THE STRUCTURE OF THE $v_2$-LOCAL ALGEBRAIC tmf RESOLUTION

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Abstract. We give a complete description of the $E_1$-term of the $v_2$-local as well as $g$-local algebraic tmf resolution.

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1. Introduction

Let $bo$ denote the connective real $K$-theory spectrum. Mahowald and his collaborators used the $bo$ resolution (aka the $bo$-based Adams spectral sequence) to study stable homotopy groups to great effect. Specifically, they computed the image of the $J$-homomorphism [DM89], proved the 2-primary height 1 telescope conjecture [Mah81], [LM87], computed the unstable $v_1$-periodic homotopy groups of spheres [Mah82], and applied homotopy theoretic methods to a variety of geometric problems [DGM81].
The spectrum $bo$ has two distinct advantages that lend itself to these applications at the prime 2. Firstly, $\pi_0 bo$ is torsion free and $\pi_1 bo$ is Bott periodic (i.e. $v_1$-torsion free), so it is equipped to detect the zeroth and first layers of the chromatic filtration. Secondly, $v_1$-periodic homotopy at the prime 2 is more complicated than at odd primes, and this is witnessed by the elements $\eta$ and $\eta^2$ generating additional anomalous torsion [Ada66]. These elements and their $v_1$-multiples are detected by the $bo$-Hurewicz homomorphism

$$\pi_* \to \pi_* bo.$$ 

At chromatic height 2, the 2-primary stable stems have a vast collection of anomalous torsion, and a significant portion of this $v_2$-periodic torsion is detected by the spectrum $tmf$ of topological modular forms (see [BMQ21]). As such the $tmf$ resolution represents a significant upgrade to the $bo$ resolution. Indeed, partial analysis of the $tmf$ resolution has resulted in numerous powerful results [BHMM08], [BHMM20], [BBB+21], [BMQ21].

For a spectrum $X$, the $tmf$ resolution of $X$ is the tower of cofiber sequences

\begin{equation}
X \leftarrow \Sigma^{-1} tmf \wedge X \leftarrow \Sigma^{-2} tmf^{\wedge 2} \wedge X \leftarrow \cdots
\end{equation}

Here $tmf$ is the cofiber of the unit

$$S \to tmf \to tmf.$$ 

Applying $\pi_*$ to the tower above results in the $tmf$-based Adams spectral sequence

$$tmf E_{n,t}^2(X) = \pi_t(tmf \wedge tmf^{\wedge n} \wedge X) \Rightarrow \pi_{t-n} X.$$ 

Ultimately, the successful applications of the $tmf$-resolution so far have been limited by our ability to compute the $E_2$-page of the $tmf$-based Adams spectral sequence — computations to date have relied on computations of the $E_1$-page in certain regions. Unlike the $bo$ case, we are not able to completely compute this $E_1$ page for $X = S$. The goal of this paper is to make a significant step towards rectifying this deficiency.

The computations of the $E_1$-page that have been successfully performed used the classical Adams spectral sequence. We focus our attention at the prime 2. Recall that for a connective spectrum $Y$, the mod 2 Adams spectral sequence (ASS) takes the form

$$ass E_2^{s,t}(Y) = \Ext^{s,t}_{A_2}(\mathbb{F}_2, H_* Y) \Rightarrow \pi_{t-s} Y_*$$

where $H_*$ denotes mod 2 homology and $A_*$ is the dual Steenrod algebra. The $E_1$-term of the $tmf$-resolution than can then itself be approached by computing the ASS’s

$$ass E_2^{s,t}(tmf \wedge tmf^{\wedge n} \wedge X) \Rightarrow \pi_{t-n}(tmf \wedge tmf^{\wedge n} \wedge X) = tmf E_1^{s,t-n}(X).$$ 

In practice, given the computation of the $E_2$-pages, these Adams spectral sequences can be completely computed, as the majority of the differentials can be deduced
from the Adams spectral sequence for tmf (as computed in [BR22]). The tmf-resolution can then be studied through the Miller square [Mil81]

\[ \text{ass}\ E_{s,t}^2(\text{tmf} \wedge \text{tmf}^n \wedge X) \rightarrow \text{ASS} \text{tmf} E_1^{n,t-s} (X) \]

Here, the left side of the square is the algebraic tmf-resolution, the analog of the tmf-resolution obtained by applying Ext to (1.1). The starting point is therefore the computation of the \( E_1 \)-page of the algebraic tmf-resolution of the sphere

\[ \text{ass}\ E_{s,t}^2(\text{tmf} \wedge \text{tmf}^n). \]

Analogous to the case of the bo-resolution and the \( BP(2) \)-resolution [Mah81] [Cul19], we propose the following conjecture.

**Conjecture 1.2.** The map

\[ \text{ass} E_{s,t}^2(\text{tmf} \wedge \text{tmf}^n) \rightarrow v_2^{-1} \text{ass} E_{s,t}^2(\text{tmf} \wedge \text{tmf}^n) \]

is injective for \( s > 0 \).

This conjecture is consistent with computations in low degrees (see, for instance, [BOSS19]). It implies a good-evil decomposition of the tmf-resolution of the sphere, analogous to that of [BBB+20], [BBB+21].

In this paper we give a complete computation of

\[ v_2^{-1} \text{ass} E_{s,t}^2(\text{tmf} \wedge \text{tmf}^n). \]

We now summarize the main results.

For a graded Hopf algebra \( \Gamma \) over \( k \), let \( \mathcal{D}_\Gamma \) denote Hovey’s stable homotopy category of \( \Gamma \)-comodules. Briefly, \( \mathcal{D}_\Gamma \) is similar to the derived category, with the chief difference that weak equivalences are defined to be the \( \pi_{*,*}^\Gamma \)-isomorphisms, where for a \( \Gamma \)-comodule \( M \), the homotopy groups \( \pi_{*,*}^\Gamma \) are defined to be

\[ \pi_{k,l}^\Gamma (M) := \text{Ext}_{k}^{k,s+n}(k, M). \]

For \( M \in \mathcal{D}_\Gamma \), we let \( \Sigma^{n,s} M \) denote a shift in internal degree by \( s + n \) and in cohomological degree by \( s \), so we have

\[ \pi_{k,l}^\Gamma (\Sigma^{n,s} M) = \pi_{k-n,t-s}^\Gamma (M) \]

and

\[ [\Sigma^{n,s} k, M]_{\Gamma} = \pi_{n,s}^\Gamma (M). \]

For a spectrum \( X \), we shall let

\[ \mathcal{X} \in \mathcal{D}_A \]

denote the object associated to the mod 2 homology \( H_*X \). In this notation the ASS takes the form

\[ \text{ass} E_{s,t}^2(X) = \pi_{t-s,s}^A(\mathcal{X}) \Rightarrow \pi_{t-s} \mathcal{X}^\wedge. \]
Since \( \text{tmf} = (A/\mathbb{A}(2)) \) \([\text{Mat16}]\), where \( A(2) \) is the subalgebra of the mod 2 Steenrod algebra generated by \( Sq^1, Sq^2, \) and \( Sq^4 \), we have a change of rings isomorphism

\[
\pi^*_{A,*}(\text{tmf} \otimes M) \cong \pi^*_{A(2),*}(M)
\]

for any \( M \in D_{A,*} \). Therefore the \( E_1 \)-term of the algebraic tmf-resolution takes the form

\[
\pi^*_{A,*}(\text{tmf} \otimes M) \cong \pi^*_{A(2),*}(\text{tmf}^{\otimes n})
\]

There is a decomposition \([BHHM08]\)

\[
tmf^{\otimes n} \cong \bigoplus_{i_1, \ldots, i_n > 0} \Sigma^{8(i_1 + \cdots + i_n)} \text{bo}_{i_1} \otimes \cdots \otimes \text{bo}_{i_n}
\]

in \( D_{A(2),*} \), where \( \text{bo}_i \) denotes the homology of the \( i \)th bo-Brown-Gitler spectrum (see Section 2).

For an object \( M \in D_{A(2),*} \), the localization \( v_2^{-1}M \) denotes the localization of \( M \) with respect to the element

\[
v_2^8 \in \pi^{A(2)}_{48,8}(\mathbb{F}_2),
\]

so we have

\[
v_2^{-1} \pi^*_{A,*}(\text{tmf} \otimes M) \cong \pi^*_{A(2),*}(v_2^{-1} \text{tmf}^{\otimes n})
\]

We will prove

**Theorem 1.5** (see Corollary 8.6 and 2.9). There are equivalences in \( D_{A(2)} \).

\[
v_2^{-1} \text{bo}_{2j} \cong \Sigma^{8j} v_2^{-1} \text{bo}_j \oplus \Sigma^{8j+8,1} v_2^{-1} \text{bo}_{j-1},
v_2^{-1} \text{bo}_{2j+1} \cong v_2^{-1} \Sigma^{8j} \text{bo}_j \otimes \text{bo}_1.
\]

The splittings of (1.4) and Theorem 1.5 inductively imply that in \( D_{A(2)} \), the objects \( v_2^{-1} \text{tmf}^{\otimes n} \) split as a wedge of bigraded suspensions of \( v_2^{-1} \text{bo}_1^{\otimes k} \). We are left with identifying these explicitly.

