Detectors in homotopy theory

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An analogy:

Particle physics:

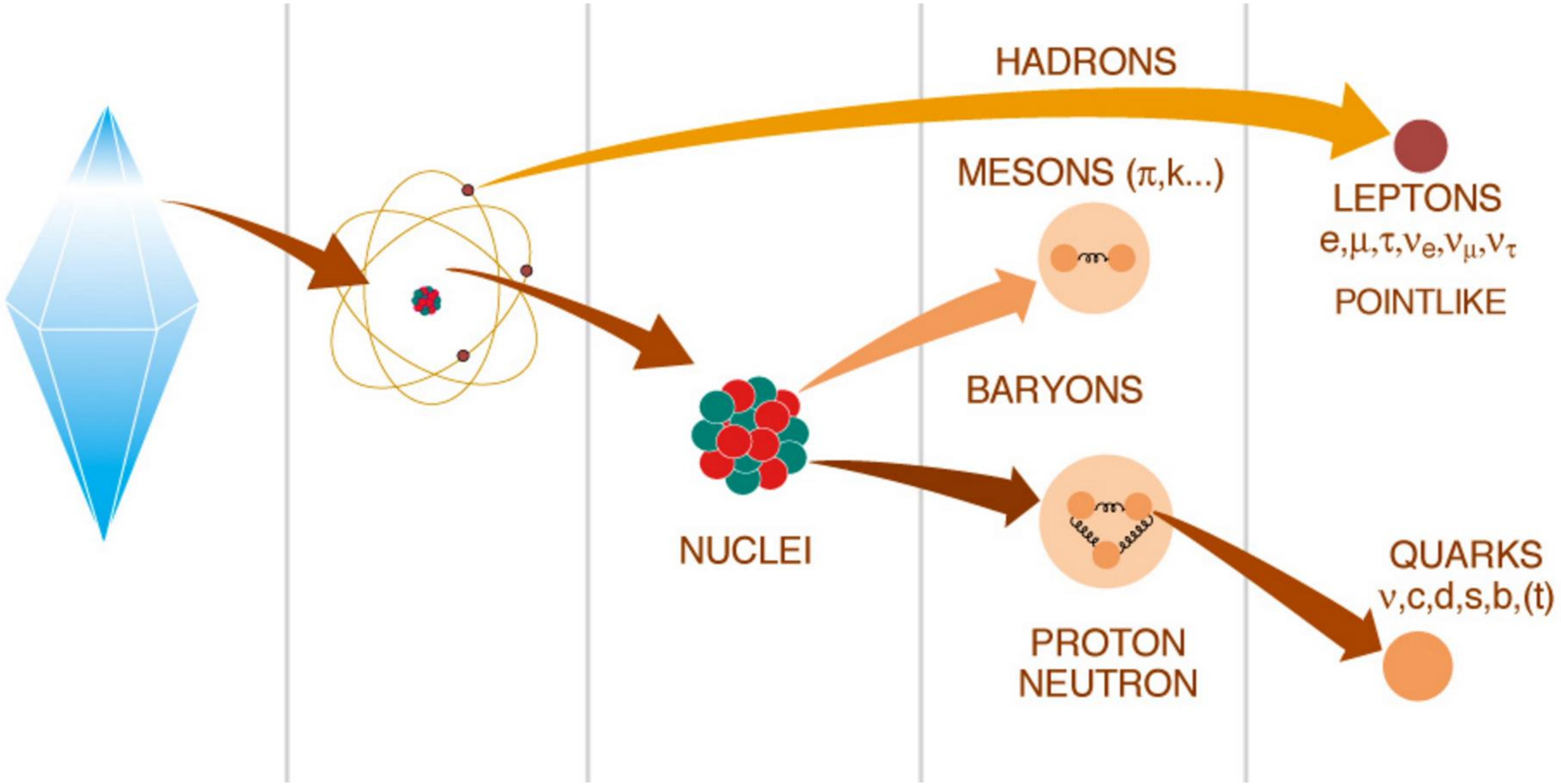
- All matter is built from elementary particles
- Goal: Discover all fundamental particles
- Tool: Massive accelerators and detectors [LHC]

Homotopy theory:

- Topological spaces (up to homotopy) are built by attaching together disks (of varying dimensions)
- Goal: Compute all attaching maps (homotopy groups of spheres)
- Tool: Massive spectral sequences

  [Adams spectral sequence]

Matter: built out of elementary particles



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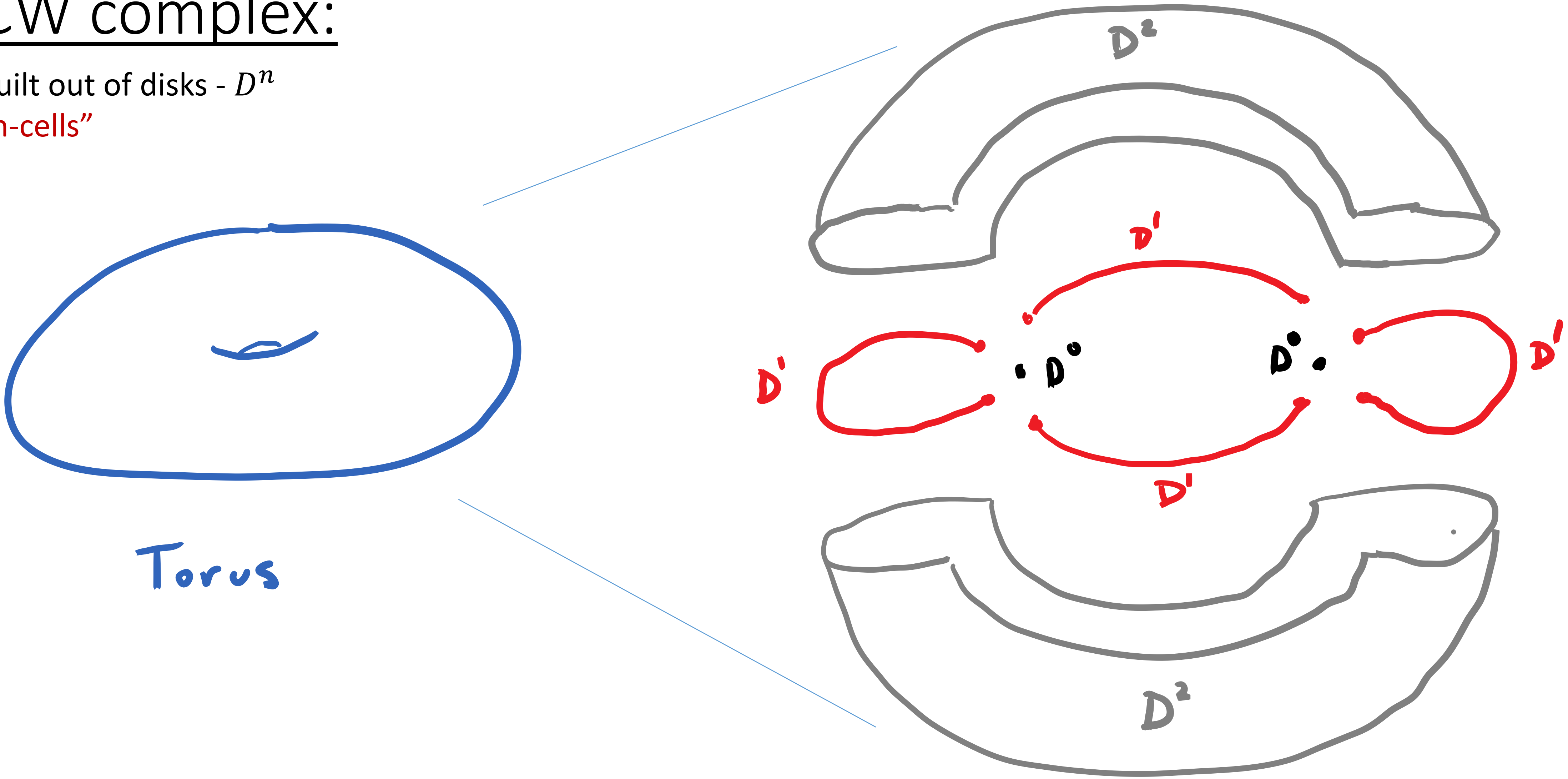
CW complex:

Built out of disks - D^n

"n-cells"



CW complexes

- Theorem:

  Every topological space is (weakly) homotopy equivalent to a CW complex.

- CW complexes have the form $X = \bigcup_n X^n$

$$X^0 = \{set\ of\ points\}$$

$$X^1 = X^0 \cup_\partial \{set\ of\ intervals\}$$

$$X^2 = X^1 \cup_\partial \{set\ of\ disks\}$$

$$\vdots$$

$$X^{i+1} = X^i \cup_\partial \{set\ of\ D^{i+1}\}$$

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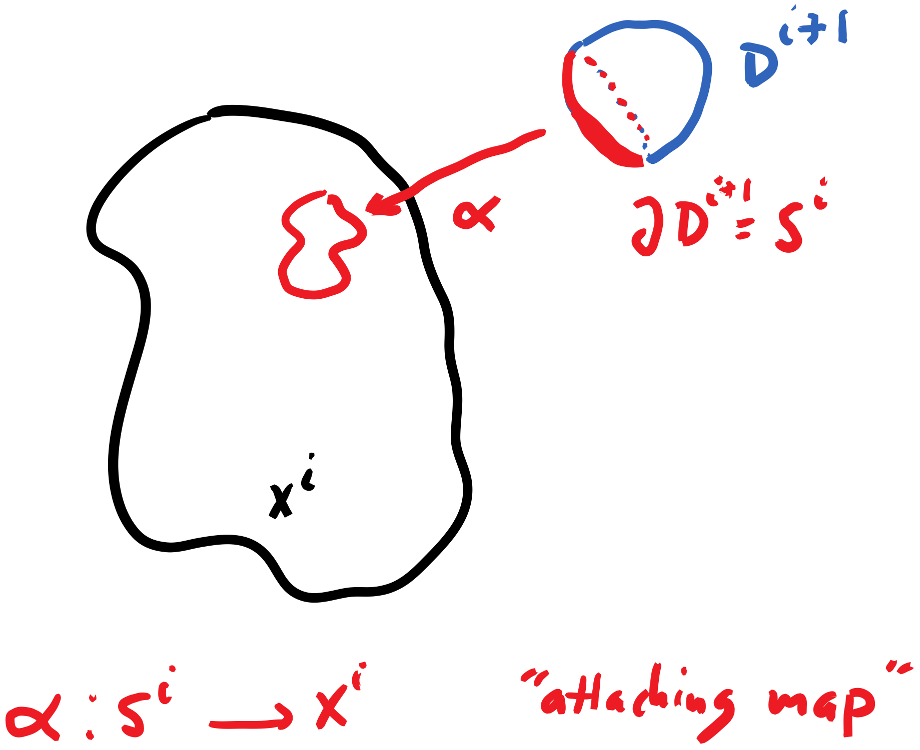


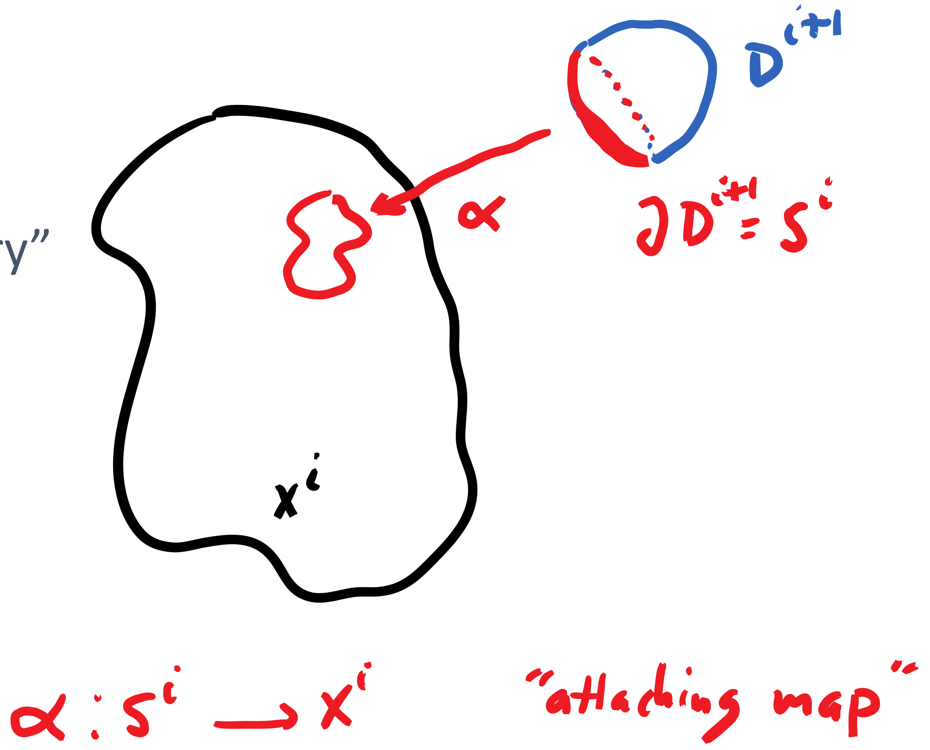
CW complexes

Inductively, the CW complex X is determined up to homotopy by the *homotopy classes* of the attaching maps

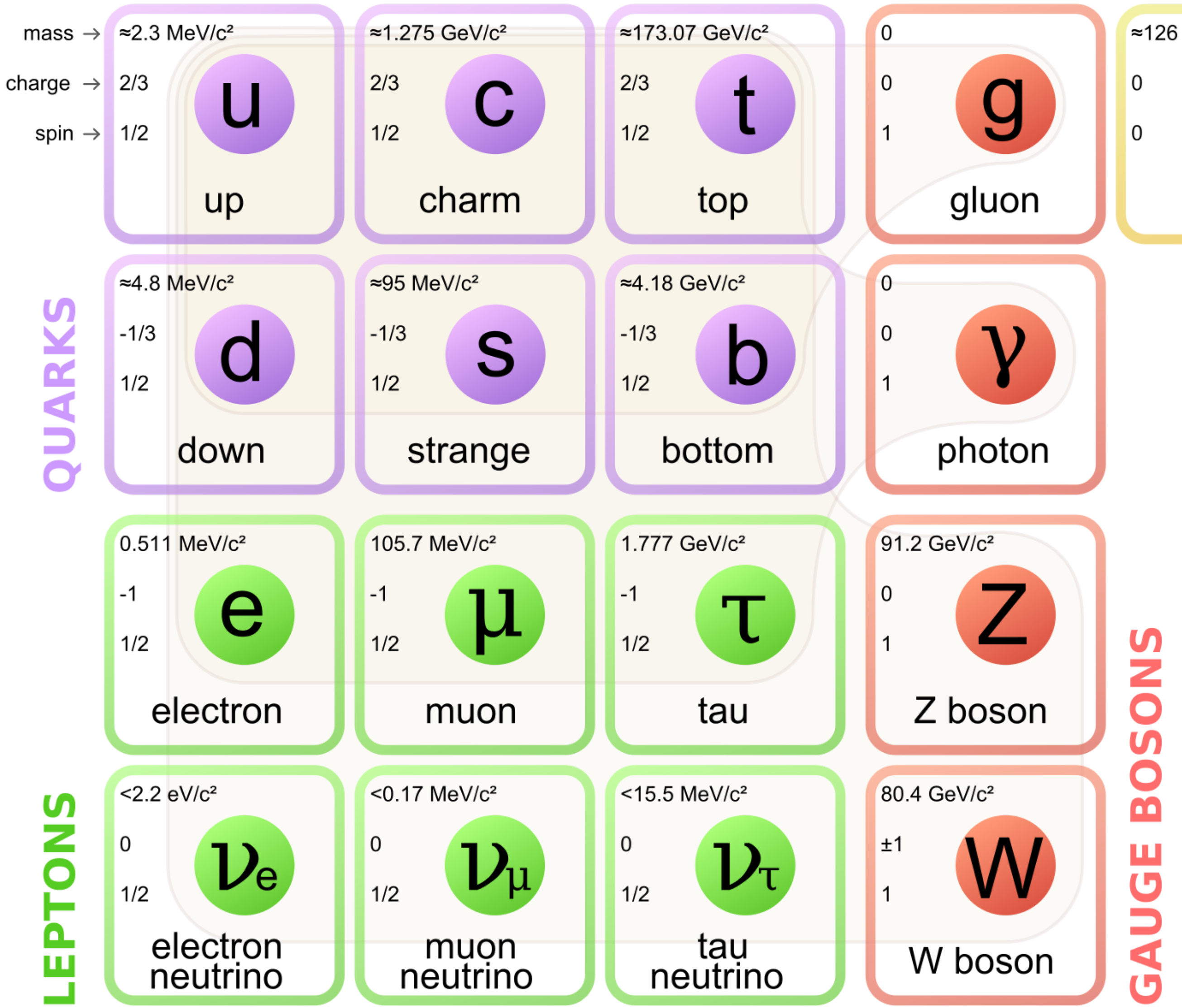
$\alpha \in \pi_i(X^i)$

CW-complexes/homotopy = "matter of geometry"

Building blocks – elements of homotopy groups



Elementary particles: complicated



[Wikipedia commons]

Homotopy groups of spheres: also complicated

$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Computation: Serre, Toda, …
Chart: Hatcher

Down to business…

- For the rest of this talk, all CW complexes are finite, connected, with fixed basepoint.

