gives rise to the  $v_n$ -periodic Goodwillie spectral sequence (which converges when  $X$  is  $\Phi_n$ -good)

$$v_n^{-1}{}^{gss}E_1^{t,*}(X) = \pi_t \mathcal{L}ie(\Sigma^{-1}\Sigma^\infty X)_{T(n)} \Rightarrow v_n^{-1}\pi_{t+1}^\wedge(X).$$

Arone and Mahowald [AM99] showed that in the case where  $X = S^d$  (and  $d$  is odd if  $p$  is odd),  $v_n^{-1}{}^{gss}E_1^{t,k}(S^d) = 0$  unless  $k = p^i \leq p^n$ , and they use this to prove that spheres are  $\Phi_n$ -good. The  $v_1$ -periodic GSS was computed for  $S^d$  by Mahowald [Mah82] for  $p = 2$  and Thompson [Tho90] for  $p$  odd.

As  $T(n)$ -local homotopy groups are largely incomputable at present for  $n > 1$ , one may alternatively consider the  $K(n)$ -local Goodwillie spectral sequence

$${}_{K(n)}^{gss}E_1^{t,*} = \pi_t \mathcal{L}ie(\Sigma^{-1}\Sigma^\infty X)_{K(n)} \Rightarrow \pi_{t+1}\Phi_n(X)_{K(n)}.$$

The  $K(2)$ -local GSS for  $S^3$  and  $p \geq 5$  was computed by Wang in [Wan15].

In general, the homotopy groups of the  $K(n)$ -localization of a spectrum  $Z$  may be computed by its  $K(n)$ -local Adams-Novikov spectral sequence, which by the Morava change of rings theorem [Mor85] takes the form

$$H_c^s(\mathbb{G}_n; (E_n)_t Z) \Rightarrow \pi_{t-s} Z_{K(n)}.$$

Here  $(E_n)_* Z$  is the (completed) Morava  $E$ -homology, and  $\mathbb{G}_n$  is the  $n$ th (extended) Morava stabilizer group. Thus the input needed to study the  $K(n)$ -local GSS is  $(E_n)_* \mathcal{L}ie(Z)$ .

The Morava  $E$ -theory of  $\mathcal{L}ie(Z)$  was computed by Brantner [Bra17] in the case where  $(E_n)_*Z$  is flat over  $(E_n)_*$ . We briefly summarize his result. Let  $\Delta$  denote the Dyer-Lashof algebra for Morava  $E$ -theory, which acts on the Morava  $E$ -cohomology of any space. The algebra  $\Delta$  was shown by Rezk to be Koszul [Rez17]. Define the algebra of *Hecke operations*  $\mathcal{H}^{\mathcal{L}ie}$  to be the Koszul dual algebra of  $\Delta$  (in the sense of [Pri70]). For simplicity, assume  $p$  is odd. Then Brantner showed that  $(E_n)_*\mathcal{L}ie(Z)$  is the free complete Hecke-Lie algebra on  $(E_n)_*Z$ :

$$(E_n)_*\mathcal{L}ie(Z) = [\mathcal{H}^{\mathcal{L}ie} \otimes_{(E_n)_*} \mathcal{L}ie^{(E_n)_*}((E_n)_*Z)]_I^\wedge.$$

In the case of  $n = 2$ , the algebra  $\mathcal{H}^{\mathcal{L}ie}$ , and the Morava  $E$ -theory  $(E_2)_*\Phi_2(S^{2i+1})$ , was determined by Zhu [Zhu18].

The first non-trivial differentials in the  $K(n)$ -local GSS are given by analogs of the formula (4.3). In the case of  $n = 1$  and  $p$  odd, Kjaer used this to compute the  $v_1$ -periodic GSS in its entirety for  $X$  a simply connected finite  $H$ -space [Kja19]. By comparing his results with the work of Bousfield [Bou99], Kjaer established that for  $p$  odd, all finite  $H$ -spaces are  $\Phi_1$ -good. This suggests that the higher chromatic analogs of the right-hand column of (5.1) should be better behaved than the higher chromatic analogs of the left-hand column. Progress on the study of  $T(n)$ -local Hopf algebras is being made in ongoing work of Brantner, Hahn, Heuts, and Yuan, who have proposed that it may be the case that *all* loop spaces are  $\Phi_n$ -good.