To this end we will introduce an object

\[
\text{TMF}_0(3) \in D_{A(2)}
\]

which serves as an algebraic version of the tmf-module \( \text{TMF}_0(3) \) (the theory of topological modular forms associated to the congruence subgroup \( \Gamma_0(3) < SL_2(\mathbb{Z}) \)), and prove

**Theorem 1.6** (Proposition 5.1 and 5.2). There are splittings in \( D_{A(2)} \).

\[
v_2^{-1} \text{bo}_1^{\otimes 3} \cong 2 \Sigma^{16,1} v_2^{-1} \text{bo}_1 \oplus \Sigma^{24,2} \text{TMF}_0(3),\n\text{TMF}_0(3) \otimes \text{bo}_1 \cong \Sigma^{24,3} \text{TMF}_0(3) \oplus \Sigma^{40,6} \text{TMF}_0(3).
\]

The splittings of Theorem 1.6 imply that the objects \( v_2^{-1} \text{bo}_1^{\otimes k} \) split in \( D_{A(2)} \), as a direct sum of bigraded suspensions of copies of \( v_2^{-1} \text{bo}_1, v_2^{-1} \text{bo}_1, v_2^{-1} \text{bo}_1^{\otimes 2} \), and \( \text{TMF}_0(3) \).

Putting this all together, we have the following theorem (see Corollary 8.7 for a more precise formulation).

\[\text{TMF}_0(3)\]
**Theorem.** There is a splitting of
\[ v_2^{-1\text{tmf}} \otimes n \in \mathcal{D}_{A(2)}^v. \]
into a well-described sum of various bigraded suspensions of
\[ \bullet \ v_2^{-1}F_2, \]
\[ \bullet \ v_2^{-1}bo_1, \]
\[ \bullet \ v_2^{-1}bo_1^\otimes 2, \]
\[ \bullet \ \text{TMF}_0(3). \]

The most subtle step to all of this is the first equivalence of Theorem 1.5. Indeed an explicit exact sequence (see (2.5)) of [BHHM08] implies that \( v_2^{-1bo_2} \) is built from \( v_2^{-1}\Sigma^8bo \) and \( v_2^{-1}\Sigma^8+81bo_1 \) in \( \mathcal{D}_{A(2)}^v \). The hard part is showing that the attaching map between these two components is trivial. This is accomplished by showing that if this attaching map is non-trivial, then it is non-trivial after \( g \)-localization where \( g \) is the generator of \( \pi_{20,4}(F_2) \). We then prove the \( g \)-local attaching map is trivial (see Corollary 8.5 and Theorem 9.3), strengthening the results of [BBT21].

**Theorem.** There is a splitting of
\[ g^{-1\text{tmf}} \otimes n \in \mathcal{D}_{A(2)}^v. \]
into a well-described sum of various bigraded suspensions of
\[ \bullet \ g^{-1}F_2, \]
\[ \bullet \ g^{-1}bo_1, \]
\[ \bullet \ g^{-1}bo_1^\otimes 2. \]

The \( v_2 \)-local results of this paper may be applied to understand the TMF-resolution, where
\[ \text{TMF} = \text{tmf}[\Delta^{-1}]. \]

Namely, there are localized ASS’s
\[ \pi_{*,*}^{A(2)}, (v_2^{-1\text{tmf}} \otimes X) \Rightarrow \pi_*(\text{TMF} \wedge \text{TMF}^{\wedge 8} \wedge X)_2^v. \]

Our \( v_2 \)-local results also may be used to understand the \( v_2 \)-localized algebraic tmf resolution
\[ v_2^{-1}A^{A(2)}, (\text{tmf} \otimes M) \Rightarrow v_2^{-1}\pi^{A*}(M). \]

Here, the \( v_2 \)-localized Ext groups \( v_2^{-1}\pi^{A*} \) are as defined in [MS87].

The \( g \)-local results of this paper may be applied to understand \( g \)-local Ext over the Steenrod algebra, using the \( g \)-local algebraic tmf-resolution
\[ \pi_{*,*}^{A(2)}, (g^{-1\text{tmf}} \otimes M) \Rightarrow g^{-1}\pi^{A*}(M). \]
Organization of the paper. In Section 2 we reduce the study of tmf to the bo-Brown-Gitler comodules bo. We review exact sequences which relate these comodules to bo⊗k. Upon v_2-localization, we show that these exact sequences give complete decompositions of v_2^{-1}bo in terms of bigraded suspensions of v_2^{-1}bo⊗k for various k, provided certain obstructions ∂_j vanish for j' ≤ j/2.

In Section 3 we review the structure of π^{A(2)}_*(bo⊗k) for 0 ≤ k ≤ 4. These will form the computational input for the rest of the paper.

In Section 4 we construct TMF_0(3) ∈ D_A(2), our algebraic analog of TMF_0(3), and establish some basic properties.

In Section 5 we prove a few key splitting theorems that inductively give complete decompositions of bo⊗k ∈ D_A(2), into indecomposable summands. Provided the obstructions ∂_j vanish, we therefore get complete decompositions of v_2^{-1}bo.

In Section 6 we define certain generating functions which conveniently allow for algebraic computation of the putative decompositions of v_2^{-1}bo.

In Section 7 we explain the analogs of the v_2-local decompositions of bo and bo⊗k in the g-local category. The decompositions of g^{-1}bo depend on the vanishing of certain obstructions ∂_j.

Section 8 we prove our main result: the obstructions ∂_j and ∂'_j vanish for all j. This results in a complete decomposition of v_2^{-1}tmf⊗n and g^{-1}tmf⊗n.

In Section 9 we relate our g-local results to the computations of Bhattacharya, Bobkova, and Thomas [BBT21], providing a strengthening of their results.

In Appendix A we discuss a stable splitting of bo∧3 and its relationship with Theorem 1.6.

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2. bo-Brown-Gitler comodules

In this section we reduce the analysis of v_2^{-1}tmf⊗n to the analysis of v_2-local bo-Brown-Gitler comodules. These are A_1-comodules which are the homology of the bo-Brown-Gitler spectra constructed by [GJM86]. Mahowald used integral Brown-Gitler spectra to analyze the bo resolution [Mah81]. The bo-Brown-Gitler comodules play a similar role in the algebraic tmf resolution [BHHM08], [MR09], [DM10], [BOSS19], [BHHM20], [BMQ21].

Endow the mod 2 homology of bo

\[ \text{bo} \cong A//A(1)_* = \mathbb{F}_2[\zeta_4, \zeta_2, \zeta_3, \ldots] \]
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(2.1) $\text{wt}(\zeta_i) = 2^{i-1}$.

The $i$th bo-$Brown$-$Gitler$ comodule is the submodule $bo_i \subset A/A(1)_*$ spanned by monomials of weight less than or equal to $4i$.

For an object $M \in D_{A(2)*}$, let $DM = \text{Hom}_{F_2}(M, F_2)$ be its $F_2$-linear dual. We record the following useful result.

**Proposition 2.2.** There is an equivalence $v_2^{-1}Dbo_1 \cong \Sigma^{-16,-1}v_2^{-1}bo_1$.

**Proof.** This follows from the short exact sequence

$0 \to bo_1 \to A(2)//A(1)_* \to \Sigma^{17}Dbo_1 \to 0$.

□

Our interest in the bo-$Brown$-$Gitler$ comodules stems from the fact that there is a splitting of $A(2)_*$-comodules [BHHM08, Cor. 5.5]:

(2.3) $\text{tmf} \cong \bigoplus_{i \geq 0} \Sigma^{8i}bo_i$

where $\Sigma^{8j}bo_j$ is spanned by the monomials of

$\text{tmf} = A//A(2)_* = F_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots]$ of weight $8j$. We therefore have a splitting of $A(2)_*$-comodules

(2.4) $\text{tmf}^\otimes n \cong \bigoplus_{i_1, \ldots, i_n > 0} \Sigma^{8(i_1 + \cdots + i_n)}bo_{i_1} \otimes \cdots \otimes bo_{i_n}$.

The object

$\Sigma^{8(i_1 + \cdots + i_n)}bo_{i_1} \otimes \cdots \otimes bo_{i_n} \in D_{A(2)*}$

can be inductively built from $bo_i^{\otimes k}$ by means of a set of exact sequences of $A(2)_*$-comodules which relate the $bo_i$’s [BHHM08, Sec. 7]:

(2.5) $0 \to \Sigma^{8j}bo_j \to bo_{2j} \to A(2)//A(1)_* \otimes \text{tmf}_{j-1} \to \Sigma^{8j+9}bo_{j-1} \to 0$,

(2.6) $0 \to \Sigma^{8j}bo_j \otimes bo_1 \to bo_{2j+1} \to A(2)//A(1)_* \otimes \text{tmf}_{j-1} \to 0$.

Here, $\text{tmf}_j$ is the $j$th $tmf$-$Brown$-$Gitler$ comodule — it is the subcomodule of $\text{tmf}$ spanned by monomials of weight less than or equal to $8j$.

**Remark 2.7.** Technically speaking, as is addressed in [BHHM08, Sec. 7], the comodules $A(2)//A(1)_* \otimes \text{tmf}_{j-1}$ in the above exact sequences have to be given a slightly different $A(2)_*$-comodule structure from the standard one arising from the tensor product. However, this
different comodule structure ends up being Ext-isomorphic to the standard one. As the analysis of this paper only requires
\[ v_2^{-1} A(2) / A(1) \otimes \text{tmf}_{j-1} \simeq 0, \]
\[ g^{-1} A(2) / A(1) \otimes \text{tmf}_{j-1} \simeq 0, \]
and these equivalences hold for the non-standard comodule structures, the reader can safely ignore this subtlety.