- We will discuss the simpler problem of classifying such CW complexes up to *stable equivalence* [still hard!]:

$$X \simeq_{st} Y \quad \Leftrightarrow \quad \Sigma^N X \simeq \Sigma^N Y \quad N \gg 0$$ [define Σ]

- Stable homotopy category of these:

Morphisms: $[X, Y]^{st} = [\Sigma^N X, \Sigma^N Y] \quad \mathrm{N} \gg 0 \quad (\textit{these stabilize})$

- Stable equivalence of CW complex depends on stable attaching maps

$$\alpha \in \pi_n^{st}(X) \coloneqq [S^n, X]^{st}$$ “stable homotopy groups of X”

Stable homotopy groups of spheres:

$\pi_i(S^n)$

$n \downarrow$ \ $i \longrightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
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5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
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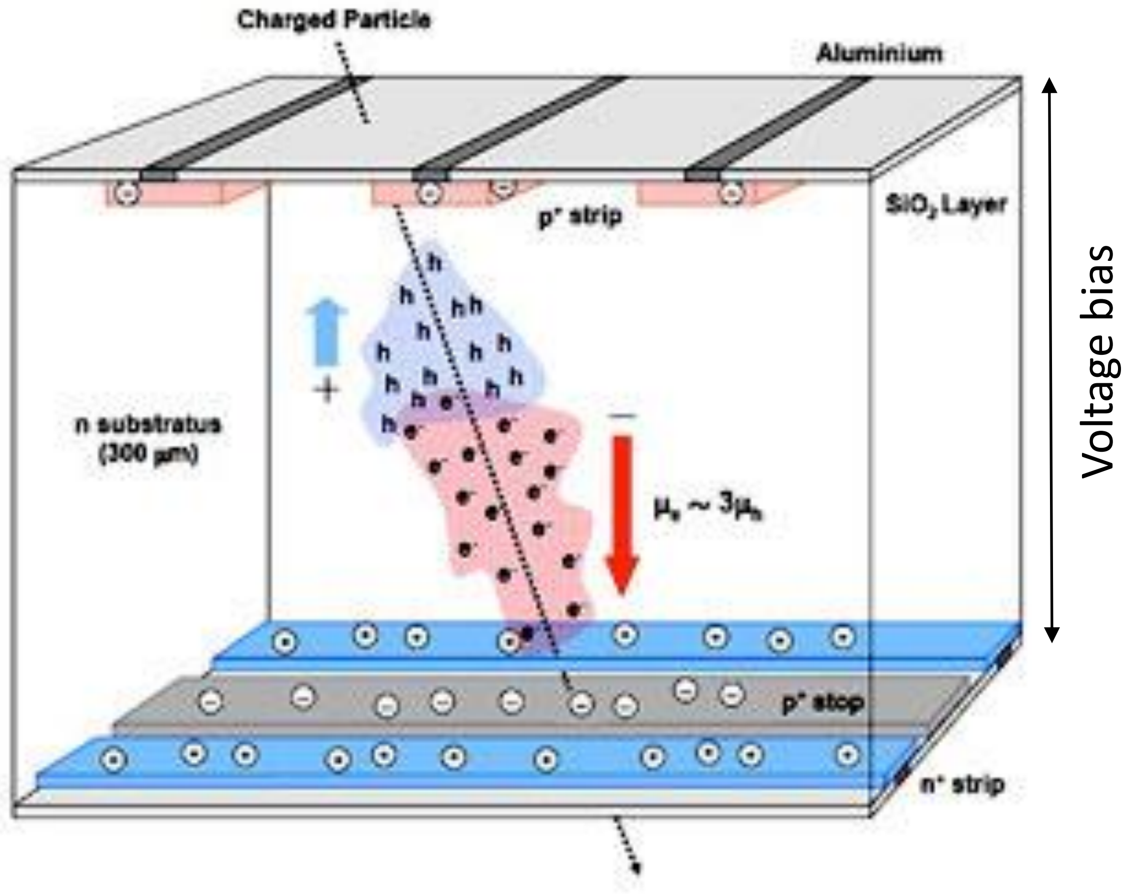
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

$\pi_n^{st}(S^n)$ $\pi_{n+1}^{st}(S^n)$ $\pi_{n+2}^{st}(S^n)$ $\pi_{n+3}^{st}(S^n)$ $\pi_{n+4}^{st}(S^n)$

Basic particle detectors:



- Takeaway:

  Use simple particle (electron) to detect more exotic particles

[AMS-02 experiment]

Goal: build a detector of (stable) homotopy groups

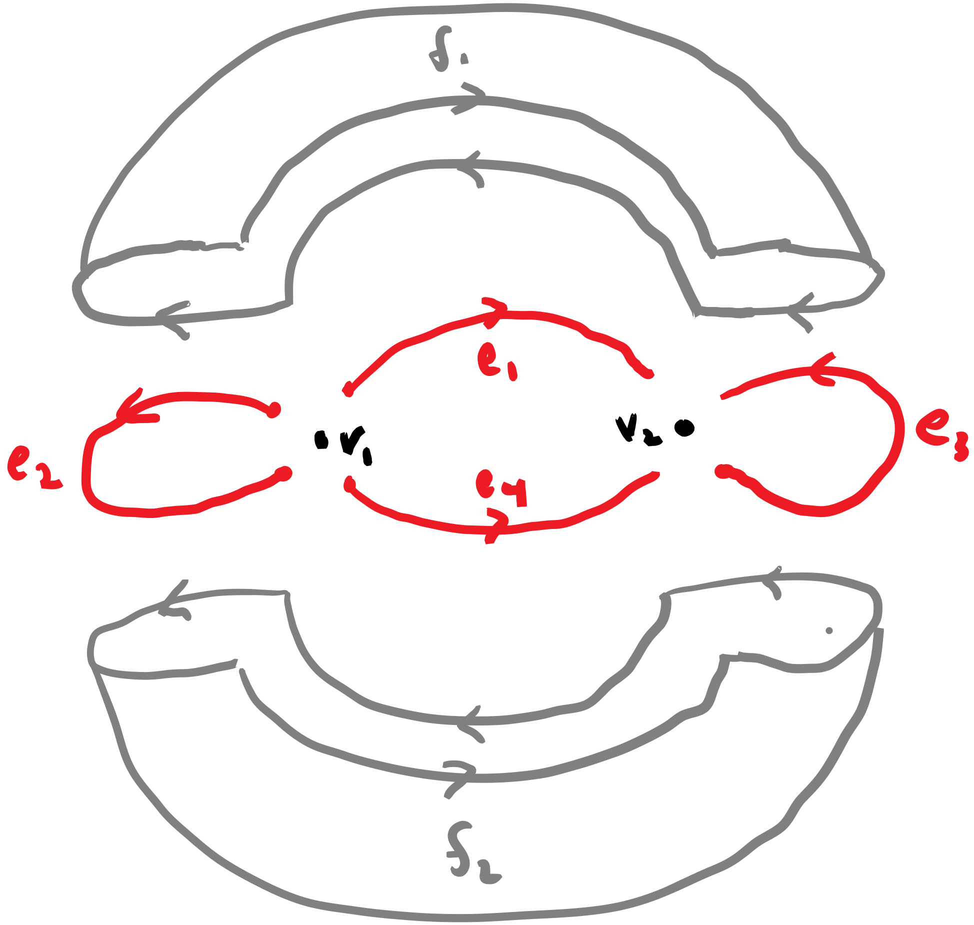
- Homology – a simple computable approximation of homotopy groups

Hurewicz homomorphism: $\pi_*^{st}(X) \to H_*(X)$ [typically not an iso!]

- Homology classes will be the "electrons" in our detector which detect elements of $\pi_*^{st}(X)$

Cellular homology

Form a chain complex – basis given by cells – differential given by degrees of attaching maps:



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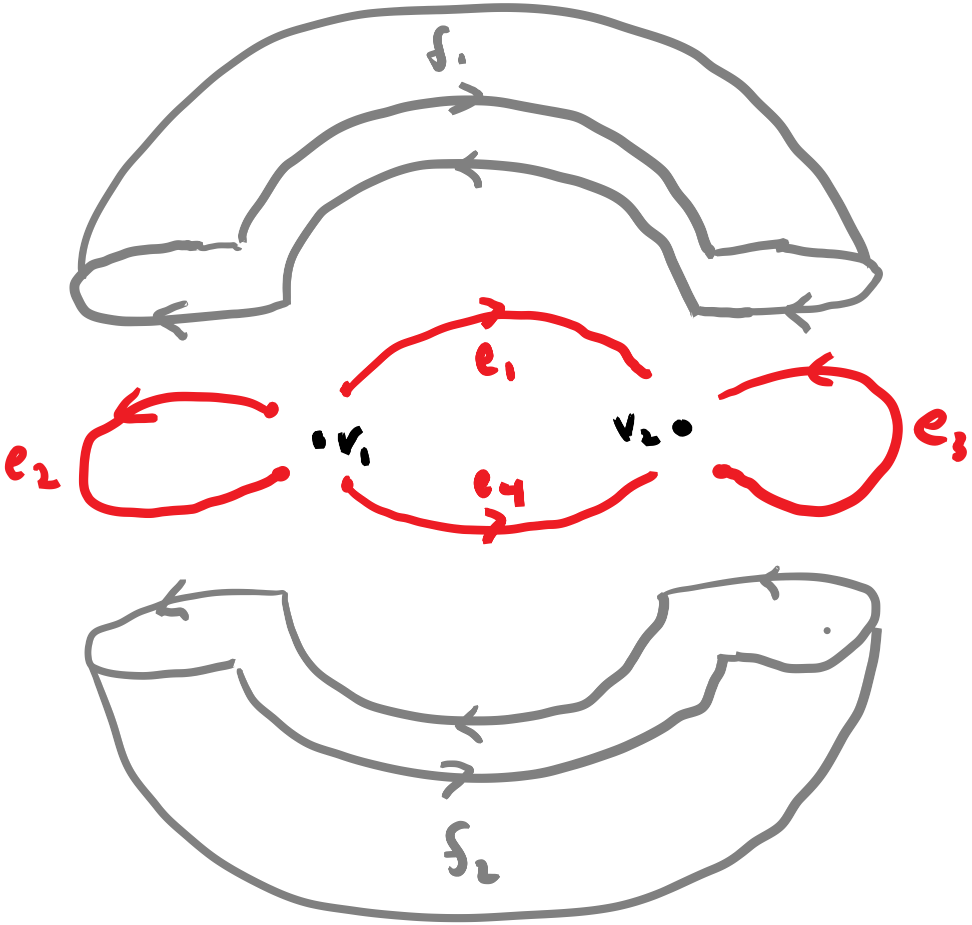

dim	0	1	2
$C_*(X)$	v_1 • v_2 •	e_1 • e_2 • e_3 • e_4 •	• f_1 • f_2

Cellular homology

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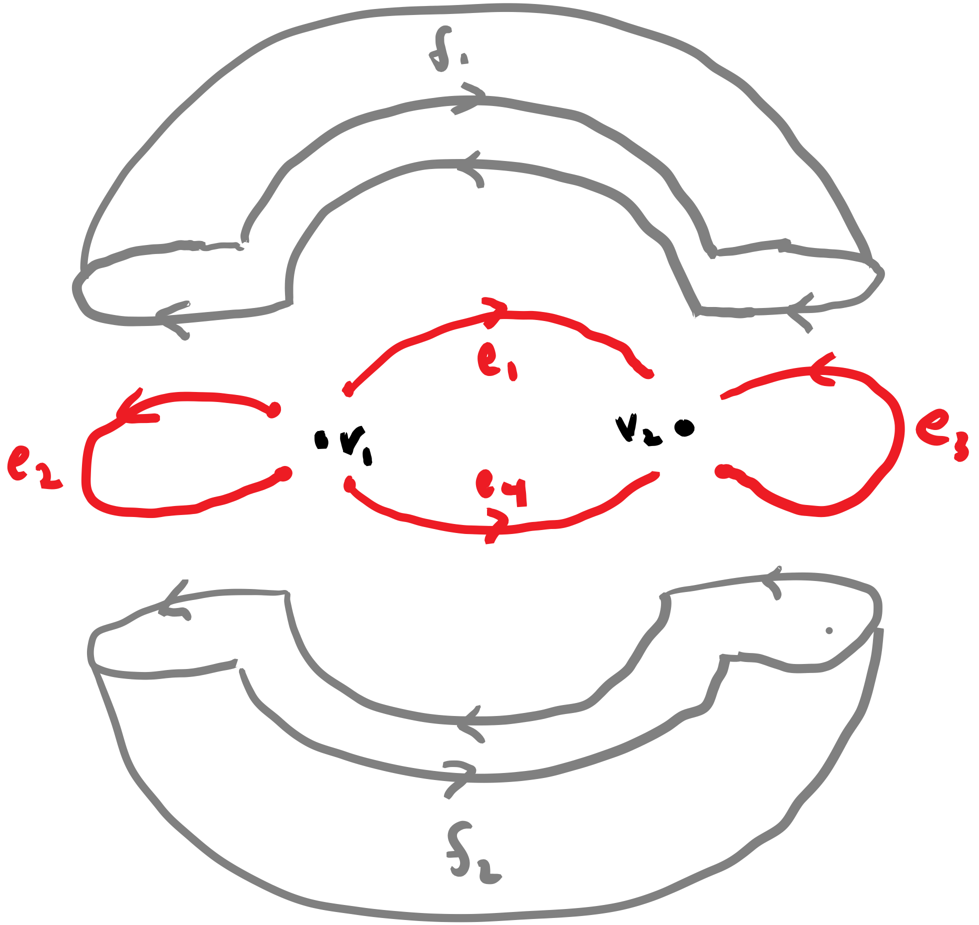