## REFERENCES

- [AC11] Greg Arone and Michael Ching, *Operads and chain rules for the calculus of functors*, *Astérisque* (2011), no. 338, vi+158. MR 2840569
- [AC15] Gregory Arone and Michael Ching, *A classification of Taylor towers of functors of spaces and spectra*, *Adv. Math.* **272** (2015), 471–552. MR 3303239
- [AC20] Omar Antolín Camarena, *The mod 2 homology of free spectral Lie algebras*, *Trans. Amer. Math. Soc.* **373** (2020), no. 9, 6301–6319. MR 4155178
- [AK98] Greg Arone and Marja Kankaanrinta, *A functorial model for iterated Snaitth splitting with applications to calculus of functors*, *Stable and unstable homotopy* (Toronto, ON, 1996), *Fields Inst. Commun.*, vol. 19, Amer. Math. Soc., Providence, RI, 1998, pp. 1–30. MR 1622334
- [AM99] Greg Arone and Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, *Invent. Math.* **135** (1999), no. 3, 743–788. MR 1669268
- [BC70] A. K. Bousfield and E. B. Curtis, *A spectral sequence for the homotopy of nice spaces*, *Trans. Amer. Math. Soc.* **151** (1970), 457–479. MR 267586
- [BCK<sup>+</sup>66] A. K. Bousfield, E. B. Curtis, D. M. Kan, D. G. Quillen, D. L. Rector, and J. W. Schlesinger, *The mod  $-p$  lower central series and the Adams spectral sequence*, *Topology* **5** (1966), 331–342. MR 199862
- [Beh12] Mark Behrens, *The Goodwillie tower and the EHP sequence*, *Mem. Amer. Math. Soc.* **218** (2012), no. 1026, xii+90. MR 2976788
- [BHLS23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank,  *$K$ -theoretic counterexamples to Ravenel’s telescope conjecture*, 2023.
- [Bou79] A. K. Bousfield, *The localization of spectra with respect to homology*, *Topology* **18** (1979), no. 4, 257–281. MR 551009
- [Bou99] ———, *The  $K$ -theory localizations and  $v_1$ -periodic homotopy groups of  $H$ -spaces*, *Topology* **38** (1999), no. 6, 1239–1264. MR 1690156
- [BR20a] Mark Behrens and Charles Rezk, *The Bousfield-Kuhn functor and topological André-Quillen cohomology*, *Invent. Math.* **220** (2020), no. 3, 949–1022. MR 4094969