Since
\[ v_2^{-1} A(2) / A(1) \otimes \text{tmf}_{j-1} \simeq 0, \]
The exact sequences (2.5) and (2.6) give rise to a cofiber sequence in \( D_{A(2)} \).
\begin{equation}
\Sigma^{8j} v_2^{-1} \text{bo}_j \to v_2^{-1} \text{bo}_{2j} \to \Sigma^{8j+8,1} v_2^{-1} \text{bo}_{j-1}
\end{equation}
and an equivalence
\begin{equation}
\Sigma^{8j} v_2^{-1} \text{bo}_j \otimes \text{bo}_1 \simeq v_2^{-1} \text{bo}_{2j+1}.
\end{equation}
Thus, (2.8) and (2.9) inductively build \( v_2^{-1} \text{bo}_k \in D_{A(2)} \).

The connecting homomorphism of the cofiber sequence (2.8)
\begin{equation}
\partial_j : v_2^{-1} \Sigma^{8j+8,1} \text{bo}_{j-1} \to v_2^{-1} \Sigma^{8j+1,-1} \text{bo}_j
\end{equation}
is the obstruction to the cofiber sequence being split. We will prove in Section 8 that the connecting homomorphism \( \partial_j = 0 \) for all \( j \), so we have
\begin{equation}
v_2^{-1} \text{bo}_{2j} \simeq v_2^{-1} \Sigma^{8j} \text{bo}_j \oplus v_2^{-1} \Sigma^{8j+8,1} \text{bo}_{j-1}.
\end{equation}

3. THE GROUPS \( \pi_{*,*}^{A(2)}(\text{bo}^k) \)

In the previous section we related the comodules \( \text{bo}_j \) to the comodules \( \text{bo}_1^k \). We now review the structure of \( \pi_{*,*}^{A(2)}(\text{bo}^k) \) for \( 0 \leq k \leq 4 \).

In order to give names to the \( v_0 \)-torsion-free generators of \( \pi_{*,*}^{A(2)}(\text{bo}^k) \), we review the corresponding \( v_0 \)-local computations. The entire structure of the \( v_0 \)-local algebraic tmf resolution is given in [BMQ21] (see also [BOSS19]).

Observe that we have
\begin{equation}
v_0^{-1} \pi_{*,*}^{A(2)}(F_2) = \mathbb{F}_2[c_1, c_2].
\end{equation}
Note that \( c_4, c_6 \in (\text{tmf}_*)_\mathbb{Q} \) are detected in the \( v_0 \)-localized ASS by \( v_1^4 \) and \( v_0^3 v_2^2 \), respectively.

We have (regarding \( \text{bo}_1 \) as a subcomodule of \( A \otimes A(2)_* \))
\[ v_0^{-1} \pi_{*,*}^{A(2)}(\text{bo}_1) = \mathbb{F}_2[v_0^\pm, v_1^4, v_2^2, \xi_1, \xi_2] \]
We therefore have an isomorphism
\[(3.2) \quad \nu_1^{-1} \pi_*^{(2)}(\mathbb{H}^k) \cong \mathbb{F}_2[v_0^+, v_1^4, v_2^2] \otimes \mathbb{F}_2 \{\bar{\xi}_1, \bar{\xi}_2\}^k.\]

To make for more compact notation, we will use bars to denote elements of tensor powers:
\[(3.3) \quad x_1 \cdots x_n := x_1 \otimes \cdots \otimes x_n.\]

\[\pi_*^{(2)}(\mathbb{F}_2) : (\text{Figure 3.1})\]

All of the elements are \(c_4 = v_1^4\)-periodic, and \(v_2^8\)-periodic. Exactly one \(v_1^4\) multiple of each element is displayed with the \(\bullet\) replaced by a \(\circ\). Observe the wedge pattern beginning in \(t-s = 35\). This pattern is infinite, propagated horizontally by \(h_{2,1}\)-multiplication and vertically by \(v_1\)-multiplication. Here, \(h_{2,1}\) is the name of the generator in the May spectral sequence of bidegree \((t-s, s) = (5, 1)\), and \(h_{2,1}^2 = g.\)

\[\pi_*^{(2)}(\mathbb{F}_2) \otimes k, \quad k = 1, 2, 3, 4 : (\text{Figures 3.2, 3.3, 3.4, 3.5})\]

Every element is \(v_2^8\)-periodic. However, unlike \(\pi_*^{(2)}(\mathbb{F}_2)\), not every element of these Ext groups is \(v_1^4\)-periodic. Rather, it is the case that either an element \(x \in \text{Ext}_2^{(2), (\mathbb{H}^k)}\) satisfies \(v_1^4x = 0\), or it is \(v_1^4\)-periodic. Each of the \(v_1^4\)-periodic elements fit into families which look like shifted and truncated copies of \(\pi_*^{(1)}(\mathbb{F}_2)\), and are labeled with a \(\circ\). We have only included the beginning of these \(v_1^4\)-periodic patterns in the chart. The other generators are labeled with a \(\bullet\). A \(\square\) indicates a polynomial algebra \(\mathbb{F}_2[h_{2,1}]\). Elements which are \(v_0\)-torsion-free are named in these charts using \((3.2)\), in the bar notation of \((3.3)\).

4. AN ALGEBRAIC MODEL OF TMF\(_0(3)\)

The spectrum \(\text{TMF}_0(3)\) is an analog of TMF associated to the moduli of elliptic curves with with \(\Gamma_0(3)\)-structures introduced and studied by Mahowald and Rezk [MR09]. In fact, Mahowald and Rezk proposed three different connective spectra whose \(E(2)\)-localizations are \(\text{TMF}_0(3)\) (also see [DM10]).

We will emulate [MR09, DM10] in the category of \(\text{D}_{A(2)}\) to construct the \(\text{TMF}_0(3)\).

Lemma 4.1. The composite
\[\Sigma^6.2 \mathbb{F}_2 \xrightarrow{h_2^2} \mathbb{F}_2 \hookrightarrow \Sigma^7 \text{D}_{\mathbb{H}_1}\]
extends to a map
\[\tilde{h}_2^2 : \Sigma^6.2 \mathbb{H}_1 \to \Sigma^7 \text{D}_{\mathbb{H}_1} .\]

Our algebraic model of \(\text{TMF}_0(3)\) is defined to be
\[\text{TMF}_0(3) := v_2^{-1}(\Sigma^{24.3} \text{D}_{\mathbb{H}_1} \cup_{h_2^2} \Sigma^{24.4} \text{D}_{\mathbb{H}_1}).\]

Figure 4.1 shows a computation of the homotopy of \(\text{D}_{\mathbb{H}_1} \cup_{h_2^2} \Sigma^{24.3} \text{D}_{\mathbb{H}_1}\). In this figure, the solid dots correspond to \(\text{D}_{\mathbb{H}_1}\) and the open dots correspond to \(\text{D}_{\mathbb{H}_1}\).

One convenient way of accessing the homotopy of \(\text{D}_{\mathbb{H}_1}\) is from the short exact
Figure 3.1. $\pi_{s_*}^{A(2)} (\mathbb{F}_2)$. 
Figure 3.2. $\pi_{*,*}^{A(2), (\text{bo}_1)}$. 
$\Ext_{A(2)}(\mathbf{b}_1 \otimes^2 \mathbf{b}_2)$
Figure 3.4. $\pi_{\ast \ast}^{A(2)} (\text{bo}_1 \otimes 3)$. 
Figure 3.5. $\pi_{A^{(2)\ast}} (\mathcal{B}^{\otimes 4})$. 
sequence in the proof of Proposition 2.2. A chart of $\pi^{A(2)*}_{*,*}(\text{TMF}_0(3))$ is displayed in Figure 4.2.

**Lemma 4.2.** Any map

$$f : \text{TMF}_0(3) \to \text{TMF}_0(3)$$

which is the identity on $\pi^{A(2)*}_{0,0}$ is an equivalence.

**Proof.** Let $1_{\text{TMF}_0(3)} \in \pi^{A(2)*}_{0,0}(\text{TMF}_0(3))$ denote the generator. The $\pi^{A(2)*}_{*,*}(\mathbb{F}_2)$-module structure implies $f$ is the identity on $g \cdot 1_{\text{TMF}_0(3)}$ and $v_2^4 h_1$. It follows from $h_2$ linearity that $f$ is the identity on $x_{17}$ (see Figure 4.2). Therefore $f$ is the identity on $v_2^4 h_1 x_{17}$. It follows from $h_0, h_1, h_2, v_0 v_2, v_0 v_2^2, v_0^2 h_1$, and $g$ multiplications. It then follows that $f$ is a $\pi^{A(2)*}_{*,*}$-isomorphism. □

We have the following algebraic version of the Recognition Principle of Davis-Mahowald-Rezk (see [MR09, Prop. 7.2]).