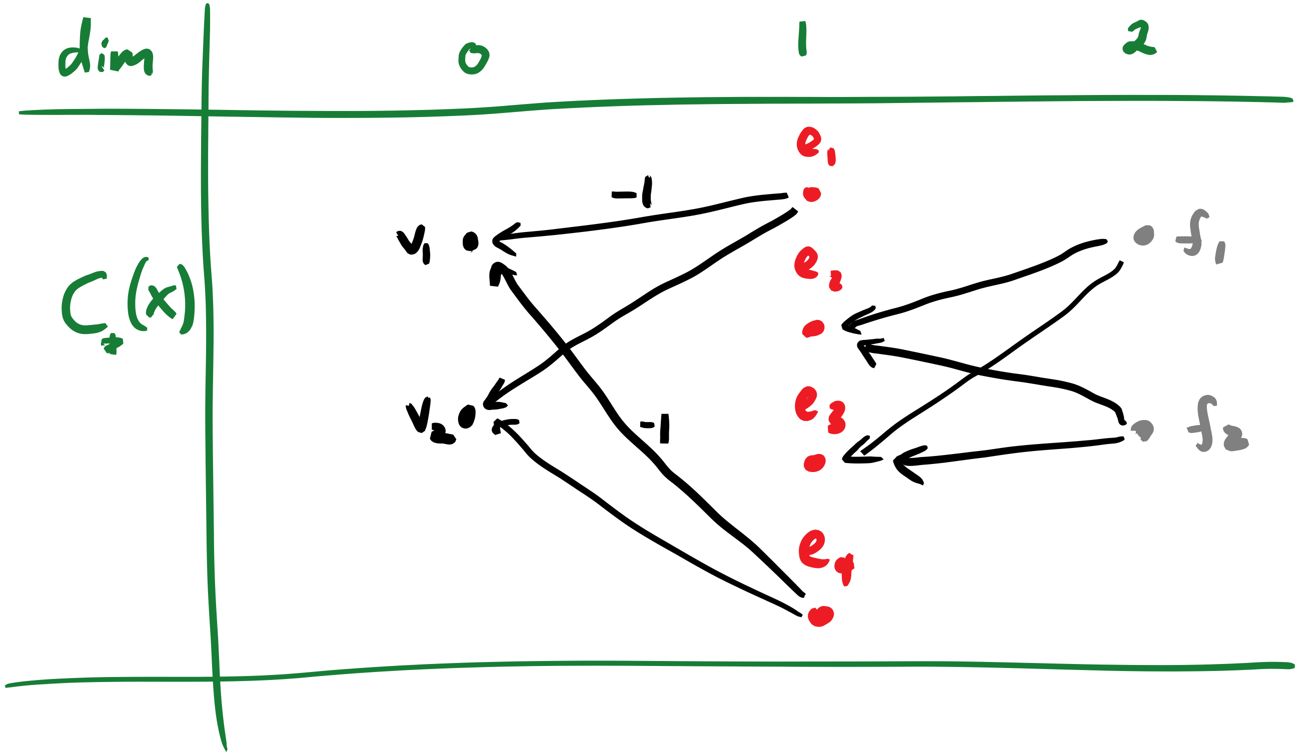

["Graph" notation]

Cellular homology

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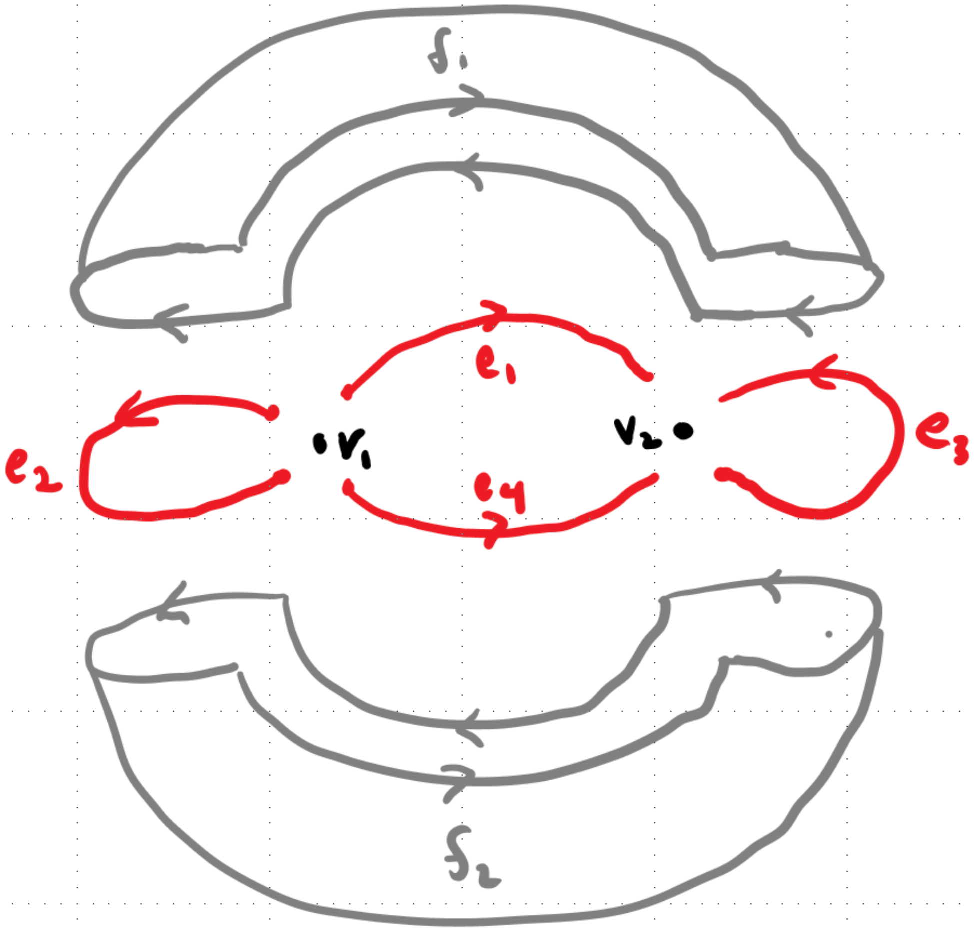

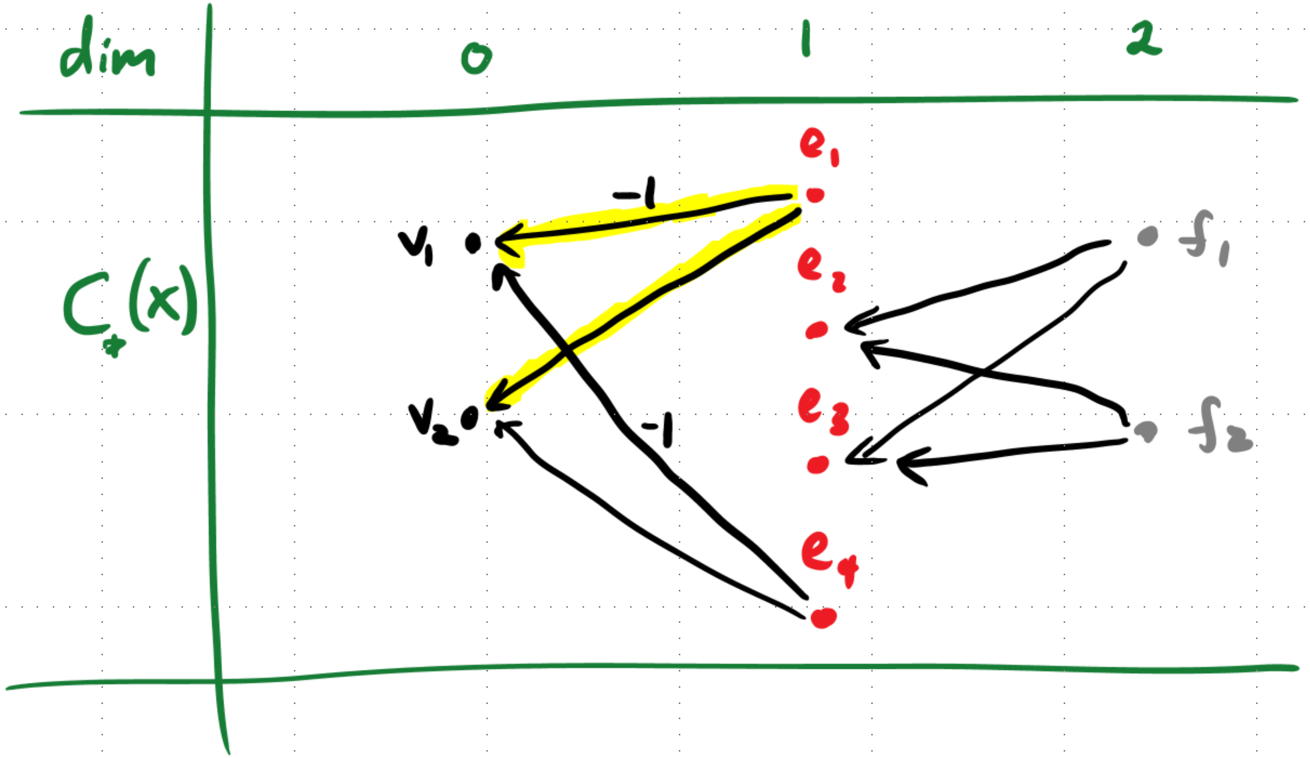

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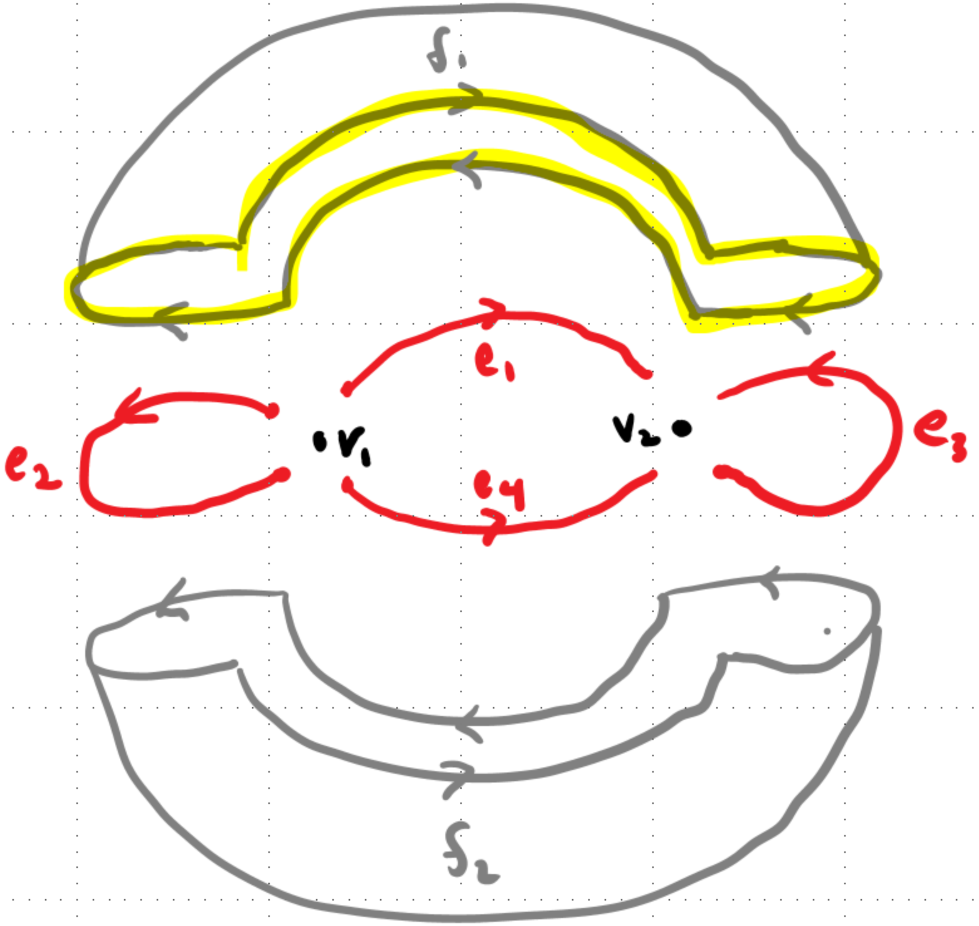