- [BR20b] ———, *Spectral algebra models of unstable  $v_n$ -periodic homotopy theory*, Bousfield classes and Ohkawa's theorem, Springer Proc. Math. Stat., vol. 309, Springer, Singapore, [2020] ©2020, pp. 275–323. MR 4100660
- [Bra17] David Lukas Benjamin Brantner, *The Lubin-Tate Theory of Spectral Lie Algebras*, ProQuest LLC, Ann Arbor, MI, 2017, Thesis (Ph.D.)—Harvard University. MR 4172386
- [Chi05] Michael Ching, *Bar constructions for topological operads and the Goodwillie derivatives of the identity*, *Geom. Topol.* **9** (2005), 833–933. MR 2140994
- [Cur65] Edward B. Curtis, *Some relations between homotopy and homology*, *Ann. of Math. (2)* **82** (1965), 386–413. MR 184231
- [Esp22] Hadrien Espic, *Koszul duality for categories with a fixed object set*, 2022.
- [Fre09] Benoit Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics, vol. 1967, Springer-Verlag, Berlin, 2009. MR 2494775
- [Goo03] Thomas G. Goodwillie, *Calculus. III. Taylor series*, *Geom. Topol.* **7** (2003), 645–711. MR 2026544
- [Heu21] Gijts Heuts, *Lie algebras and  $v_n$ -periodic spaces*, *Ann. of Math. (2)* **193** (2021), no. 1, 223–301. MR 4199731
- [HS98] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, *Ann. of Math. (2)* **148** (1998), no. 1, 1–49. MR 1652975
- [Joh95] Brenda Johnson, *The derivatives of homotopy theory*, *Trans. Amer. Math. Soc.* **347** (1995), no. 4, 1295–1321. MR 1297532
- [Kja18] Jens Jakob Kjaer, *On the odd primary homology of free algebras over the spectral lie operad*, *J. Homotopy Relat. Struct.* **13** (2018), no. 3, 581–597. MR 3856302
- [Kja19] Jens Jakob Kjaer, *Unstable  $v_1$ -periodic homotopy of simply connected, finite  $h$ -spaces, using goodwillie calculus*, 2019.
- [Kon23] Nikolay Konovalov, *Algebraic Goodwillie spectral sequence*, 2023.
- [Kuh04] Nicholas J. Kuhn, *Tate cohomology and periodic localization of polynomial functors*, *Invent. Math.* **157** (2004), no. 2, 345–370. MR 2076926
- [Lin81] Wên Hsiung Lin, *Algebraic Kahn-Priddy theorem*, *Pacific J. Math.* **96** (1981), no. 2, 435–455. MR 637982
- [Lur17] Jacob Lurie, *Higher algebra*, 2017.
- [Mah82] Mark Mahowald, *The image of  $J$  in the EHP sequence*, *Ann. of Math. (2)* **116** (1982), no. 1, 65–112. MR 662118
- [MM65] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, *Ann. of Math. (2)* **81** (1965), 211–264. MR 174052
- [Mor85] Jack Morava, *Noetherian localisations of categories of cobordism comodules*, *Ann. of Math. (2)* **121** (1985), no. 1, 1–39. MR 782555
- [MT24] Connor Malin and Niall Taggart, *Koszul duality and a classification of stable Weiss towers*, 2024.
- [Pri70] Stewart B. Priddy, *Koszul resolutions*, *Trans. Amer. Math. Soc.* **152** (1970), 39–60. MR 265437
- [Pri73] ———, *Transfer, symmetric groups, and stable homotopy theory*, Algebraic  $K$ -theory, I: Higher  $K$ -theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., vol. Vol. 341, Springer, Berlin-New York, 1973, pp. 244–255. MR 350727
- [Qui69] Daniel Quillen, *Rational homotopy theory*, *Ann. of Math. (2)* **90** (1969), 205–295. MR 258031
- [Rec66] David L. Rector, *An unstable Adams spectral sequence*, *Topology* **5** (1966), 343–346. MR 199863
- [Rez17] Charles Rezk, *Rings of power operations for Morava  $E$ -theories are Koszul*, 2017.
- [Sam53] Hans Samelson, *A connection between the Whitehead and the Pontryagin product*, *American Journal of Mathematics* **75** (1953), no. 4, 744–752.
- [Sul77] Dennis Sullivan, *Infinitesimal computations in topology*, *Inst. Hautes Études Sci. Publ. Math.* (1977), no. 47, 269–331. MR 646078
- [Tho90] Robert D. Thompson, *The  $v_1$ -periodic homotopy groups of an unstable sphere at odd primes*, *Trans. Amer. Math. Soc.* **319** (1990), no. 2, 535–559. MR 1010890
- [Wal06] Ben C. Walter, *Rational homotopy calculus of functors*, ProQuest LLC, Ann Arbor, MI, 2006, Thesis (Ph.D.)—Brown University. MR 2709120

- [Wan15] Guozhen Wang, *Unstable chromatic homotopy theory*, ProQuest LLC, Ann Arbor, MI, 2015, Thesis (Ph.D.)–Massachusetts Institute of Technology. MR 3427198
- [Zhu18] Yifei Zhu, *Morava  $E$ -homology of Bousfield-Kuhn functors on odd-dimensional spheres*, Proc. Amer. Math. Soc. **146** (2018), no. 1, 449–458. MR 3723154