**Theorem 4.3** (Recognition Principle). Suppose that $X \in \mathcal{D}_{A(2)*}$ satisfies

$$\pi^{A(2)*}_{*,*}(X) \cong \pi^{A(2)*}_{*,*}(\text{TMF}_0(3))$$

where the above isomorphism preserves $v_0, h_1, h_2, v_4^2, v_0 v_2, v_0^2 v_2, v_0^2 h_1, v_2^4 h_1,$ and $g$ multiplications. Then there is an equivalence

$$X \cong \text{TMF}_0(3).$$

**Proof.** Let

$$x_{17} : \Sigma^{17,3} \mathbb{F}_2 \to X$$
represent the generator of $\pi^{A(2)*}_{17,3}(X)$. Since

$$\pi^{A(2)*}_{17,4}(X) = \pi^{A(2)*}_{19,4}(X) = \pi^{A(2)*}_{23,4}(X) = 0,$$

there exists an extension of $x_{17}$ to a map

$$\Sigma^{24,3} D_{b0} \to X.$$

Since

$$\pi^{A(2)*}_{23,5}(X) = \pi^{A(2)*}_{27,5}(X) = \pi^{A(2)*}_{29,5}(X) = \pi^{A(2)*}_{30,5}(X) = 0,$$

there exists a further extension of this map to a map

$$\Sigma^{24,3} D_{b0} \cup \Sigma^{24,4} b_{10} \to X.$$

The conditions on the isomorphism (4.4) imply that $X \cong v_2^{-1} X$. Thus the map above localizes to a map

$$v_2^{-1}(\Sigma^{24,3} D_{b0} \cup \Sigma^{24,4} b_{10}) \to X.$$

The conditions on the isomorphism (4.4) then force the map above to be a $\pi^{A(2)*}_{*,*}$-isomorphism. □
Figure 4.1. Computing the homotopy of $D_{\Sigma^0_1} \cup_{\Sigma^0_2} \Sigma^0_1$. 
Figure 4.2. $\pi_{\ast, \ast}^{A(2)}(\text{TMF}_0(3))$. 
For us, a weak ring object in \( D_{A(2)_*} \) is an object \( R \in D_{A(2)_*} \) with a unit 
\[ u : F_2 \to R \]
and a multiplication 
\[ m : R \otimes R \to R \]
such that the two composites
\[
\begin{align*}
R \otimes F_2 & \xrightarrow{1 \otimes u} R \otimes R \xrightarrow{m} R, \\
F_2 \otimes R & \xrightarrow{u \otimes 1} R \otimes R \xrightarrow{m} R
\end{align*}
\]
are equivalences.

**Proposition 4.5.** \( \text{TMF}_0(3) \) is a weak ring object in \( D_{A(2)_*} \).

**Proof.** We shall need to imitate the “first model” of [MR09], [DM10]. Start with the \( A_* \)-comodule \( \mathcal{Y} \) described in [DM10, Thm. 2.1(a)]. Then the method of proof for [DM10, Thm. 2.1(b)] shows that there exists a map 
\[ h_0 h_2 : \Sigma^{3,2} \mathcal{Y} \to F_2 \]
in \( D_{A_*} \), extending \( h_0 h_2 \), so we can take the cofiber
\[ X := F_2 \cup_{h_0 h_2} \Sigma^{4,1} \mathcal{Y}. \]
Regarding this cofiber as an object of \( D_{A(2)_*} \), define
\[ R := v_2^{-1} X \in D_{A(2)_*}. \]
We will show (a) \( R \simeq \text{TMF}_0(3) \) and (b) \( R \) is a ring object of \( D_{A(2)_*} \).

For (a), we will compute \( \pi^{A(2)_*}_* (R) \). To this end, we observe that the methods of the proof of [DM10, Thm. 2.1(c)] show that there is a map
\[ f : X \to A(2) \amalg A(1)_* \]
which extends the inclusion \( F_2 \hookrightarrow A(2) \amalg A(1)_* \). Let \( C \) be the cofiber of \( f \):
\[ (4.6) \]
\[ X \xrightarrow{f} A(2) \amalg A(1)_* \to C. \]
Then the proof of [DM10, Thm. 2.1(d)] shows that
\[
\pi^{A(2)_*}_* (A(2)_* \otimes C) \cong \begin{cases} 
\Sigma^4 A(2)/A(2)(\text{Sq}^4, \text{Sq}^5 \text{Sq}^1)_*, & s = 0, \\
0, & s > 0.
\end{cases}
\]
as an \( A(2)_* \)-comodule. The \( A(2)_* \)-based Adams spectral sequence for \( C \) then collapses to give an isomorphism
\[
\pi^{A(2)_*}_* (C) \cong \text{Ext}^{s+s}_* (F_2, \Sigma^4 A(2)/A(2)(\text{Sq}^4, \text{Sq}^5 \text{Sq}^1)_*). 
\]
These Ext groups were computed in [DM10, Thm. 2.9]. The cofiber sequence (4.6) gives an equivalence
\[ R \simeq \Sigma^{-1,1} v_2^{-1} C. \]
We see by inspection of Davis-Mahowald’s Ext computation alluded to above that there is an isomorphism
\[
\pi^{A(2)_*}_* (\Sigma^{-1,1} v_2^{-1} C) \cong \pi^{A(2)_*}_* (\text{TMF}_0(3)).
\]
satisfying the hypotheses of the Recognition Principle (Theorem 4.3). We deduce that there is an equivalence

$$\text{TMF}_0(3) \simeq R.$$ 

We now just need to prove $R$ is a ring object in $DA_{(3)}$. For this we imitate the proof of [DM10, Thm. 2.1(e)]. Namely, consider the composite

$$m : X \otimes X \xrightarrow{f \otimes f} A(2) \otimes A(1)_* \xrightarrow{\mu} A(2) \otimes A(1)_*.$$ 

By the cofiber sequence (4.6), the map $m$ lifts to a map

$$m : X \otimes X \to X$$

if the composite

$$X \otimes X \xrightarrow{m} A(2) \otimes A(1)_* \to C$$

is null. In the proof of [DM10, Thm. 2.1(e)], it is established using Bruner’s Ext software that

$$[X \otimes X, C]_{A(2)} = 0.$$ 

Therefore, the lift $m$ exists. Since it is a lift of $m$, it is the identity on the bottom cell. It follows that the composites

$$R \otimes F_2 \to R \otimes R \xrightarrow{m} R,$$

$$F_2 \otimes R \to F_2 \otimes R \xrightarrow{m} R$$

are equivalences. Thus $m$ gives $R$ the structure of a weak ring object. (In fact, the analog of Lemma 4.2 holds for $X$, and so $X$ is also a weak ring object.)

5. SPLITTING $\text{bo}_1 \otimes k$

In this section we prove our main $v_2$-local splitting theorems, which will be the basis of all of our subsequent $v_2$-local decomposition results.

**Proposition 5.1.** There is a splitting

$$v_2^{-1} \text{bo}_1 \otimes 3 \simeq 2\Sigma^{16,1} v_2^{-1} \text{bo}_1 \otimes \Sigma^{24,2} \text{TMF}_0(3).$$

**Proof.** Since we are working in characteristic 2, there is a decomposition

$$\text{bo}_1 \otimes 3 \simeq (\text{bo}_1 \otimes 3)^{\text{h}C_3} \oplus B$$

where $C_3$ acts by cyclically permuting the terms, and we have

$$\pi_*^{A(2)}((\text{bo}_1 \otimes 3)^{\text{h}C_3}) = \pi_*^{A(2)}((\text{bo}_1 \otimes 3)^{\text{h}C_3}).$$

It is easily checked, using the names of the generators in Figure 3.4, that there is an isomorphism

$$v_2^{-1} \pi_*^{A(2)}((\text{bo}_1 \otimes 3)^{\text{h}C_3}) \cong \pi_*^{A(2)}(\text{TMF}_0(3)).$$
A direct application of the Recognition Principle (Theorem 4.3) shows that
\[ v_2^{-1}(bo_1^\otimes 3)hC_3 \simeq \Sigma^{24,2}TMF_0(3). \]

Let
\[ x_{16} : \Sigma^{16,1}_+ F_2 \to bo_1^\otimes 2 \]
correspond to the generator of \( \pi_{16,1}^{A(2)}(bo_1^\otimes 2) \). Then the composite
\[ \Sigma^{16,1}_+ v_2^{-1}bo_1 \oplus \Sigma^{16,1}_+ v_2^{-1}bo_1 \xrightarrow{x_{16} \otimes 1 \otimes x_{16}} v_2^{-1}bo_1^\otimes 3 \to v_2^{-1}B \]
is seen to be a \( \pi_{*,*}^{A(2)} \)-isomorphism, hence an equivalence. \( \square \)

**Proposition 5.2.** There is a splitting
\[ TMF_0(3) \wedge bo_1 \simeq \Sigma^{24,3}TMF_0(3) \oplus \Sigma^{40,6}TMF_0(3). \]

**Proof.** Tensoring the splitting of Proposition 5.1 with \( bo_1 \), we have
\[ v_2^{-1}bo_1^\otimes 4 \simeq 2\Sigma^{16,1}_+ v_2^{-1}bo_1^\otimes 2 \oplus \Sigma^{24,2}TMF_0(3) \wedge bo_1. \]