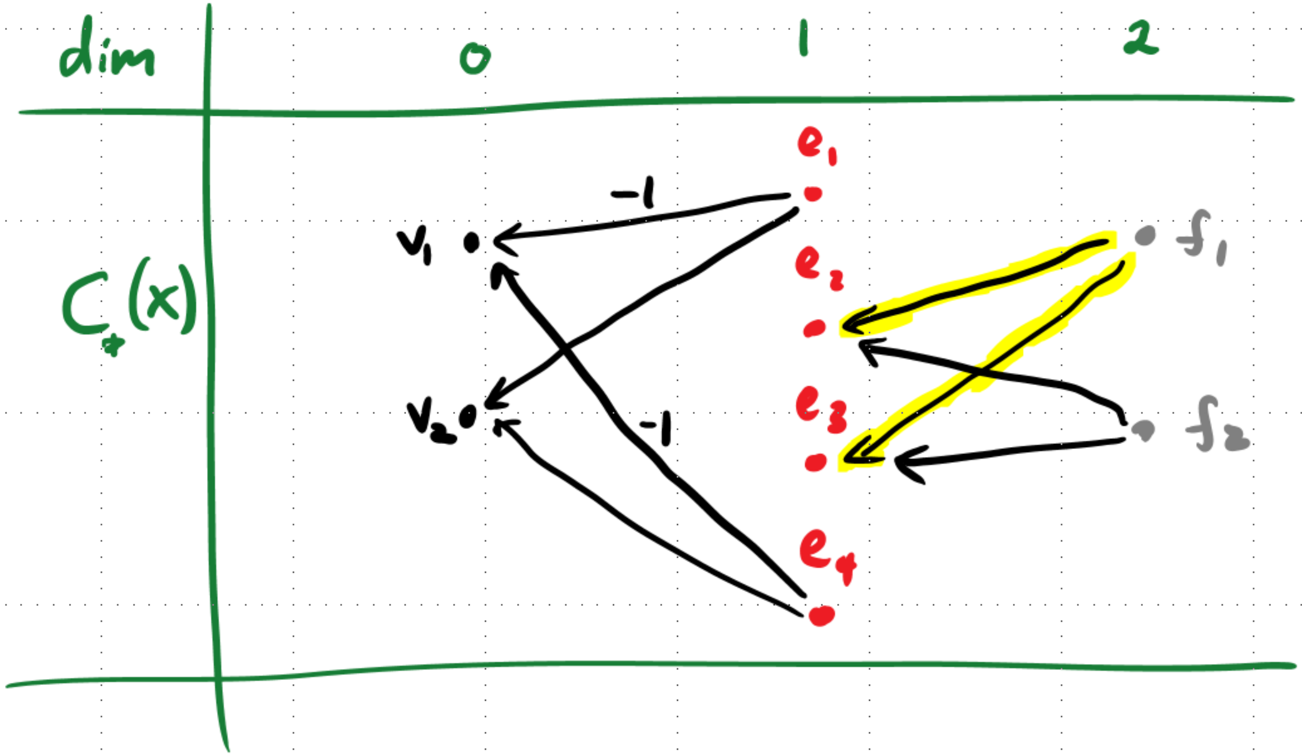

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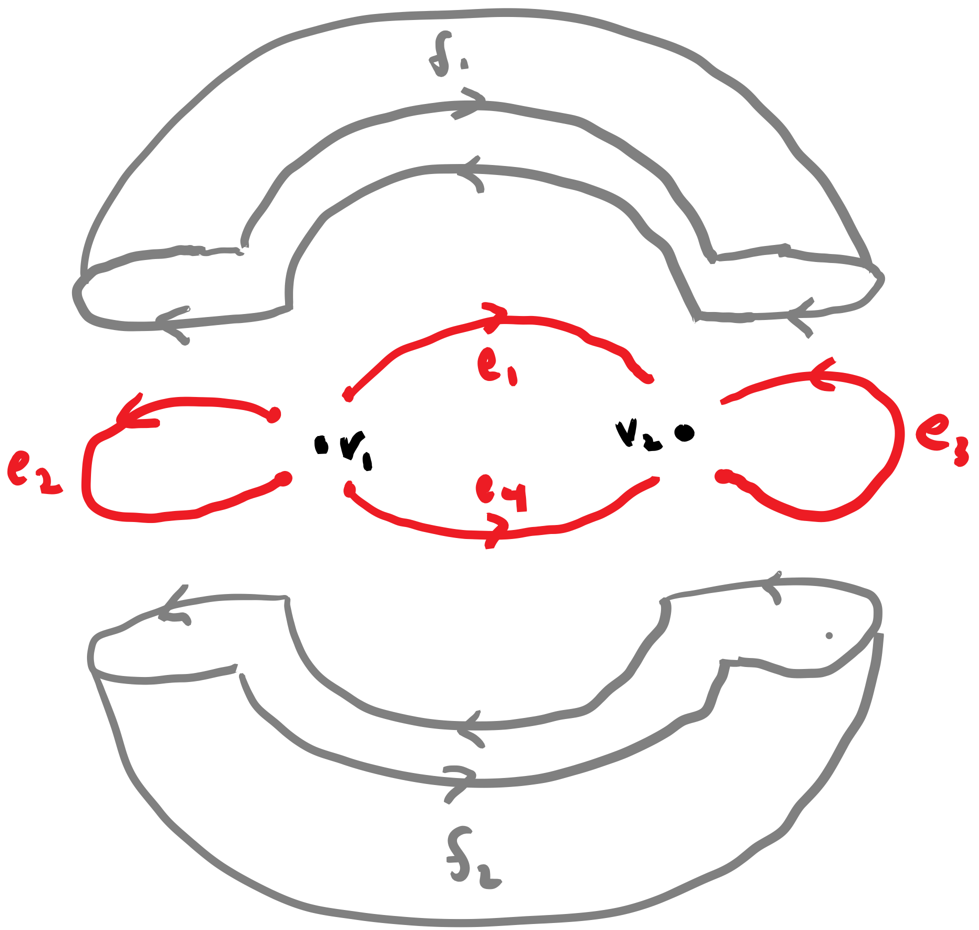
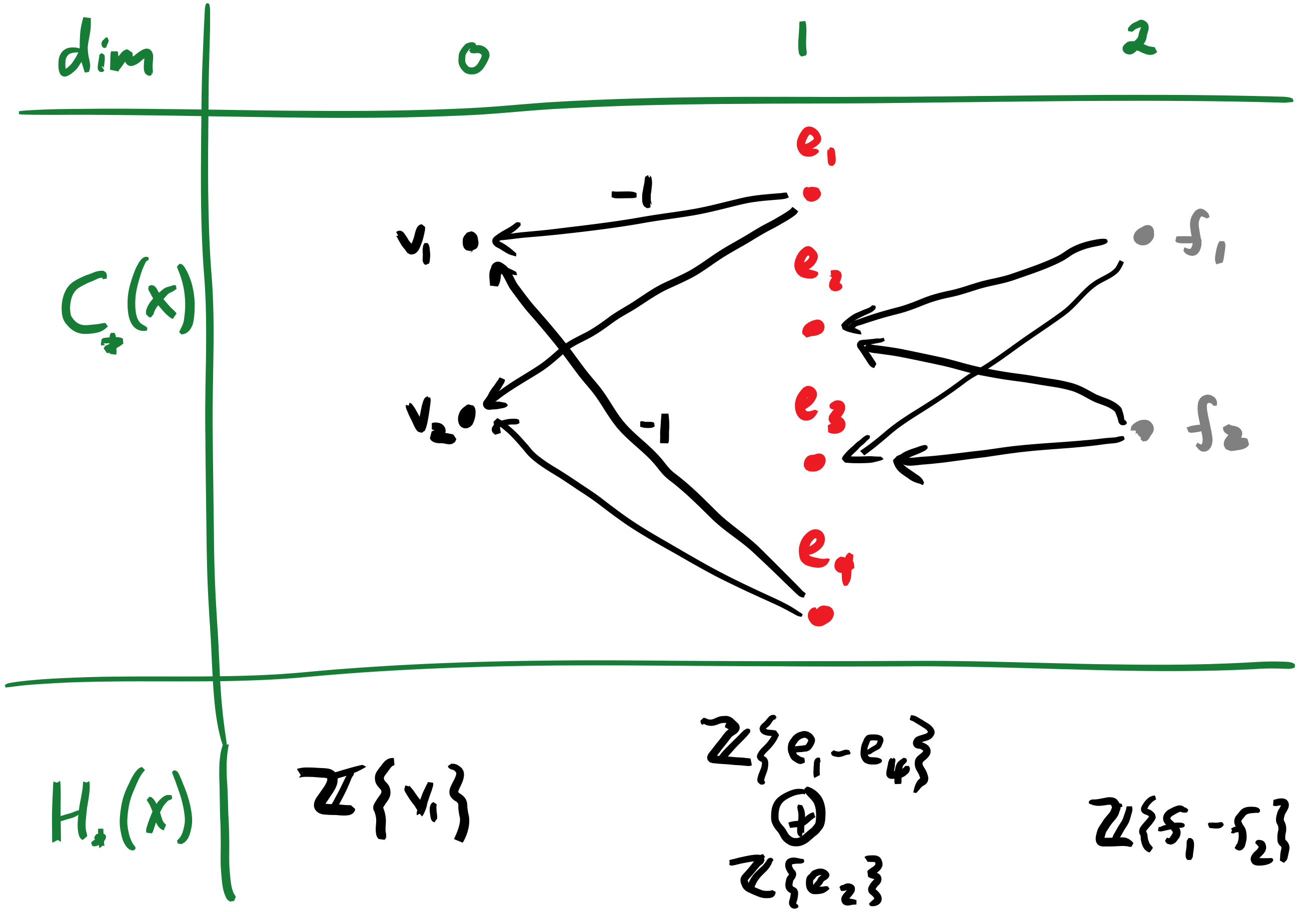




Cellular homology

$\mathbb{R}P^4 = S^4 /_{\text{antipodal}}$

$\mathbb{R}P^0 \hookrightarrow \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3 \hookrightarrow \mathbb{R}P^4$

Cellular homology

$\mathbb{RP}^4 = S^4/_{\text{antipodal}}$

$\mathbb{RP}^0 \hookrightarrow \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2 \hookrightarrow \mathbb{RP}^3 \hookrightarrow \mathbb{RP}^4$

dim	0	1	2	3	4
$C_*(\mathbb{RP}^4)$	•	• $\xleftarrow{2}$	•	• $\xleftarrow{2}$	•

Cellular homology

$\mathbb{RP}^4 = S^4/_{\text{antipodal}}$

	$\mathbb{RP}^0 \hookrightarrow$	$\mathbb{RP}^1 \hookrightarrow$	$\mathbb{RP}^2 \hookrightarrow$	$\mathbb{RP}^3 \hookrightarrow$	$\mathbb{RP}^4$
dim	0	1	2	3	4
$C_*(\mathbb{RP}^4)$	•	• $\xleftarrow{2}$	•	• $\xleftarrow{2}$	•
$H_*(\mathbb{RP}^4)$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

Cellular homology: different coefficients

Homology with coefficients in a ring R – each dot represents a copy of R instead of a copy of $\mathbb{Z}$.

[In this example, $R = \mathbb{F}_2$]

	$\mathbb{R}P^0 \hookrightarrow$	$\mathbb{R}P^1 \hookrightarrow$	$\mathbb{R}P^2 \hookrightarrow$	$\mathbb{R}P^3 \hookrightarrow$	$\mathbb{R}P^4$
dim	0	1	2	3	4
$C_\bullet(\mathbb{R}P^4)$	•	• $\xleftarrow{2 \,=\, 0}$	•	• $\xleftarrow{2 \,=\, 0}$	•
$H_\bullet(\mathbb{R}P^4; \mathbb{F}_2)$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$

BAD detector?? Does not detect degree 2 attaching maps!

Cohomology: reverse arrows

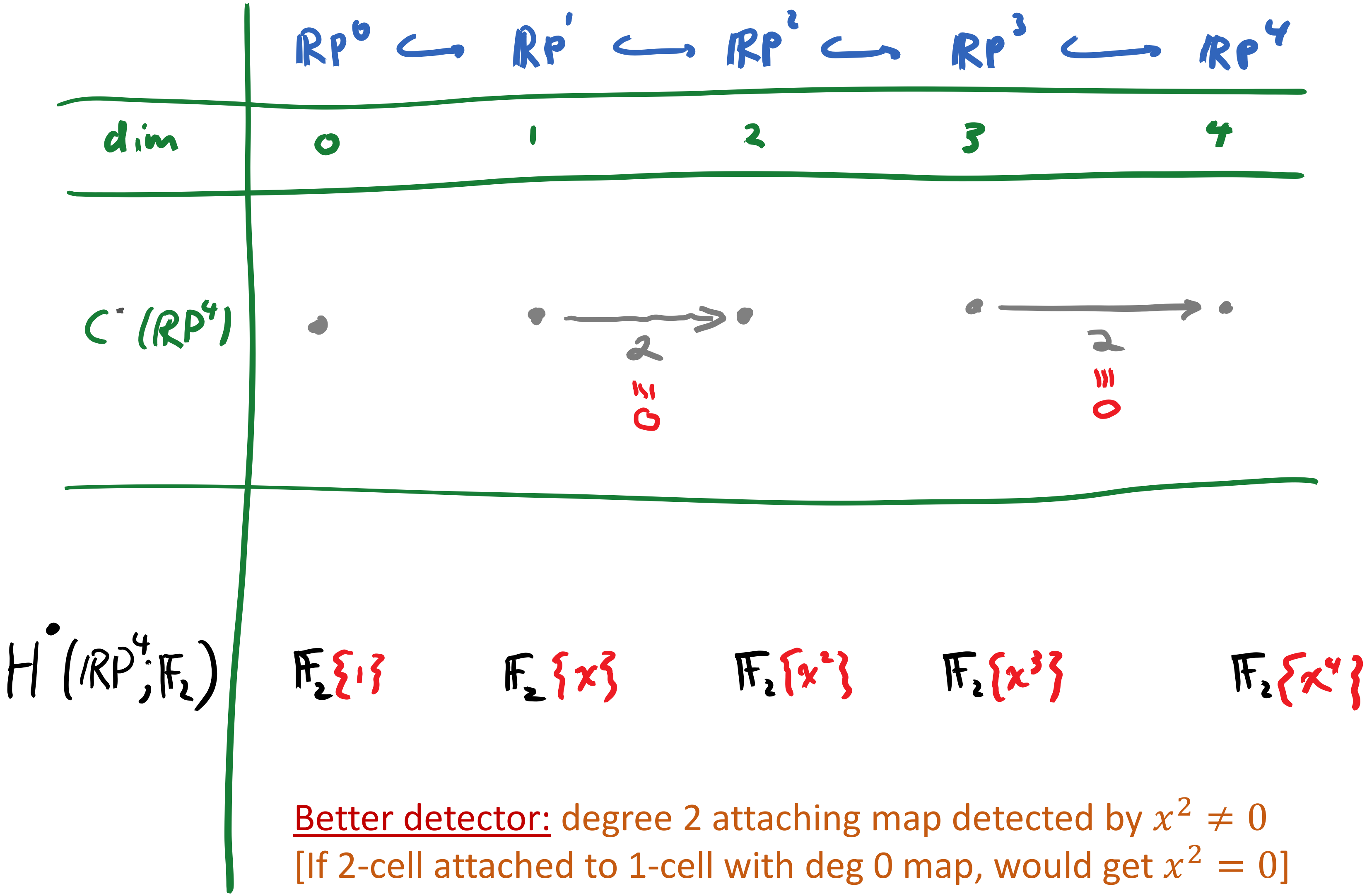
	$\mathbb{RP}^0 \hookrightarrow$	$\mathbb{RP}^1 \hookrightarrow$	$\mathbb{RP}^2 \hookrightarrow$	$\mathbb{RP}^3 \hookrightarrow$	$\mathbb{RP}^4$
dim	0	1	2	3	4
$C^\bullet(\mathbb{RP}^4)$	$\bullet$	$\bullet \xrightarrow{2}$	$\bullet$	$\bullet \xrightarrow{2}$	$\bullet$
$H^\bullet(\mathbb{RP}^4)$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$

Cohomology: cup product structure

Cohomology is a ring!

In this example,

$$H^*(\mathbb{R}P^4; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^5)$$



Better detector: degree 2 attaching map detected by $x^2 \neq 0$

[If 2-cell attached to 1-cell with deg 0 map, would get $x^2 = 0$]

Cell diagrams: a way of encoding attaching maps

X = CW complex

$$X^i = X^{i-1} \cup_\alpha D^i$$

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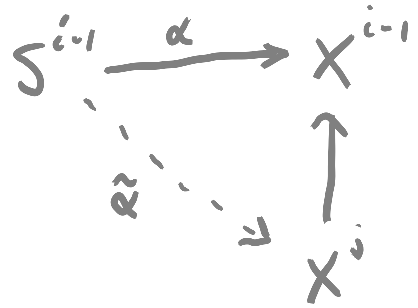


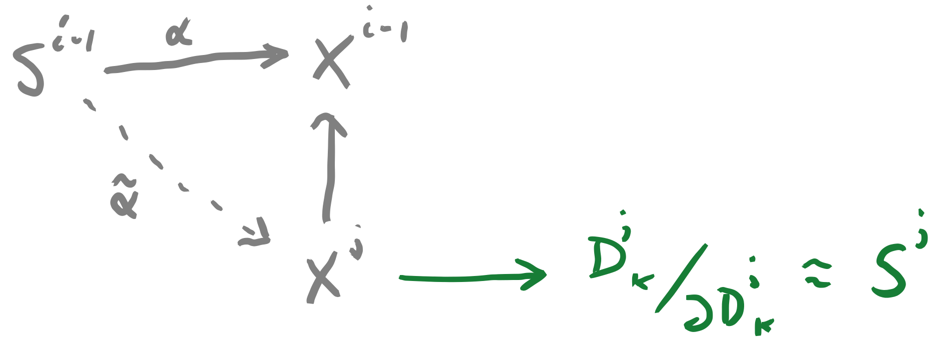
Let j be minimal so that α factors through X^j

Cell diagrams: a way of encoding attaching maps

X = CW complex

$X^i = X^{i-1} \cup_\alpha D^i$

$X^j = X^{j-1} \cup \bigcup_k D^j_k$



Suppose X has j-cells $D_1^j, D_2^j, \ldots$

For each such cell D_k^j there is a projection map

Cell diagrams: a way of encoding attaching maps

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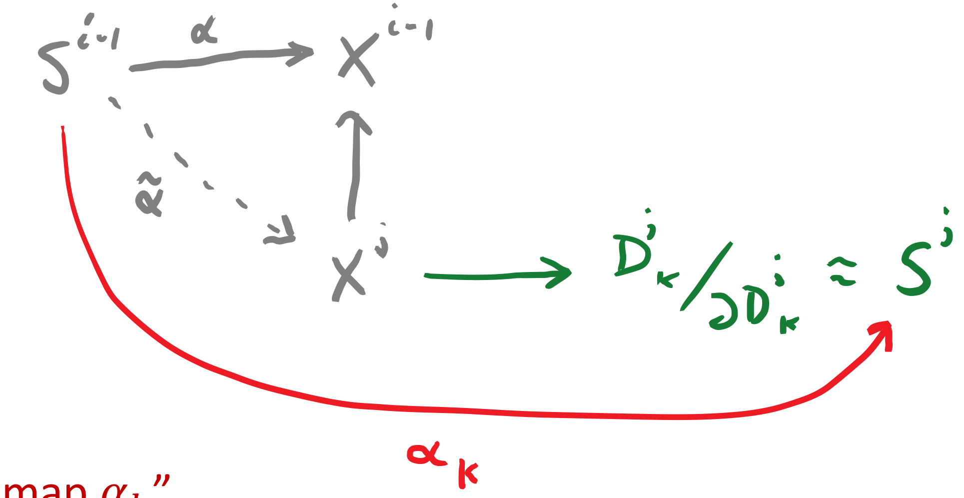

We say:

"the i-cell attaches to the j-cell D_k^j with attaching map α_k"

Cell diagrams: a way of encoding attaching maps

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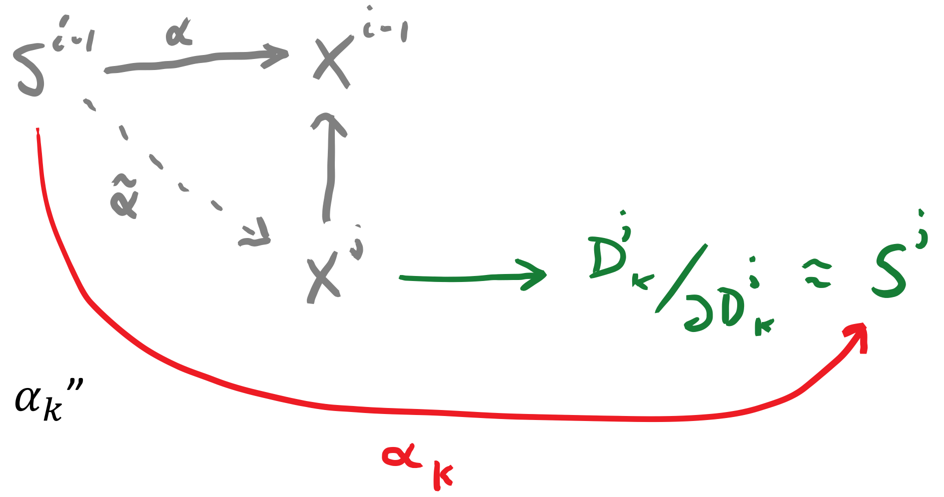

We say:

"the i-cell attaches to the j-cell D_k^j with attaching map α_k"

Note:

- A given cell can attach nontrivially to many other cells
- In this way, the stable equivalence class of X is essentially determined by the collection of all its attaching maps $\alpha_k \in \pi_{i-1}^{st}(S^j)$
- This is why I asserted that the stable homotopy groups of spheres are the "elementary particles" which comprise CW complexes

Cell diagrams: a way of encoding attaching maps

X = CW complex

$X^i = X^{i-1} \cup_\alpha D^i$

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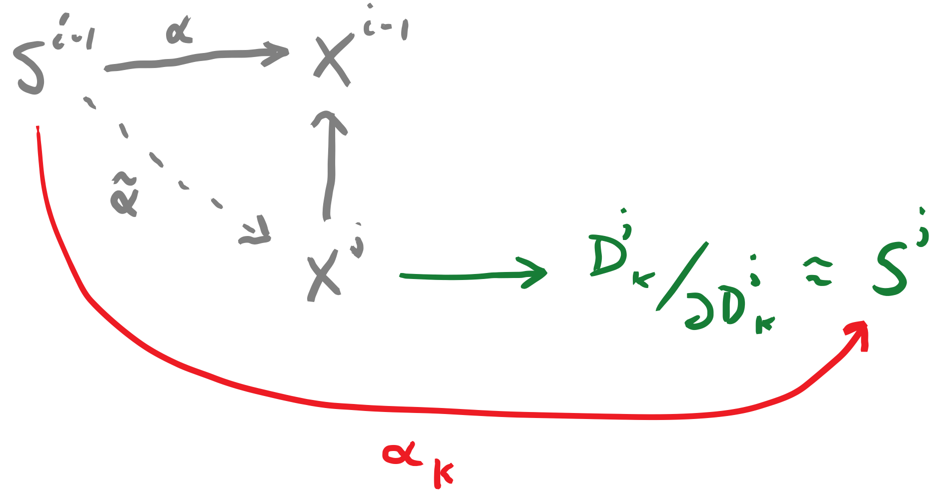
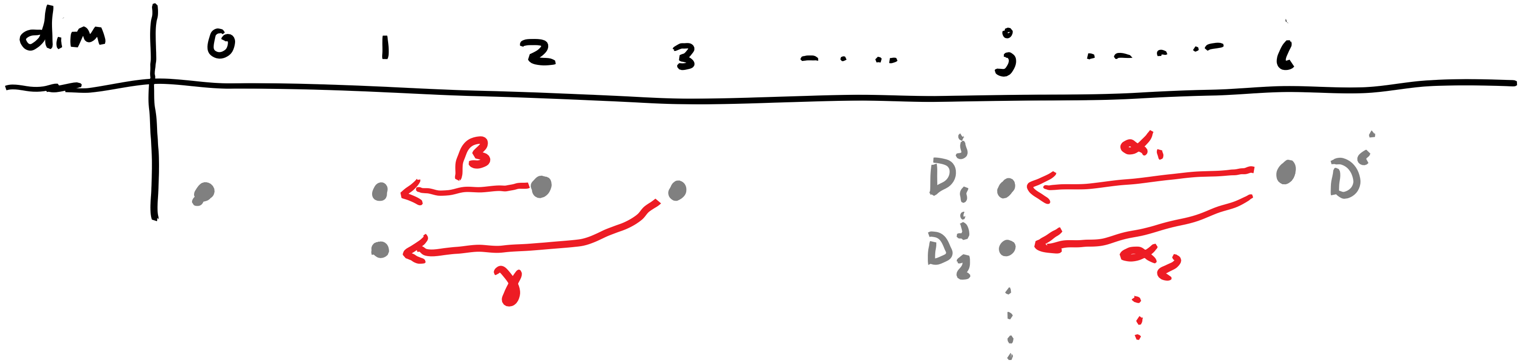


Cell diagram: "Refinement of Cellular chain complex"

1) Draw one dot for each cell

2) Draw arrows labelled by attaching maps



[examples: $\mathbb{C}P^2, \mathbb{R}P^4$]

Steenrod operations: more structure on mod 2 cohomology

Theorem: (Steenrod)

There are natural homomorphisms ($i \geq 0$)

$$Sq^i : H^n(X; \mathbb{F}_2) \to H^{n+i}(X; \mathbb{F}_2)$$

- $Sq^i(x) = \begin{cases} x, & i = 0 \\ ?, & 1 \leq i \leq n-1 \\ x^2, & i = n \\ 0 & i > n \end{cases}$

Steenrod operations sometimes detect attaching maps!

[examples: $\mathbb{C}P^2, \mathbb{R}P^4$]

- $Sq^i Sq^j = \sum_k \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$ [Adem relations]

Steenrod algebra = algebra of these operators

$\mathcal{A} := \mathbb{F}_2 \langle Sq^i : i > 0 \rangle$ / Adem relations

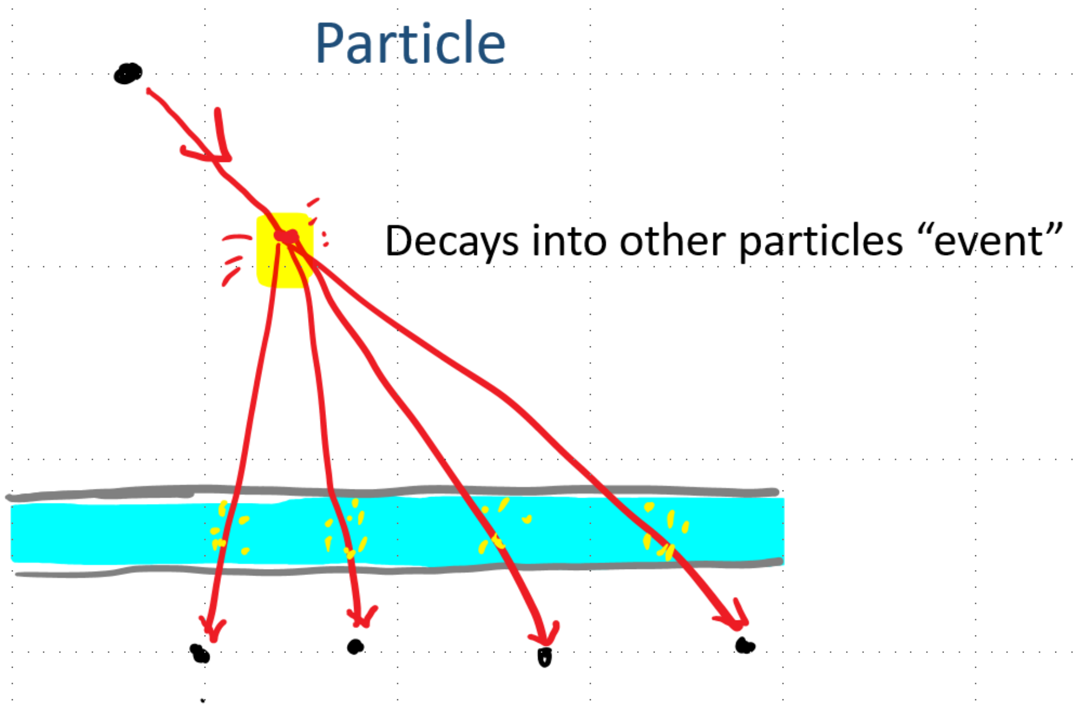
Similar operations on $H^*(X; \mathbb{F}_p)$

For the rest of this talk, all cohomology reduced, with $\mathbb{F}_2$-coefficients!

$$H^*X \coloneqq \widetilde{H}^*(X; \mathbb{F}_2)$$

Game plan to build our detector…

Silicon particle detector:

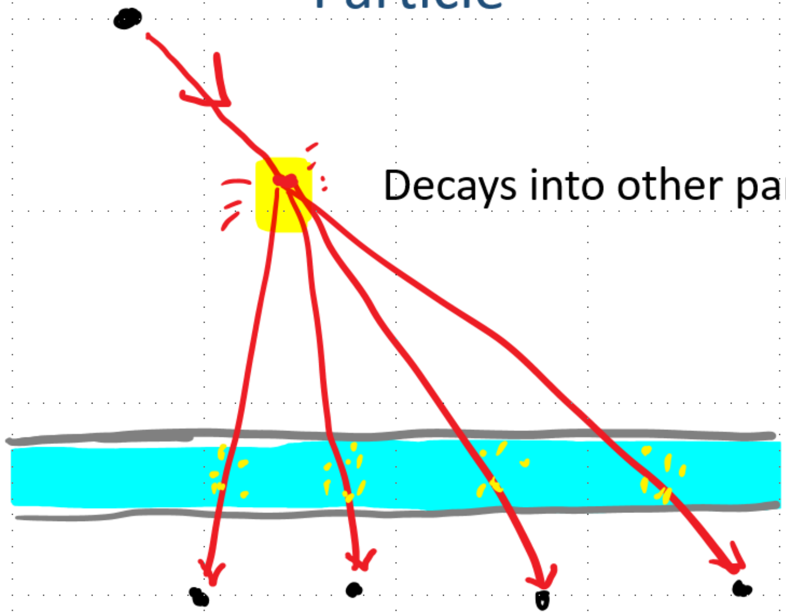


"signal" = collection of signals in xy-channels

Game plan to build our detector…

Silicon particle detector:

Particle



Decays into other particles "event"

"signal" = collection of signals in xy-channels

Our homotopy detector:

$f : Y \to X$ (stable homotopy class)

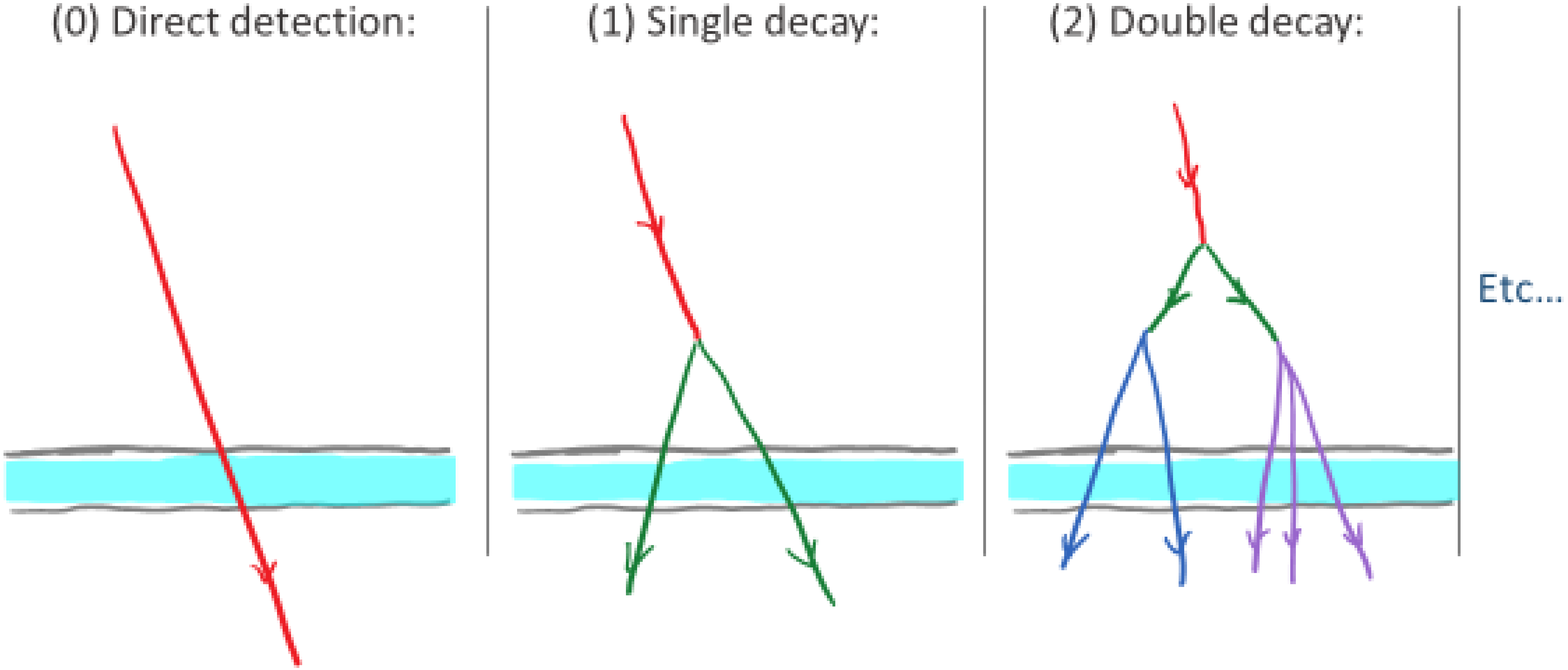
"event"

$[f] \in Ext_{\mathcal{A}}^{s}(H^{*+s}X, H^{*}Y)$

"homology signal"

Particle detector event channels

Many different possibilities for decay channels for a given particle

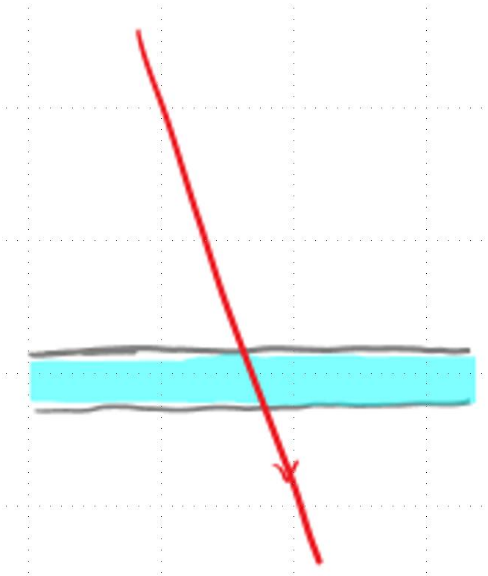


Homology event channels

Given:

$$f: Y \to X$$

(0) Direct detection



Suppose the induced map

$$f^*: H^*X \to H^*Y$$

is nonzero.