Examination of \( \pi_{*,*}^{A(2)}(bo_1^\otimes 4) \) (Figure 3.5) reveals that
\[ \pi_{*,*}^{A(2)}(v_2^{-1}bo_1^\otimes 4) \simeq 2\pi_{*,*}^{A(2)}(\Sigma^{16,1}_+ v_2^{-1}bo_1^\otimes 2) \oplus \pi_{*,*}^{A(2)}(\Sigma^{18,5}_+ TMF_0(3)) \oplus \pi_{*,*}^{A(2)}(\Sigma^{64,8}_+ TMF_0(3)). \]

It follows that there is an isomorphism
\[ \pi_{*,*}^{A(2)}(TMF_0(3) \wedge bo_1) \cong \pi_{*,*}^{A(2)}(\Sigma^{24,3}TMF_0(3)) \oplus \pi_{*,*}^{A(2)}(\Sigma^{40,6}TMF_0(3)). \]

Moreover, one can check form the \( \pi_{*,*}^{A(2)}(F_2) \)-module structure of \( \pi_{*,*}^{A(2)}(bo_1^\otimes 4) \) that the isomorphism preserves multiplication by
\[ v_0, v_1^4, v_0v_2, v_2^8, h_1, h_2, g, v_2^2h_1. \]

The map
\[ \Sigma^{24,3}_+ F_2 \oplus \Sigma^{40,6}_+ F_2 \to TMF_0(3) \wedge bo_1 \]
which maps the two generators in gives rise to a map of \( TMF_0(3) \)-modules
\[ \Sigma^{24,3}TMF_0(3) \oplus \Sigma^{40,6}TMF_0(3) \to TMF_0(3) \wedge bo_1. \]

One can then use \( \pi_{*,*}^{A(2)}(F_2) \)-module structures to determine that this map is an isomorphism on \( \pi_{*,*}^{A(2)} \). \( \square \)

**Remark 5.3.** Propositions 5.1 and 5.2 allow one to inductively compute a splitting of \( v_2^{-1}bo_1^\otimes k \) in \( D_{A(2)} \), as a sum of suspensions of \( v_2^{-1}bo_1 \), \( v_2^{-1}bo_1^\otimes 2 \), and \( TMF_0(3) \). For example, we have
\[ v_2^{-1}bo_1^\otimes 4 \simeq (2\Sigma^{16,1}_+ v_2^{-1}bo_1 \oplus \Sigma^{24,2}TMF_0(3)) \otimes bo_1 \]
\[ 2\Sigma^{16,1}_+ v_2^{-1}bo_1^\otimes 2 \oplus \Sigma^{24,2}TMF_0(3) \otimes bo_1 \]
\[ 2\Sigma^{16,1}_+ v_2^{-1}bo_1^\otimes 2 \oplus \Sigma^{48,5}_+ TMF_0(3) \oplus \Sigma^{64,8}_+ TMF_0(3). \]
In the next case, we can further simplify the answer using $v_2^5$ periodicity.

$$v_2^{-1}b_2^6 \simeq (2\Sigma^{16,1} v_2^{-1}b_0^2 + \Sigma^{48,5} \text{TMF}_0(3)) \oplus b_0^1,$$

$$\simeq 2\Sigma^{16,1} v_2^{-1}b_2^3 \oplus \Sigma^{48,5} \text{TMF}_0(3) \oplus b_0^1 \oplus \Sigma^{64,8} \text{TMF}_0(3) \oplus b_0^1,$$

$$\simeq 4\Sigma^{32,2} v_2^{-1}b_0^2 \oplus 2\Sigma^{40,3} \text{TMF}_0(3) \oplus \Sigma^{72,8} \text{TMF}_0(3) \oplus 2\Sigma^{88,11} \text{TMF}_0(3) \oplus \Sigma^{104,14} \text{TMF}_0(3),$$

$$\simeq 4\Sigma^{32,2} v_2^{-1}b_0^1 \oplus 2\Sigma^{24} \text{TMF}_0(3) \oplus 4\Sigma^{40,3} \text{TMF}_0(3) \oplus 4\Sigma^{56,6} \text{TMF}_0(3).$$

We similarly may compute

(5.4) $$v_2^{-1}b_0^6 \simeq 4\Sigma^{32,2} v_2^{-1}b_0^2 \oplus 4\Sigma^{16,2} \text{TMF}_0(3) \oplus 5\Sigma^{64,6} \text{TMF}_0(3) \oplus 5\Sigma^{32,1} \text{TMF}_0(3) \oplus 4\Sigma^{48,4} \text{TMF}_0(3).$$

Finally, we will find the following splitting to be useful.

**Proposition 5.5.** There is a splitting

$$\text{TMF}_0(3)^{\otimes 2} \simeq \text{TMF}_0(3) \oplus \Sigma^{0,-1} \text{TMF}_0(3) \oplus \Sigma^{16,2} \text{TMF}_0(3) \oplus \Sigma^{32,5} \text{TMF}_0(3).$$

**Proof.** Smashing the splitting of Proposition 5.1 with itself, and applying Proposition 5.2 and periodicity, we have

$$v_2^{-1}b_0^6 \simeq 4\Sigma^{32,2} b_0^2 \oplus 4\Sigma^{40,3} b_0 \oplus \text{TMF}_0(3) \oplus \Sigma^{48,4} \text{TMF}_0(3)^{\otimes 2},$$

$$\simeq 4\Sigma^{32,2} b_0^2 \oplus 4\Sigma^{64,6} \text{TMF}_0(3) \oplus 4\Sigma^{80,9} \text{TMF}_0(3) \oplus \Sigma^{48,4} \text{TMF}_0(3)^{\otimes 2},$$

$$\simeq 4\Sigma^{32,2} b_0^2 \oplus 4\Sigma^{64,6} \text{TMF}_0(3) \oplus 4\Sigma^{32,1} \text{TMF}_0(3) \oplus \Sigma^{48,4} \text{TMF}_0(3)^{\otimes 2}.$$
6. Generating functions

In this section we will describe a useful combinatorial way of computing decompositions of \( v_2^{-1} \text{bo}_1 \otimes k \) and \( v_2^{-1} \text{bo}_1 \).

We will represent the objects of \( D_{A(2)} \) of the form

\[
\sum_{i} s^i t^i x^i y^i = \sum_{i} s^i t^i x^i y^i + \sum_{i} t^i s^i x^i y^i + \sum_{i} s^i t^i x^i y^i.
\]

Propositions 5.1, 5.2, and \( v_2 \)-periodicity impose some relations on this polynomial ring — we therefore work in the quotient ring

\[
R := \mathbb{Z}[s, t, x, y] / (x^3 = 2t^2 s x + t^3 s^2 y, xy = t^3 s^3 y + t^5 s^6 y, t^6 s^8 = 1).
\]

Note that these relations imply

\[
y^2 = y + s^{-1} y + t^2 s^2 y + t^4 s^5 y.
\]

This relation reflects the splitting of Prop 7.3.

We may use the relations of \( R \) to reduce \( x^k \) to a sum of monomials whose terms are of the form \( t^i s^j x^i y^j \). These reduced forms of \( x^k \) correspond to splittings of \( v_2^{-1} \text{bo}_1 \). For example, the splitting (5.4) corresponds to the expression

\[
x^6 = 5s^6 t^8 y + s^4 t^6 y + s^3 t^4 y + 5st^2 y + 4s^2 t^4 x^2
\]

in \( R \). Table 1 shows the reduced forms of \( x^k \) in \( R \) for \( k \leq 16 \).

In light of Propositions 2.2 we can also compute the duals of objects of the form (6.1) represented as an element of \( R \) via the ring map:

\[
D : R \to R
\]

\[
t \mapsto t^{-1}
\]

\[
s \mapsto s^{-1}
\]

\[
x \mapsto t^{-2} s \cdot x
\]

\[
y \mapsto s \cdot y
\]

Note the formula \( D(y) = sy \) is forced by the relations of \( R \). We note however that Proposition 5.1 and Proposition 2.2 can be used to deduce that \( v_2^{-1} D \text{TMF}_0(3) \simeq \Sigma^0.1 \text{TMF}_0(3) \).

Now assume that the connecting morphisms \( \partial_j \) (2.10) are trivial for for \( 1 \leq j \leq j_0 \). (We will eventually prove \( \partial_j \) is always zero in Theorem 8.1) Then we can inductively define elements of \( R \) which encode the splitting of \( v_2^{-1} \text{bo}_1 \) for \( j \leq 2j_0 + 1 \). These are the \( \text{bo}-\text{Brown-Gitler} \) polynomials, introduced in [BHHM20, Sec. 8]. Their
The structure of the $v_2$-local algebraic $tmf$ resolution

| $x^3$ | $s^2t^3y + 2st^2x$ |
| $x^4$ | $s^5t^6y + t^2y + 2st^2x^2$ |
| $x^5$ | $s^6t^7y + 4s^3t^5y + t^3y + 4st^4x$ |
| $x^6$ | $5s^6t^8y + s^4t^6y + s^3t^6y + 5st^4y + 4st^4x^2$ |
| $x^7$ | $6s^7t^9y + s^6t^9y + 14s^4t^7y + s^2t^5y + 6st^5y + 8s^3t^6x$ |
| $x^8$ | $20st^{10}y + 7s^5t^6y + 7st^6y + 20s^2t^6y + st^6y + t^5y + 8s^3t^6x^2$ |
| $x^9$ | $8s^7t^{11}y + s^6t^9y + 48s^5t^9y + s^4t^7y + 8s^3t^7y + 27s^2t^7y + 27t^5y$ |

$+ 16s^4t^8x^2$

| $x^{10}$ | $s^7t^{12}y + 35s^6t^{10}y + 35s^5t^{10}y + s^4t^8y + 75s^3t^8y + 9s^2t^8y$ |

$+ 9st^6y + 75t^6y + 16s^4t^8x^2$

| $x^{11}$ | $10s^7t^{11}y + 166s^6t^{11}y + 10s^5t^{11}y + 44s^4t^9y + 110s^3t^9y + s^2t^9y$ |

$+ s^2t^7y + 110st^7y + 44t^7y + 32st^8x^2$

| $x^{12}$ | $154s^7t^{12}y + 154s^6t^{12}y + s^5t^{12}y + 11s^5t^{10}y + 276s^4t^{10}y$ |

$+ 54s^3t^{10}y + 54s^2t^8y + 276st^8y + 11t^8y + t^6y + 32s^3t^{10}x^2$

| $x^{13}$ | $584s^7t^{13}y + 65s^6t^{13}y + s^5t^{11}y + 20s^5t^{11}y + 430s^4t^{11}y$ |

$+ 12s^3t^{11}y + 12s^3t^9y + 430s^2t^9y + 208st^9y + t^8y + 65t^7y + 64s^3t^{12}x$ |

| $x^{14}$ | $638s^7t^{14}y + 13s^6t^{14}y + 77st^6y + 1014s^5t^{12}y + 273s^4t^{12}y$ |

$+ s^3t^{12}y + s^4t^{10}y + 273s^3t^{10}y + 1014s^2t^{10}y + 77st^{10}y + 13st^8y + 63s^3y$ |

$+ 64s^2t^{12}x^2$

| $x^{15}$ | $350s^7t^{15}y + s^6t^{15}y + 14s^5t^{13}y + 911s^6t^{13}y + 1652s^5t^{13}y$ |

$+ 90st^{13}y + 90s^4t^{11}y + 1652s^3t^{11}y + 911s^2t^{11}y + 14st^9y + s^2t^9y$ |

$+ 350st^9y + 2092t^9y + 128s^7t^{14}x$ |

| $x^{16}$ | $104s^7t^{16}y + 440s^6t^{14}y + 3744s^5t^{14}y + 1261s^5t^{14}y + 15s^4t^{14}y$ |

$+ 15s^3t^{14}y + 1261s^4t^{12}y + 3744s^3t^{12}y + 440s^2t^{12}y + st^{12}y + 104s^2t^{10}y$ |

$+ 256st^{10}y + 256st^{10}y + t^8y + 128s^7t^{14}x^2$

Table 1. Reduced expressions for $x^k$ in $R$ corresponding to decompositions of $v_2^{-1}bo^{2k}$.

definition comes from (2.9) and (2.11).

\[
\begin{align*}
    f_0 & := 1, \\
    f_1 & := x, \\
    f_{2j+1} & := t^j \cdot f_j, \\
    f_{2j} & := t^j f_j + t^{j+1} s \cdot f_{j-1}.
\end{align*}
\]

(6.3)

Table 2 shows reduced expressions for $f_j$ in $R$ for $j \leq 16$.

7. $g$-local computations

We will now consider the $g$-local bo-Brown-Gitler comodules, for

\[ g = h_{2,1}^4 \in \mathcal{A}^{(2)}\mathcal{F}_2. \]

The $g$-local results of this section will be crucial for the main result of Section 8.
Because the terms $A(2) \otimes \mathcal{mf}_{j-1}$ in (2.5) and (2.6) are $g$-locally acyclic in $\mathcal{D}_{A(2)}$, we have cofiber sequences

$$(7.1) \Sigma^8 j g^{-1} b_0 j \to g^{-1} b_0 2j \to \Sigma^{8j+8} g^{-1} b_0 j-1 \to \Sigma^{8j+1} g^{-1} b_0 j$$

and equivalences

$$(7.2) g^{-1} b_0 2j+1 \simeq \Sigma^{8j} g^{-1} b_0 j \otimes b_0 1.$$
ring $R$, we work in the ring

$$R' := \mathbb{Z}[s^\pm, t^\pm, x]/(x^3 = 2t^2sx).$$

By Proposition 7.4, we may encode $g$-local Spanier-Whitehead duality by the function

$$D : R' \rightarrow R'$$

$$s \mapsto s^{-1}$$

$$t \mapsto t^{-1}$$

$$x \mapsto t^{-2}s^{-1}x$$

Define elements $f'_j \in R'$ by the same inductive definition used to define the elements $f_j \in R$. A simple induction reveals the following.

**Lemma 7.5.** The elements $f'_j \in R'$ take the form

$$f'_j = \begin{cases} 
\sum_i (a_{i,j} s^{i}t^j + b_{i,j} s^{i}t^j x + c_{i,j} s^{i}t^j x^2), & j \text{ even,} \\
\sum_i (b_{i,j} s^{i}t^j x + c_{i,j} s^{i}t^j x^2), & j \text{ odd,}
\end{cases}$$

for $a_{i,j}, b_{i,j}, c_{i,j} \in \mathbb{N}$.

8. The attaching maps $\partial_j$ and $\partial'_j$

**Theorem 8.1.** The attaching maps $\partial_j$ (2.10) and $\partial'_j$ (7.1) are zero for all $j$.

**Proof.** Write the exact sequence (2.5) as a splice of two short exact sequences

$$0 \rightarrow \Sigma^{8j}b_{o,j} \rightarrow b_{o,j} \rightarrow A(2)//A(1)_* \otimes \text{tmf}_{j-1} \rightarrow \Sigma^{8j+9}b_{o,j-1} \rightarrow 0$$

and let

$$\Sigma^{8j}b_{o,j} \rightarrow b_{o,j} \rightarrow K \overset{\alpha}{\rightarrow} \Sigma^{8j+1,-1}b_{o,j}$$

$$\Sigma^{8j+8,1}b_{o,j-1} \overset{\beta}{\rightarrow} K \rightarrow A(2)//A(1)_* \otimes \text{tmf}_{j-1} \rightarrow \Sigma^{8j+9}b_{o,j-1}$$
be the cofiber sequences in $D_{A(2)}$, induced from these short exact sequences. Then we have the following commutative diagram in $D_{A(2)}$.

![Diagram]

We therefore have

(8.2) \[ g^{-1} \partial_j = v_2^{-1} \partial'_j. \]

Now, Assume inductively that $\partial_k$ and $\partial'_k$ are zero for $k < j$. Then for $k < 2j + 1$, $v_2^{-1} \text{bo}_k$ and $g^{-1} \text{bo}_k$ decomposes in $D_{A(2)}$, as a sum of terms corresponding to the terms of $f_k$ and $f'_k$, respectively. Note that we have

\[ \partial_j \in \pi_{7,2}^{A(2)}(v_2^{-1} D(\text{bo}_{j-1}) \otimes \text{bo}_j), \]
\[ \partial'_j \in \pi_{7,2}^{A(2)}(g^{-1} D(\text{bo}_{j-1}) \otimes \text{bo}_j). \]

It follows from Lemma 7.5 that

\[ D(f'_{j-1}) \cdot f'_j = \sum_i (\alpha_i s^i x + \beta_i s^j t^{-1} x^2) \]

for $\alpha_i, \beta_i \in \mathbb{N}$, and therefore

(8.3) \[ g^{-1} D(\text{bo}_{j-1}) \otimes \text{bo}_j \simeq \bigoplus_i (\alpha_i \Sigma^{0,i} g^{-1} \text{bo}_1 + \beta_i \Sigma^{-8,i} g^{-1} \text{bo}_1 \otimes^2). \]

Note that there is a map of rings

$\phi : R' \to R$

sending $s$ to $t$, $t$ to $t$, and $x$ to $x$. We have

\[ f_k \equiv \phi(f'_k) \mod y. \]

We therefore have

\[ D(f_{j-1}) \cdot f_j = \sum_i (\alpha_i s^i x + \beta_i s^j t^{-1} x^2) + \sum_{k,l} \gamma_{k,l} s^k t^l y. \]

It follows that we have

(8.4) \[ v_2^{-1} D(\text{bo}_{j-1}) \otimes \text{bo}_j \simeq \bigoplus_i (\alpha_i \Sigma^{0,i} v_2^{-1} \text{bo}_1 + \beta_i \Sigma^{-8,i} v_2^{-1} \text{bo}_1 \otimes^2) \otimes \bigoplus_{k,l} \Sigma^{8l,k} \text{TMF}_0(3). \]

Note that

\[ \pi_{8m+7, n}^{A(2)}(\text{TMF}_0(3)) = 0 \]
for all $n, m$, so the the only potential non-zero components of $\partial_j$ under the decomposition (8.4) are the components

$$\begin{align*}
(\partial_j)_{i}^{(1)} & \in \pi_{7,2-i}(\alpha_i v_2^{-1} b_{01}), \\
(\partial_j)_{i}^{(2)} & \in \pi_{15,2-i}(\beta_i v_2^{-1} b_{01}^{\otimes 2}).
\end{align*}$$

Similarly, let

$$\begin{align*}
(\partial_j')_{i}^{(1)} & \in \pi_{7,2-i}(\alpha_i g^{-1} b_{01}), \\
(\partial_j')_{i}^{(2)} & \in \pi_{15,2-i}(\beta_i g^{-1} b_{01}^{\otimes 2})
\end{align*}$$

denote the components of $\partial'_j$ under the splitting (8.3).