Define: $[f] \coloneqq f^* \in Hom_{\mathcal{A}}(H^*X, H^*Y) = Ext^0_{\mathcal{A}}(H^*X, H^*Y)$

"signal"

Homology event channels

Given:

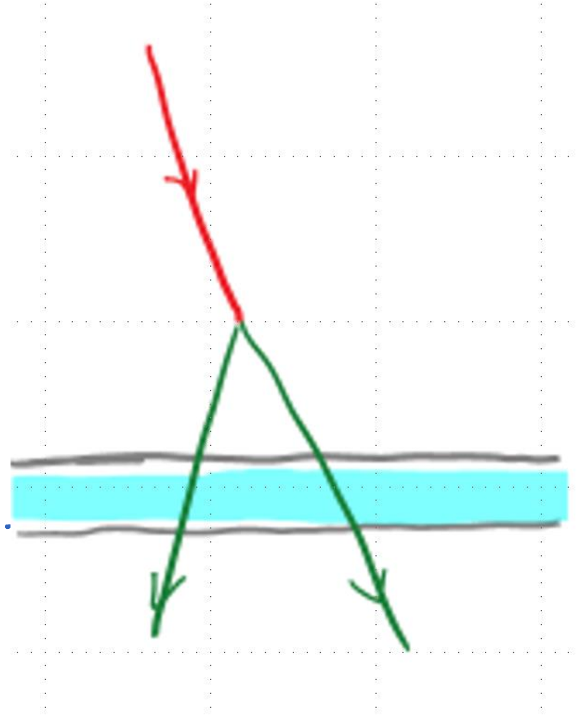
$$f: Y \to X \qquad \text{(zero on cohomology)}$$

(1) Indirect detection "single decay"

- Form a new CW complex - "mapping cone"

$$C_f \quad := X \cup_f CY$$

[picture]



- The long exact sequence

$$\cdots \to H^*X \underset{0}{\to} H^*Y \to H^{*+1}C_f \to H^{*+1}X \underset{0}{\to} H^{*+1}Y \to \cdots$$

is actually a short exact sequence:

$$0 \to H^*Y \to H^{*+1}C_f \to H^{*+1}X \to 0$$

- If this extension of $\mathcal{A}$-modules is nontrivial, get "signal":

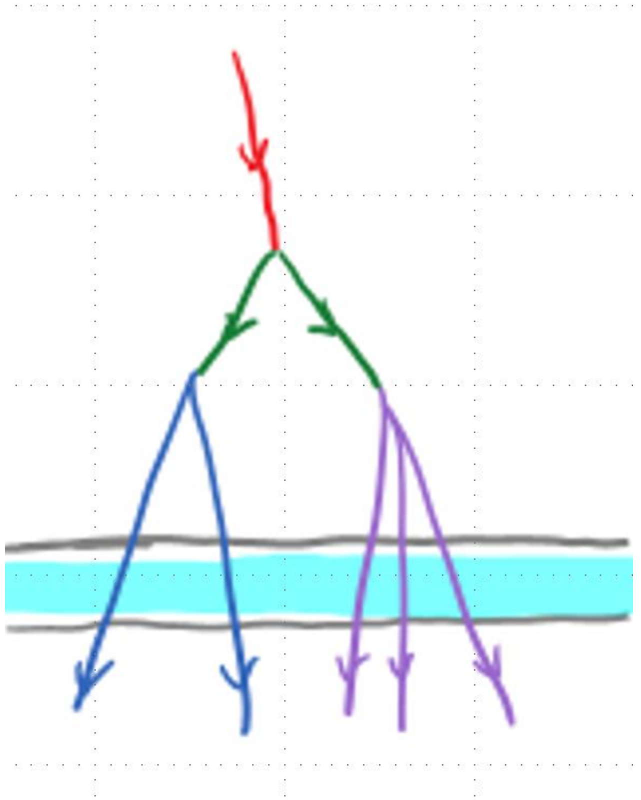
$$0 \neq [f] \in Ext^1_{\mathcal{A}}(H^{*+1}X, H^*Y)$$

Homology event channels

Given:

$f: Y \to X$ (zero on cohomology)

(s) Indirect detection "s-decays"



- Factor f into

$$Y = X_0 \underset{f_1}{\to} X_1 \underset{f_2}{\to} \cdots \underset{f_s}{\to} X_s = X$$

such that each f_i is zero on cohomology - "event"

- Get an exact sequence

$$0 \to H^*Y \to H^{*+1}C_{f_1} \to H^{*+2}C_{f_2} \to \cdots \to H^{*+s}C_{f_s} \to H^{*+s}X \to 0$$

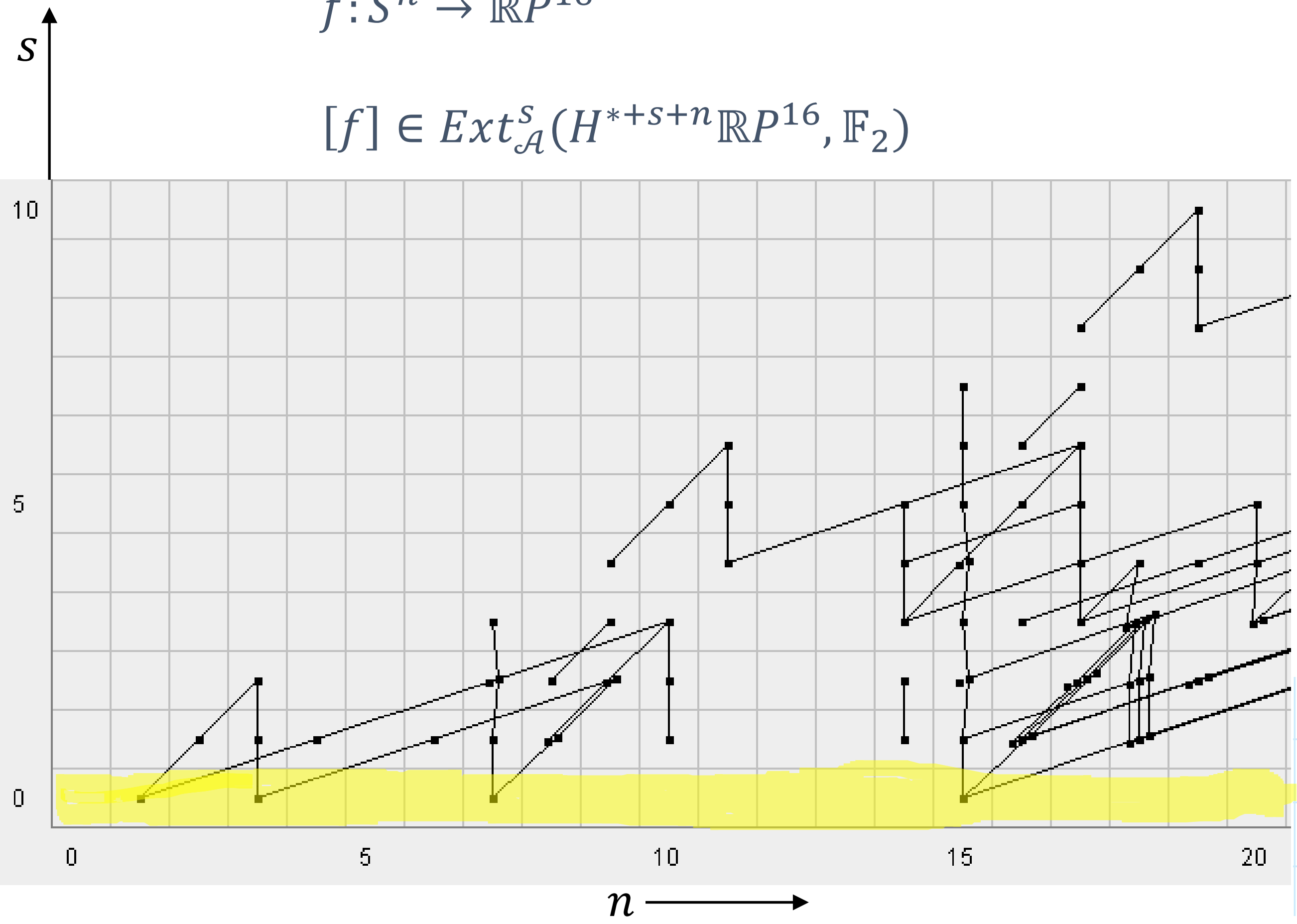
- Get a "signal"

$$[f] \in Ext^s_{\mathcal{A}}(H^{*+s}X, H^*Y)$$

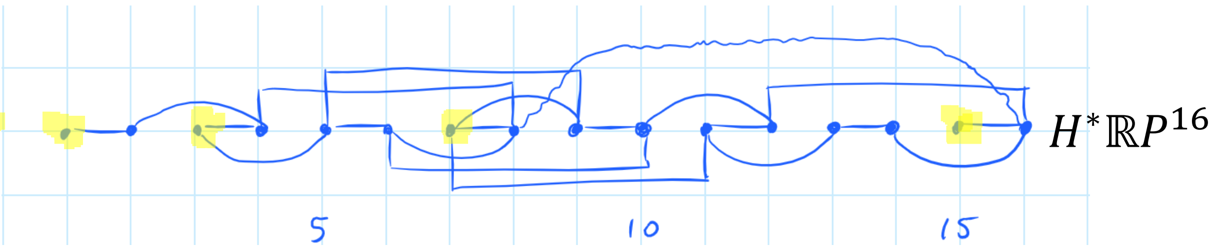
Examples of homology events, signals

$f: S^n \to \mathbb{R}P^{16}$

$[f] \in Ext^s_{\mathcal{A}}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



$Ext^0 \leftrightarrow$ cells not hit by Steenrod operations

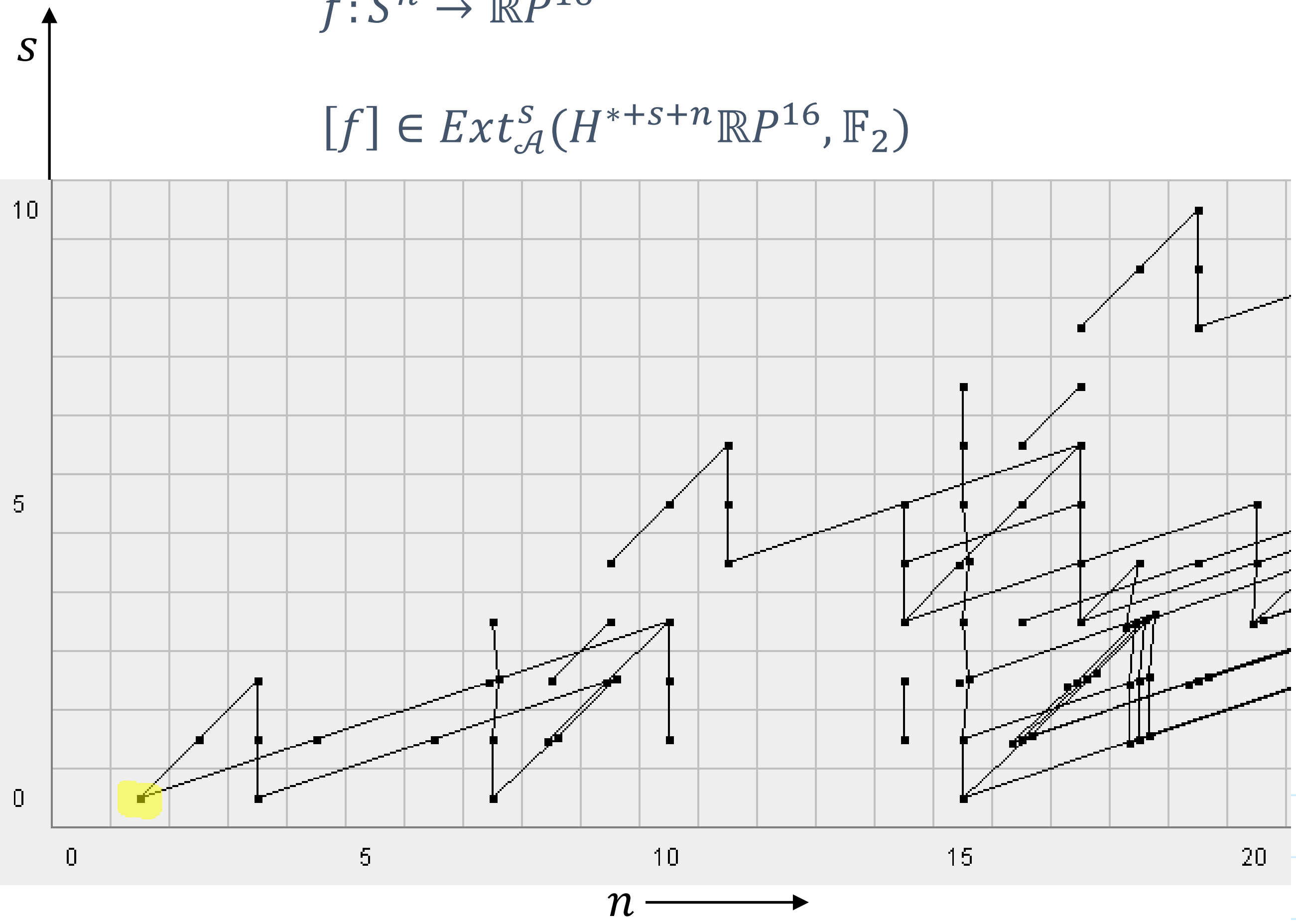


[Chart: Ext computing software Bruner/Perry]

Examples of homology events, signals

$f: S^n \to \mathbb{R}P^{16}$

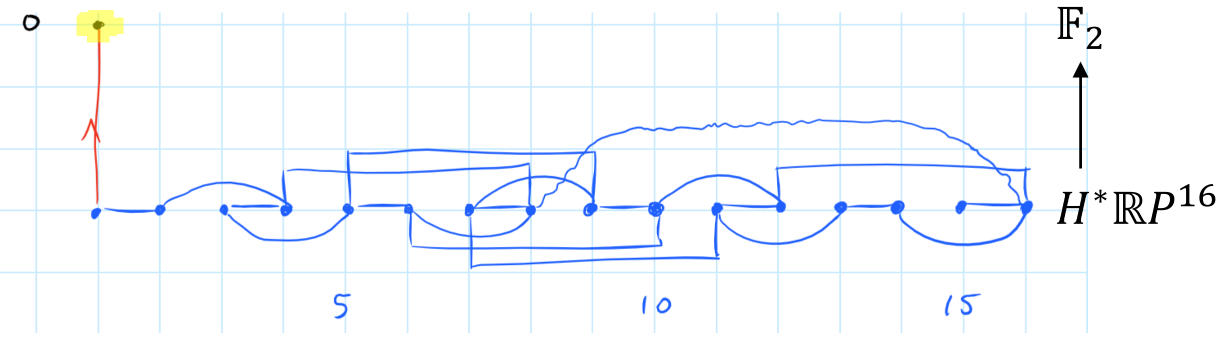
$[f] \in Ext^s_{\mathcal{A}}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



Event:

$\iota_1: S^1 \to \mathbb{R}P^{16}$

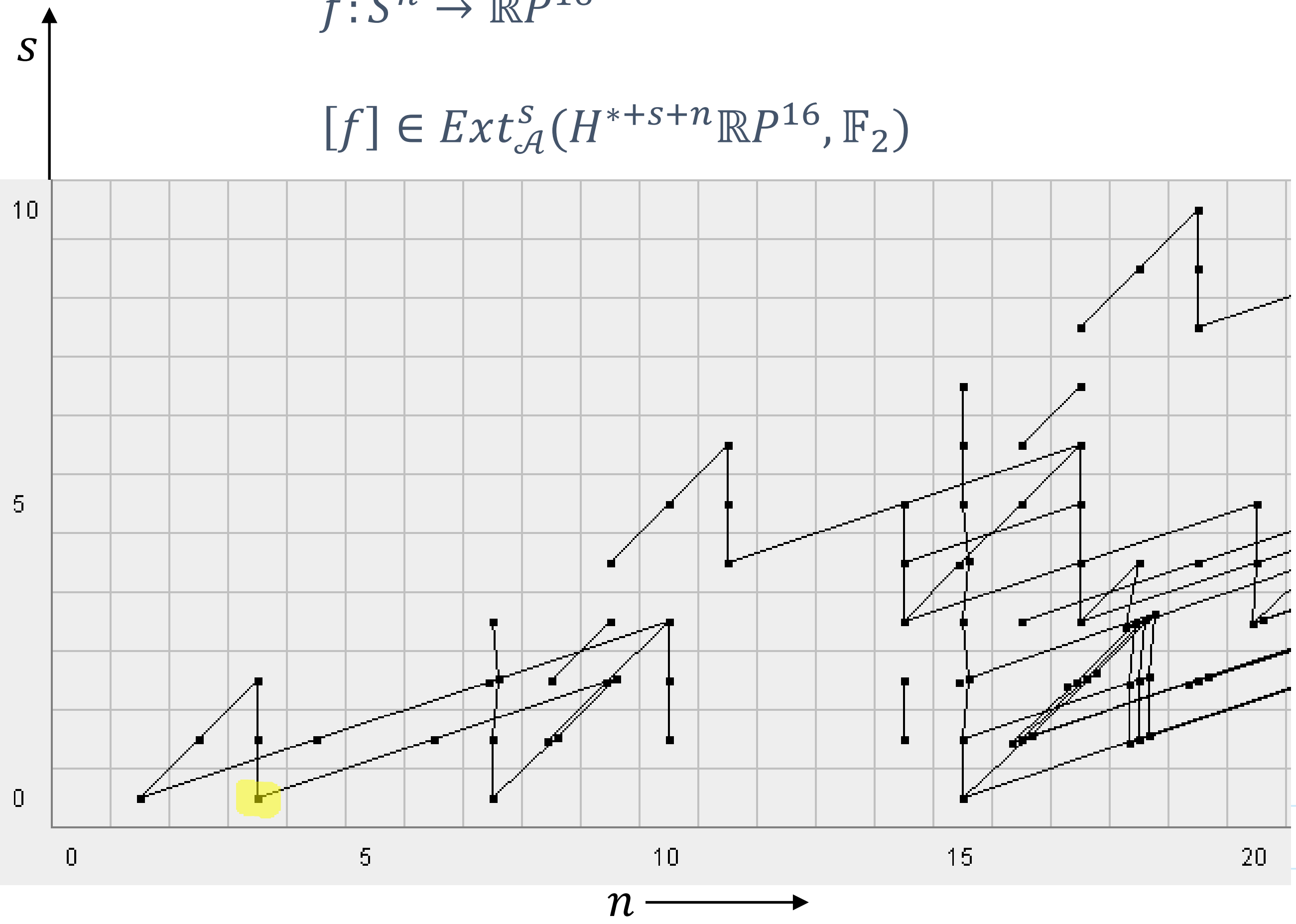
Signal:



Examples of homology events, signals

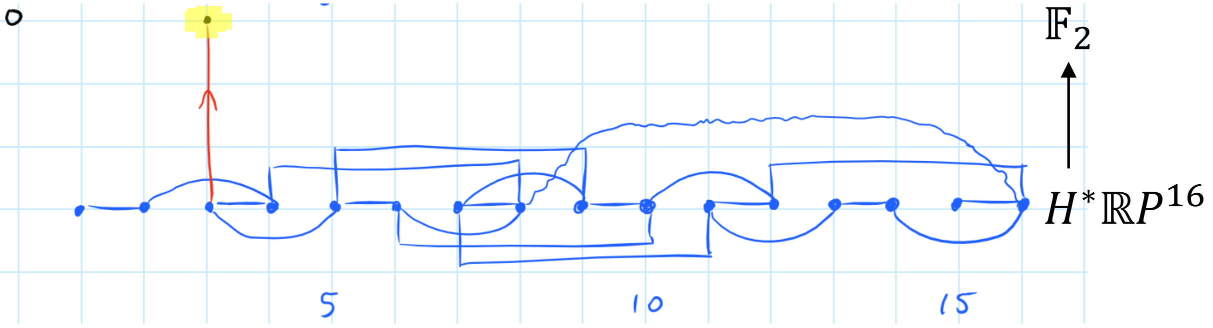
$f: S^n \to \mathbb{R}P^{16}$

$[f] \in Ext^s_{\mathcal{A}}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



Event:
$\iota_3: S^3 \to \mathbb{R}P^{16}$

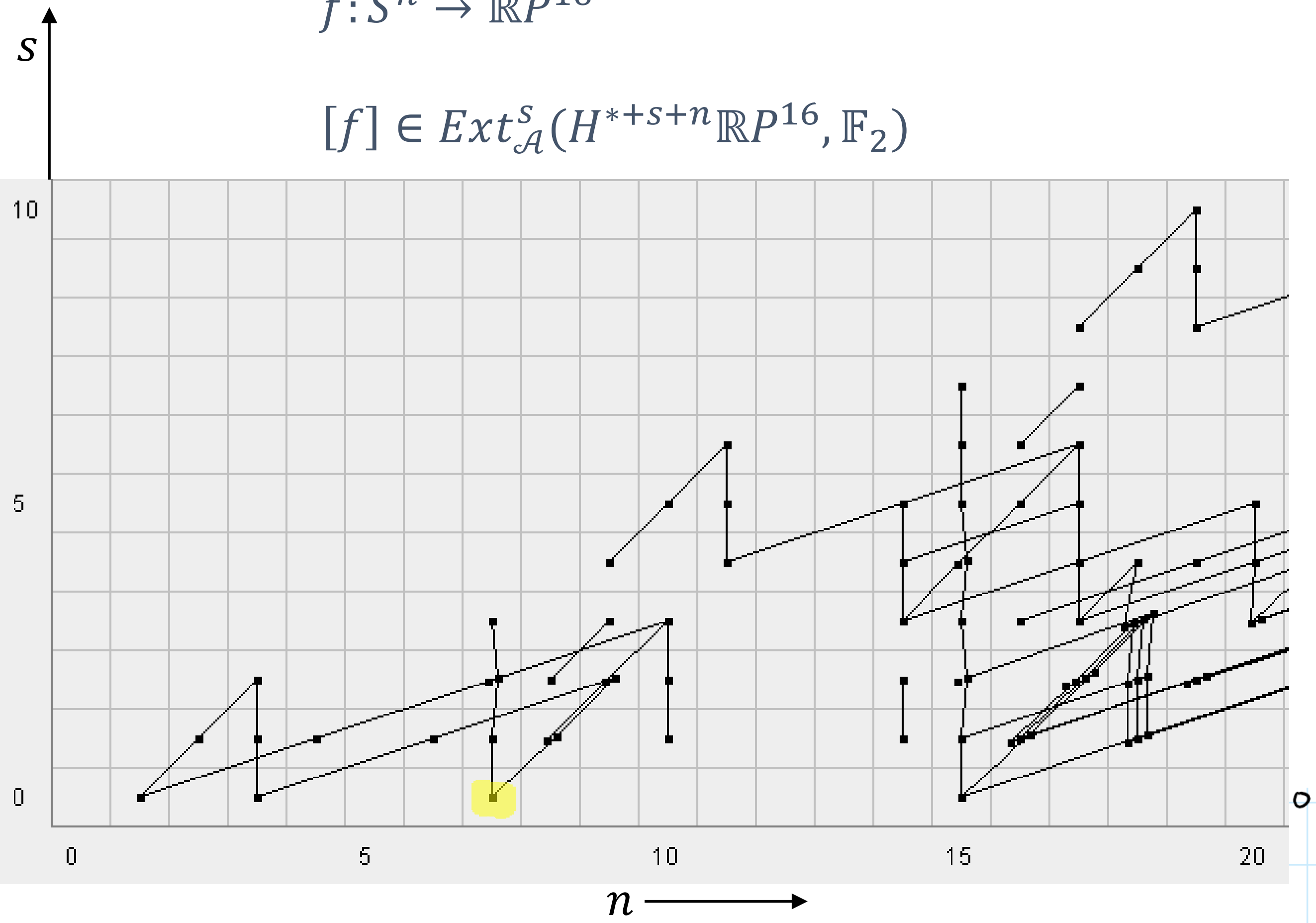
Signal:



Examples of homology events, signals

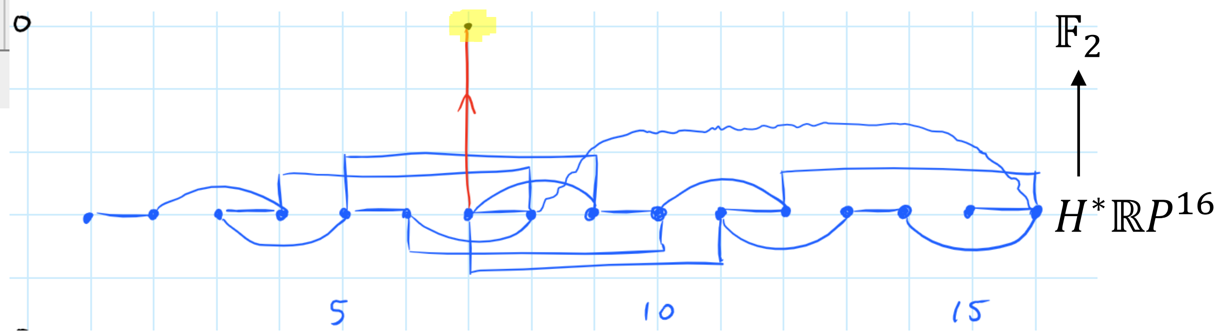
$f: S^n \to \mathbb{R}P^{16}$

$[f] \in Ext_{\mathcal{A}}^{s}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



Event:
$\iota_7: S^7 \to \mathbb{R}P^{16}$

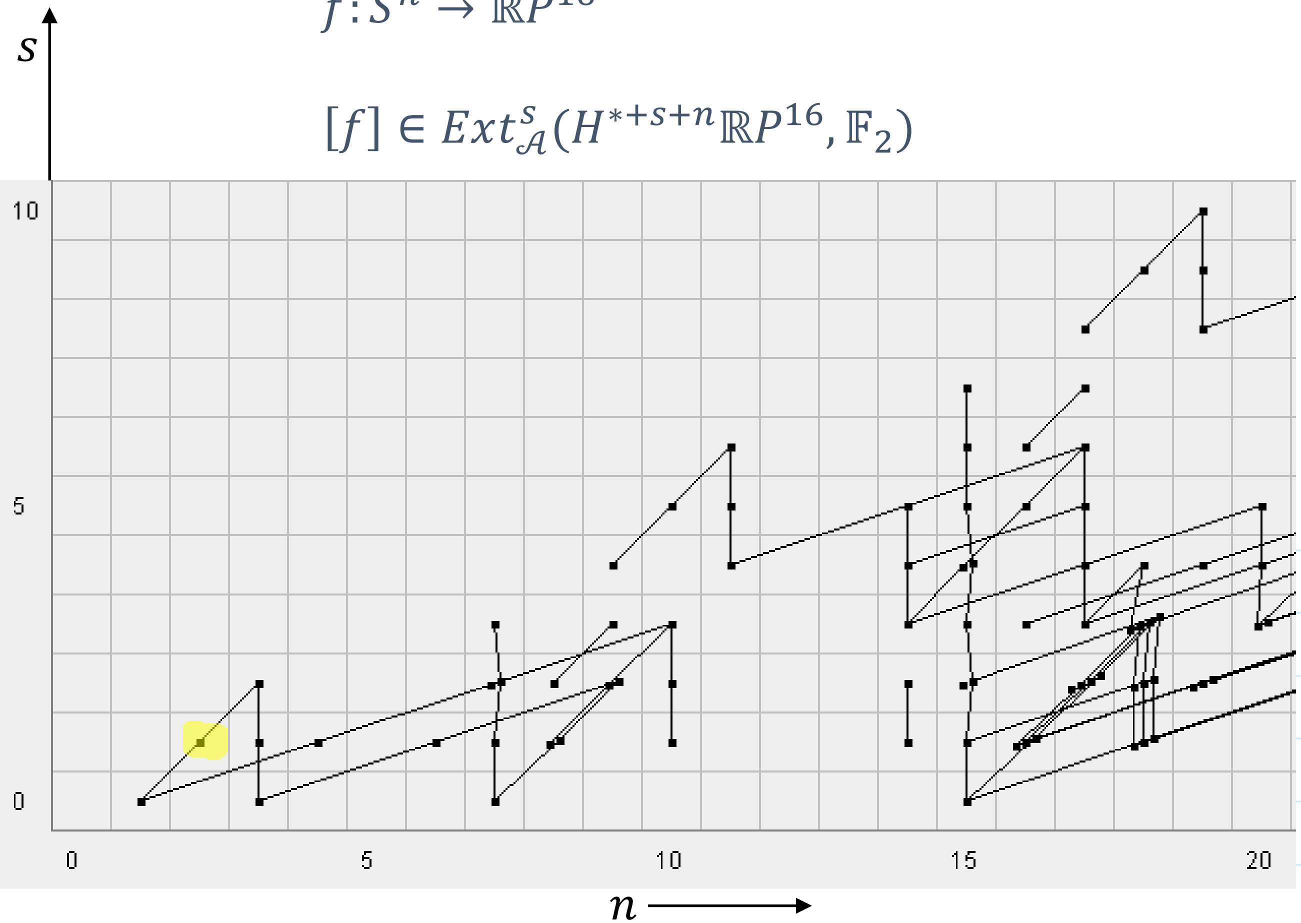
Signal:



Examples of homology events, signals

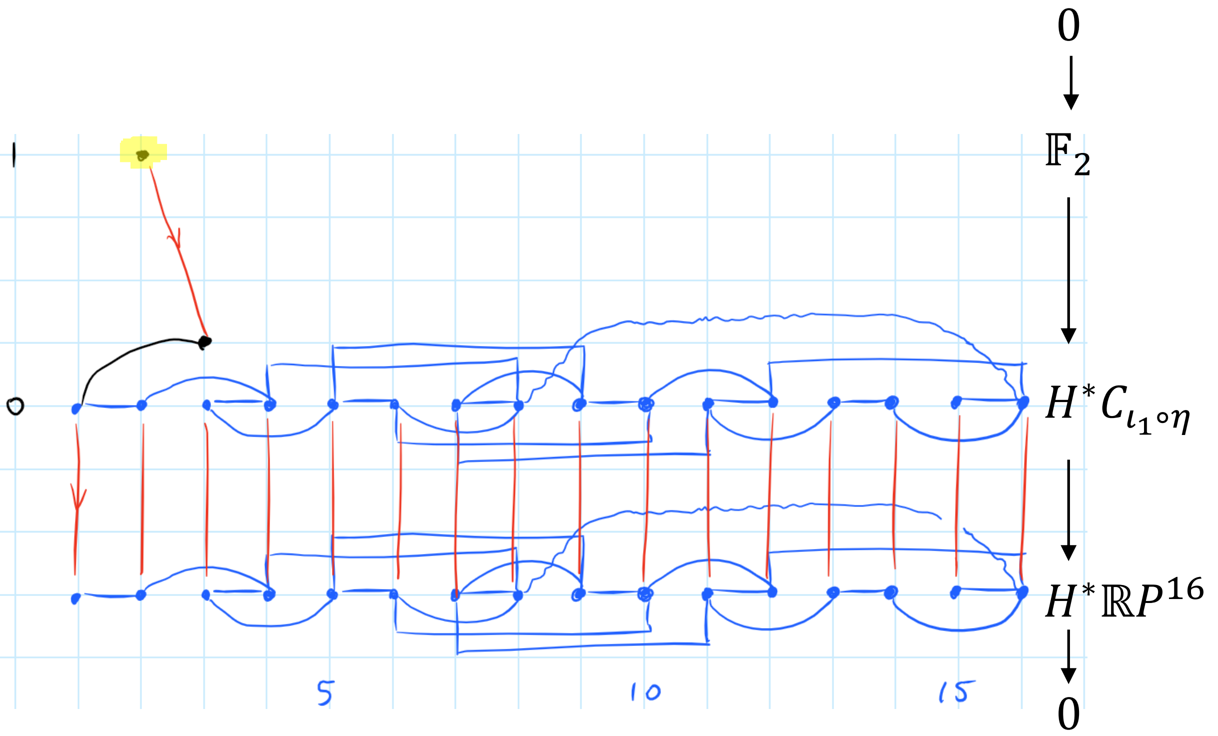
$f: S^n \to \mathbb{R}P^{16}$

$[f] \in Ext_{\mathcal{A}}^{s}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