Note that the splittings (8.3) and (8.4) are compatible under the maps

$$g^{-1} D(b_{0j-1}) \otimes b_{0j} \to v_2^{-1} g^{-1} D(b_{0j-1}) \otimes b_{0j}$$

since $g^{-1} \text{TMF}_0(3) \simeq 0$, and by (8.2) $\partial_j'$ and $\partial_j$ map to the same element of $
\pi_{7,2}^A(v_2^{-1} g^{-1} D(b_{0j-1}) \otimes b_{0j}).$

We therefore deduce that under the maps

$$\begin{align*}
\alpha_i g^{-1} b_{01} & \to \alpha_i v_2^{-1} g^{-1} b_{01} \leftarrow \alpha_i v_2^{-1} b_{01}, \\
\beta_i g^{-1} b_{01}^{\otimes 2} & \to \beta_i v_2^{-1} g^{-1} b_{01}^{\otimes 2} \leftarrow \beta_i v_2^{-1} b_{01}^{\otimes 2}
\end{align*}$$

we have

$$\begin{align*}
v_2^{-1}(\partial'_j)_{i}^{(1)} & = g^{-1}(\partial_j)_{i}^{(1)}, \\
v_2^{-1}(\partial'_j)_{i}^{(2)} & = g^{-1}(\partial_j)_{i}^{(2)}.
\end{align*}$$

However, direct inspection of $\pi_{s,s}^A(b_{01})$ and $\pi_{s,s}^A(b_{01}^{\otimes 2})$ reveals:

- The maps

$$\begin{align*}
\pi_{7,s}^A(g^{-1} b_{01}) & \to \pi_{7,s}^A(v_2^{-1} g^{-1} b_{01}) \leftarrow \pi_{7,s}^A(v_2^{-1} b_{01}), \\
\pi_{15,s}^A(g^{-1} b_{01}^{\otimes 2}) & \to \pi_{15,s}^A(v_2^{-1} g^{-1} b_{01}^{\otimes 2}) \leftarrow \pi_{15,s}^A(v_2^{-1} b_{01}^{\otimes 2})
\end{align*}$$

are injections for all $s$.

- We have

$$\begin{align*}
\pi_{7,s}^A(g^{-1} b_{01}) & = 0, \\
\pi_{15,s}^A(g^{-1} b_{01}^{\otimes 2}) & = 0
\end{align*}$$

for $s \geq 1$.

- We have

$$\begin{align*}
\pi_{7,s}^A(v_2^{-1} b_{01}) & = 0, \\
\pi_{15,s}^A(v_2^{-1} b_{01}^{\otimes 2}) & = 0
\end{align*}$$

for $s \leq 1$. 
It follows that we must have
\[
(\partial_j)_{i}^{(1)} = 0, \\
(\partial'_j)_{i}^{(1)} = 0, \\
(\partial_j)_{i}^{(2)} = 0, \\
(\partial'_j)_{i}^{(2)} = 0.
\]

**Corollary 8.5.** We have
\[
g^{-1}b_{0j} \simeq \Sigma^8j g^{-1}b_{0j} \oplus \Sigma^{8j+8,1} g^{-1}b_{0j-1}.
\]
Therefore, if we write \( f'_j \) in the form
\[
f'_j = \sum_i (a_{i,j}s^i t^j + b_{i,j}s^i t^{j-1}x + c_{i,j}s^i t^{j-2}x^2)
\]
then we have
\[
g^{-1}b_{0j} \simeq \bigoplus_i (a_{i,j} \Sigma^8 j, i g^{-1} F_2 \oplus b_{i,j} \Sigma^{8(j-1), i} g^{-1} b_{01} \oplus c_{i,j} \Sigma^{8(j-2), i} g^{-1} b_{01}^2).
\]

**Corollary 8.6.** We have
\[
v^{-1}b_{0j} \simeq \Sigma^8j v^{-1}b_{0j} \oplus \Sigma^{8j+8,1} v^{-1}b_{0j-1}.
\]
Therefore, if we write \( f_j \) in the form
\[
f_j = \sum_i (a_{i,j}s^i t^j + b_{i,j}s^i t^{j-1}x + c_{i,j}s^i t^{j-2}x^2) + \sum_{k,l} d_{j,k,l} s^k t^l y
\]
then we have
\[
v^{-1}b_{0j} \simeq \bigoplus_i (a_{i,j} \Sigma^8 j, i v^{-1} F_2 \oplus b_{i,j} \Sigma^{8(j-1), i} v^{-1} b_{01} \oplus c_{i,j} \Sigma^{8(j-2), i} v^{-1} b_{01}^2)
\]
\[\oplus \bigoplus_{k,l} d_{j,k,l} \Sigma^{8j,k,} \text{TMF}_0(3).
\]

**Corollary 8.7.** Consider the element
\[
h := tf_1w + t^2 f_2w^2 + t^3 f_3 w^3 \cdots \in R[[w]].
\]
Write the coefficient of \( w^j \) in \( h^n \) as
\[
\sum_i (a_{i,j}^{(n)} s^i t^{2j} + b_{i,j}^{(n)} s^i t^{2j-1}x + c_{i,j}^{(n)} s^i t^{2j-2}x^2) + \sum_{j,k,l} d_{j,k,l}^{(n)} s^k t^l y
\]
then the weight \( 8j \) summand of \( v^{-1} \text{tmf}^n \) decomposes as
\[\bigoplus (a_{i,j}^{(n)} \Sigma^{16j, i} v^{-1} F_2 \oplus b_{i,j}^{(n)} \Sigma^{16j-8, i} v^{-1} b_{01} \oplus c_{i,j}^{(n)} \Sigma^{16j-16, i} v^{-1} b_{01}^2)
\]
\[\oplus \bigoplus_{k,l} d_{j,k,l}^{(n)} \Sigma^{8j,k,} \text{TMF}_0(3).
\]
9. Applications to the $g$-local algebraic $\text{tmf}$ resolution

Consider the quotient Hopf algebra $C_\ast := \mathbb{F}_2[\zeta]/(\zeta^4)$ of $A(2)_\ast$, with

$$\pi_{C_\ast}^*(\mathbb{F}_2) = \mathbb{F}_2[v_1, h_{2,1}].$$

The second author, Bobkova, and Thomas computed the $P_2^1$-Margolis homology of the $\text{tmf}$-resolution, and in the process computed the structure of $A / A(2)^{\otimes n}$ as $C_\ast$-comodules. From this one can read off the Ext groups

$$h_{-1}^2 C_\ast^*(\text{tmf}^{\otimes n})$$

(see [BMQ21 Thm. 3.12]).

The groups $h_{-1}^2 C_\ast^*$ are closely related to the groups $g_{-1}^1 A(2)^*_\ast$. In [BMQ21 Cor. 3.11], it is proven that for $M \in D_{A(2)_\ast}$, there is a $v_8^2$ Bockstein spectral sequence

$$h_{-1}^2 C_\ast^*(M) \otimes \mathbb{F}_2[v_8^2] \Rightarrow g_{-1}^1 A(2)^*_\ast(M).$$

In this section we would like to explain how Corollary 8.5 can be used to compute $g_{-1}^1 A(2)^*_\ast(\text{tmf}^{\otimes n})$. By relating this to [BBT21], we will show that in the case of $M = \text{tmf}^{\otimes n}$, the spectral sequence (9.1) collapses (Theorem 9.3).

We follow [BMQ21] in our summary of the results of [BBT21]. The coaction of $C_\ast$ is encoded in the dual action of the algebra $E[Q_1, P_2^1]$ on $\text{tmf}^{\otimes n}$. Define elements

$$x_{i,j} = 1 \otimes \cdots \otimes 1 \otimes \zeta_{i+3} \otimes 1 \otimes \cdots \otimes 1,$$

$$t_{i,j} = 1 \otimes \cdots \otimes 1 \otimes \zeta_{i+1}^4 \otimes 1 \otimes \cdots \otimes 1$$

in $\text{tmf}^{\otimes n}$.

For an ordered set

$$J = ((i_1, j_1), \ldots, (i_k, j_k))$$

of multi-indices, let

$$|J| := k$$

denote the number of pairs of indices it contains. Define linearly independent sets of elements

$$\mathcal{T}_J \subset \text{tmf}^{\otimes n}$$

inductively as follows. Define

$$\mathcal{T}_{(i,j)} = \{x_{i,j}\}.$$

For $J$ as above with $|J|$ odd, define

$$\mathcal{T}_{J, (i,j)} = \{z \cdot x_{i,j} \}_{z \in \mathcal{T}_J},$$

$$\mathcal{T}_{J, (i,j), (i', j')} = \{Q_1(z \cdot x_{i,j})x_{i', j'} \}_{z \in \mathcal{T}_J} \cup \{Q_1(z \cdot x_{i', j'})x_{i,j} \}_{z \in \mathcal{T}_J}.$$

Let

$$N_J \subset \text{tmf}^{\otimes n}$$

denote the $\mathbb{F}_2$-subspace with basis

$$Q_1 \mathcal{T}_J := \{Q_1(z) \}_{z \in \mathcal{T}_J}.$$
While the set $T_J$ depends on the ordering of $J$, the subspace $N_J$ does not.

Finally, for a set of pairs of indices

$$J = \{(i_1, j_1), \ldots, (i_k, j_k)\}$$

as before, define

$$x_J t_J := x_{i_1, j_1} t_{i_1, j_1} \cdots x_{i_k, j_k} t_{i_k, j_k}.$$  

The following is can be read off of the computations of [BBT21].

**Theorem 9.2** (Bhattacharya-Bobkova-Thomas). As modules over $\mathbb{F}_2[h^{\pm}_{2,1}, v_1]$, we have

$$h^{-1} \pi^*_{+^*}(1m)^{\otimes n}) = \mathbb{F}_2[h^{\pm}_{2,1}] \otimes \mathbb{F}_2[v_1]\{x_{J'} t_{J'}\}_{J \cap J' = \emptyset}$$

$$\bigoplus_{|J| \text{ odd}} N_J \{x_{J'} t_{J'}\}_{J \cap J' = \emptyset}$$

$$\bigoplus_{|J| \text{ even}} \mathbb{F}_2[v_1]/v_1^2 \otimes N_J \{x_{J'} t_{J'}\}_{J \cap J' = \emptyset}$$

where $J$ and $J'$ range over the subsets of

$$\{(i, j) : 1 \leq i, 1 \leq j \leq n\}$$

and $v_1$ acts trivially on $N_J$ for $|J|$ odd.

We now explain how the equivalences

$$g^{-1}b_{2j} \simeq \Sigma^{8j} g^{-1}b_{2j} \oplus \Sigma^{8j+8,1} g^{-1}b_{2j-1},$$

$$g^{-1}b_{2j+1} \simeq \Sigma^{8j} g^{-1}b_{2j} \otimes b_{1}$$

are related to Theorem 9.2. This analysis comes from the definitions of the maps of (2.5) and (2.6) in [BHHM08]. For a set $J$ of indices of the form

$$J = \{(i_1, 1), \ldots, (i_k, 1)\},$$

define $J + \Delta$ to be the set

$$J + \Delta = \{(i_1 + 1, 1), \ldots, (i_k + 1, 1)\}.$$  

Then the induced maps on homotopy are determined by:

$$\pi^{A(2),*}_* (\Sigma^{8j} g^{-1} b_{2j}) \rightarrow \pi^{A(2),*}_* (g^{-1} b_{2j})$$

$$N_J \{x_{J'} t_{J'}\} \mapsto N_{J+\Delta} \{x_{J'+\Delta} t_{J'+\Delta}\}$$

$$\pi^{A(2),*}_* (\Sigma^{8j+8,1} g^{-1} b_{2j-1} \rightarrow \pi^{A(2),*}_* (g^{-1} b_{2j})$$

$$N_J \{x_{J'} t_{J'}\} \mapsto h_{2,1} : N_{J+\Delta} \{x_{1,1} t_{1,1} t_{J'+\Delta} \}

$$\pi^{A(2),*}_* (\Sigma^{8j} g^{-1} b_{2j} \otimes b_{1}) = \pi^{A(2),*}_* (g^{-1} b_{2j+1})$$

$$N_J \cup \{(1,2)\} \{x_{J'} t_{J'}\} \mapsto N_{(J+\Delta) \cup (1,2)} \{x_{J'+\Delta} t_{J'+\Delta}\}.$$
We have (with $g = h_{2,1}^4$)
\[
\pi_{*,*}^{A(2)}(g^{-1}F_2) = F_2[h_{2,1}^\pm, v_1, v_2^8],
\]
\[
\pi_{*,*}^{A(2)}(g^{-1}bo_1) = F_2[h_{2,1}^\pm, v_1, v_2^8]/(v_1)^{(t_1,1)},
\]
\[
\pi_{*,*}^{A(2)}(g^{-1}bo_1^{\otimes 2}) = F_2[h_{2,1}^\pm, v_1, v_2^8]/(v_2^1)\{Q_1(x_1,1,x_1,2)\}.
\]

Corollary 8.5 therefore implies the following extension of Theorem 9.2.

**Theorem 9.3.** As modules over $F_2[h_{2,1}^\pm, v_1, v_2^8]$, we have
\[
g^{-1}\pi_{*,*}^{A(2)}(tmf^{\otimes n}) = F_2[h_{2,1}^\pm, v_2^8] \otimes \left( F_2[v_1]|\{x_j t_{j'}\}_J \right)_{|J|\text{ odd}} \oplus \left( F_2[v_1]|/v_1^2 \otimes N_J\{x_j t_{j'}\}_{|J'|=\emptyset} \right)
\]
where $J$ and $J'$ range over the subsets of
\[
\{(i,j) : 1 \leq i, 1 \leq j \leq n\}
\]
and $v_1$ acts trivially on $N_J$ for $|J|$ odd.

**Appendix A. A splitting of $bo_1^{\wedge 3}$**

The $v_2$-local splitting of Proposition 5.1 comes from a stable splitting of $bo_1^{\wedge 3}$ induced by an idempotent decomposition of the identity element
\[
1 = f_1 + f_2 + e \in Z_{(2)}[\Sigma_3]
\]
as described in Remark A.2. More precisely, if we set
\[
F_i := \text{hocolim}\{bo_1^{\wedge 3} \xrightarrow{f_i} bo_1^{\wedge 3} \xrightarrow{f_i} \ldots\}
\]
for $i \in \{1, 2\}$ and
\[
E := \text{hocolim}\{bo_1^{\wedge 3} \xrightarrow{e} bo_1^{\wedge 3} \xrightarrow{e} \ldots\},
\]
using the evident permutation action of $\Sigma_3$ on $bo_1^{\wedge 3}$, then it is easy to see that
\[
bo_1^{\wedge 3} \simeq F_1 \vee F_2 \vee E.
\]
In fact, $F_1$, $F_2$ and $E$ are finite spectra and their mod 2 cohomology as a Steenrod module can be easily computed using the cocommutativity of Steenrod operations and a Künneth isomorphism (see [Rav92 Appendix C]). For the purposes of this paper, we only need their underlying $A(2)$-module structure which we record in the format of a Bruner module definition file [BEM17 Apdx. A] (see Figure A.1 and Figure A.2).

**Remark A.2.** In the group ring $Z_{(2)}[\Sigma_3]$, the identity element 1 can be written as a sum of idempotent elements
\[
f_1 = \frac{1 + (1 2) - (1 3) - (1 2 3)}{3}, f_2 = \frac{1 + (1 3) - (1 2) - (1 3 2)}{3} \quad \text{and} \quad e = \frac{1 + (1 2 3) + (1 3 2)}{3}.
\]
Figure A.1. The $A(2)$-module structure of $H^*(F_1) \cong H^*(F_2)$ as an input file for Bruner's program

**Remark A.3.** Note that $f_1$ and $f_2$ are conjugates and therefore, $F_1 \simeq F_2$.

Bruner's program is capable of computing the action of $\pi_{*,*}^{A(2)}(\mathbb{F}_2)$ on $\pi_{*,*}^{A(2)}(M^\vee)$, where $M^\vee$ is the $\mathbb{F}_2$-linear dual of a finite $A(2)$-module $M$. Therefore, it can be used for verifying the details necessary in the proof of Proposition 5.1 and Proposition 5.2.

**Remark A.4.** Using Bruner's program and Figure 4.2, one can easily verify

$$v_2^{-1} \pi_{*,*}^{A(2)}(H_*(E)) \cong \pi_{*,*}^{A(2)}(\Sigma^{24.2}\text{TMF}_0(3)).$$

Then by Theorem 4.3, we get $\Sigma^{24.2}\text{TMF}_0(3) \simeq v_2^{-1}H_*(E)$ in $D_{A(2)}$.

**Remark A.5 (A different proof of Proposition 5.1).** Let $M_1$ denote the first integral Brown-Gitler module. It consists of three $\mathbb{F}_2$-generators $\{x_0, x_2, x_3\}$ where $|x_i| = i$ such that

$$Sq^2(x_0) = x_2 \text{ and } Sq^1(x_2) = x_3.$$
It is tedious but straightforward to check that there is a short exact sequence
\[ 0 \to H^*(\Sigma^{17} \text{bo}_1) \to \Sigma^4 A(2) \otimes A(1) \otimes M_1 \to H^* E \to 0 \]
of $A(2)$-modules. This short exact sequence translates into an $D_{A(2)}$-equivalence
\[ v_2^{-1} H_*(F_1) \cong H_*(F_2) \cong \Sigma^{16} v_2^{-1} \text{bo}_1 \]
which, along with Remark A.4 and (A.1), gives yet another proof of Proposition 5.1.

![Table](image)
References


[BBT21] Prasit Bhattacharya, Irina Bobkova, and Brian Thomas, The \( P_{1/2} \) Margolis homology of connective topological modular forms, Homology Homotopy Appl. 23 (2021), no. 2, 379–402. MR 4319993


[BHHM08] M. Behrens, M. Hill, M. J. Hopkins, and M. Mahowald, On the existence of a \( v_{32} \)-self map on \( M(1,4) \) at the prime 2, Homology Homotopy Appl. 10 (2008), no. 3, 45–84. MR 2475617


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