Event:
$\iota_1 \circ \eta: S^2 \to \mathbb{R}P^{16}$

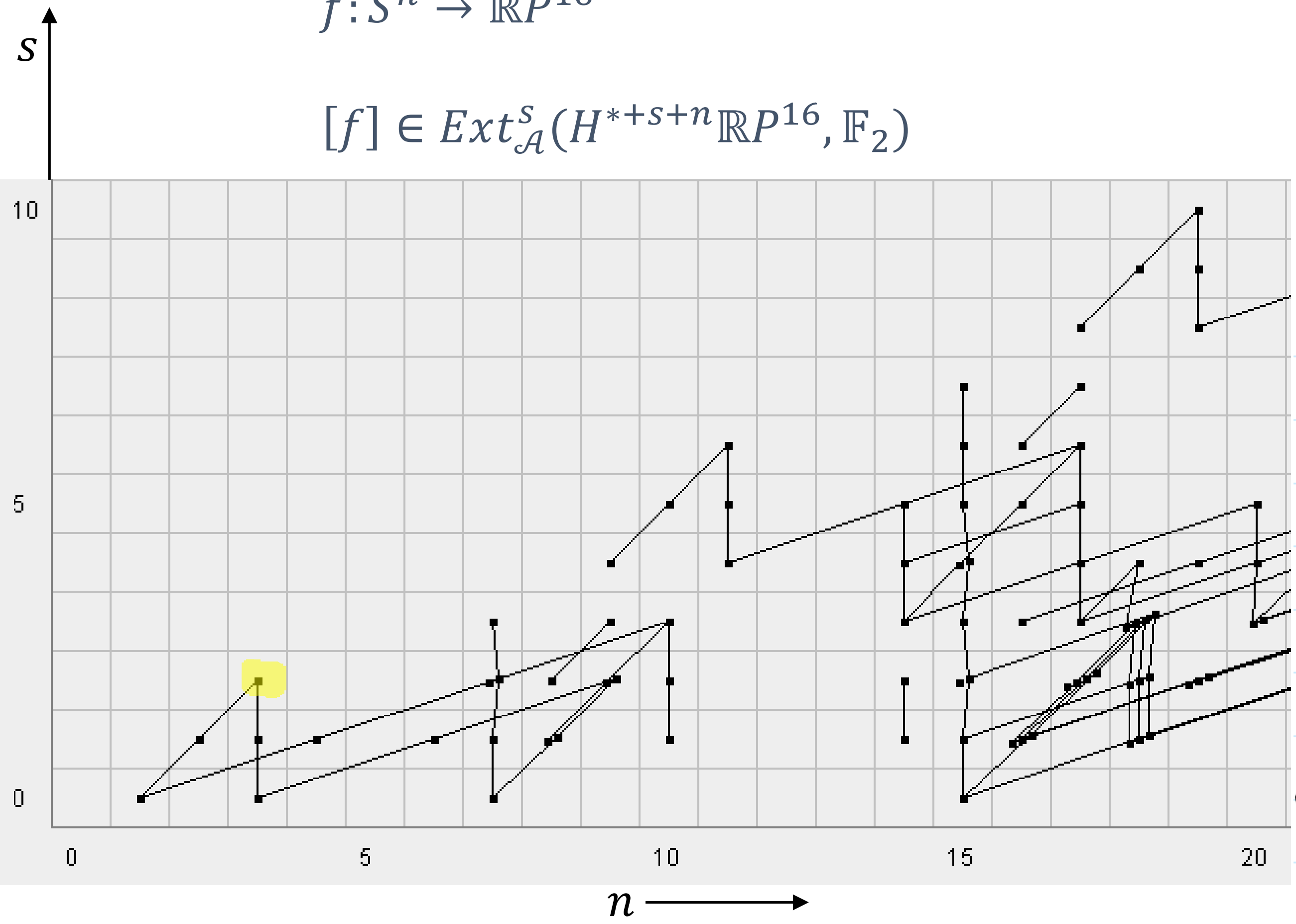
Signal:



Examples of homology events, signals

$f: S^n \to \mathbb{R}P^{16}$

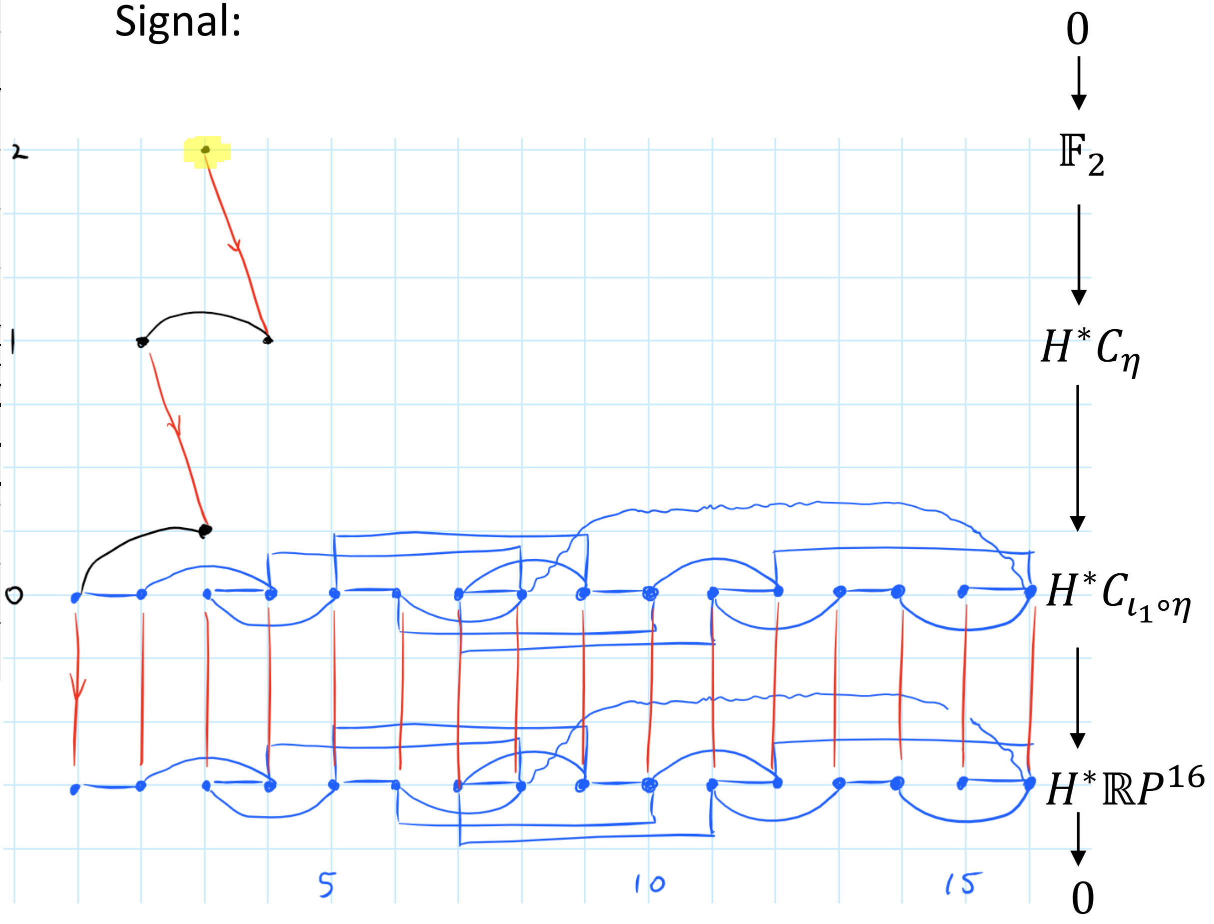
$[f] \in Ext_{\mathcal{A}}^{s}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



Event:

$S^3 \xrightarrow[\eta]{} S^2 \xrightarrow[\iota_1 \circ \eta]{} \mathbb{R}P^{16}$

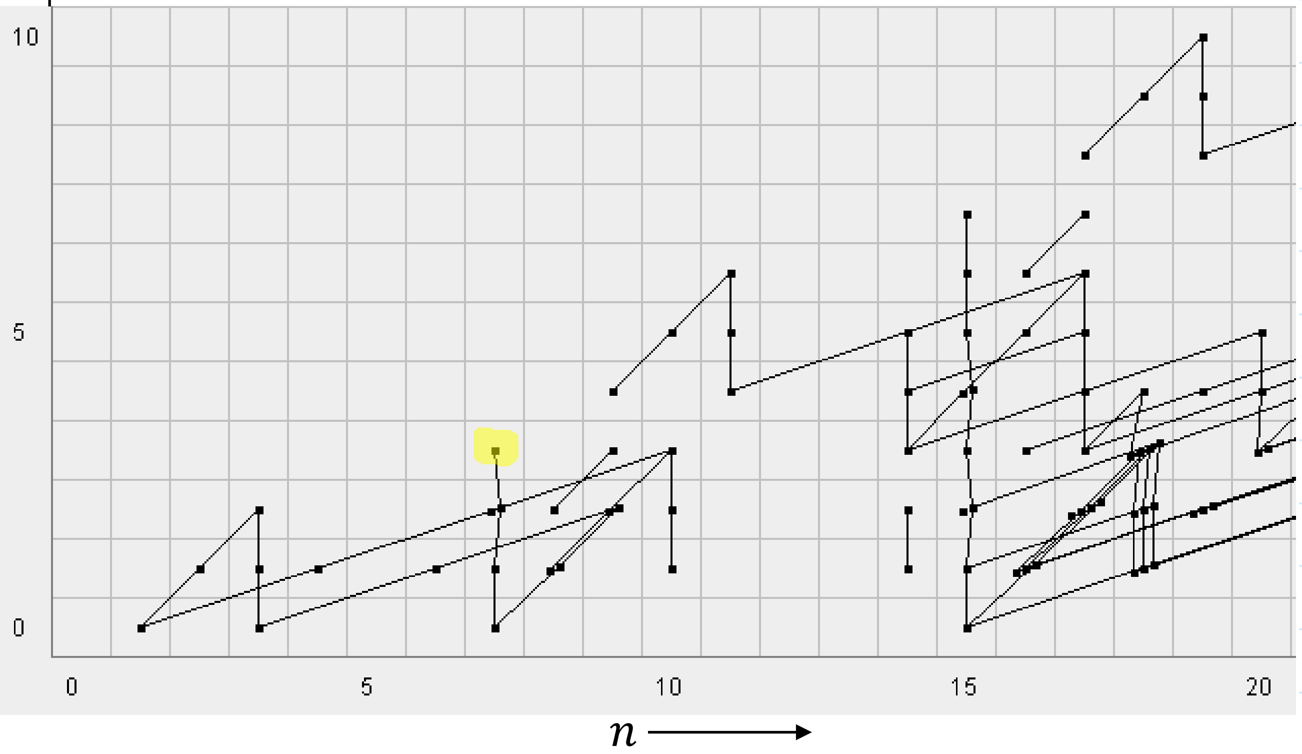
Signal:



Examples of homology events, signals

$f: S^n \to \mathbb{R}P^{16}$

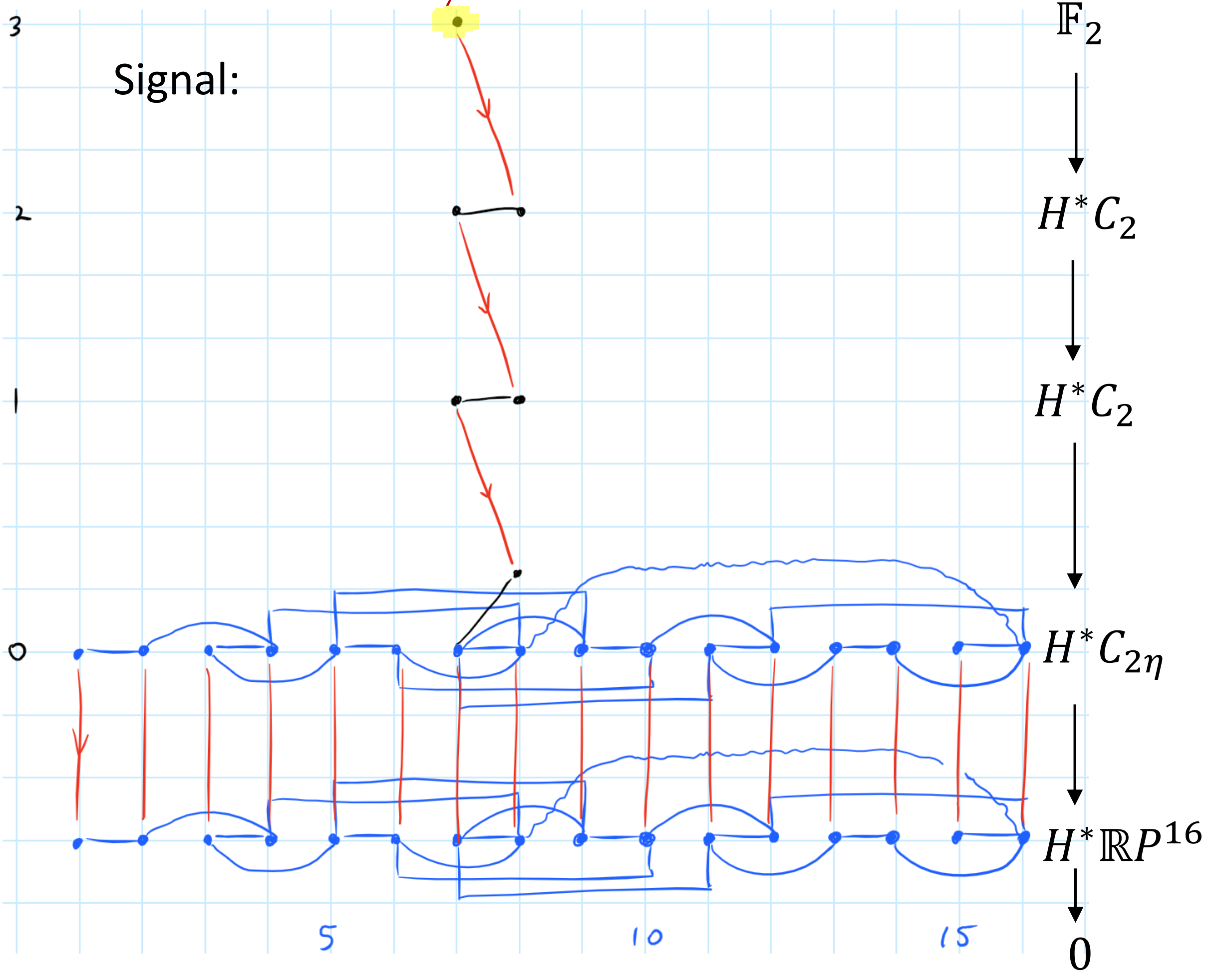
$[f] \in Ext^s_{\mathcal{A}}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



Event:

$S^7 \xrightarrow{2} S^7 \xrightarrow{2} S^7 \xrightarrow{2\iota_7} \mathbb{R}P^{16}$

Signal:



Two problems

- "noise":

  $f: Y \to X$ could be null homotopic, and yet produce a nonzero signal
  $$0 \neq [f] \in Ext_{\mathcal{A}}^{s}(H^{*+s}X, H^{*}Y)$$

- "physically impossible signals":

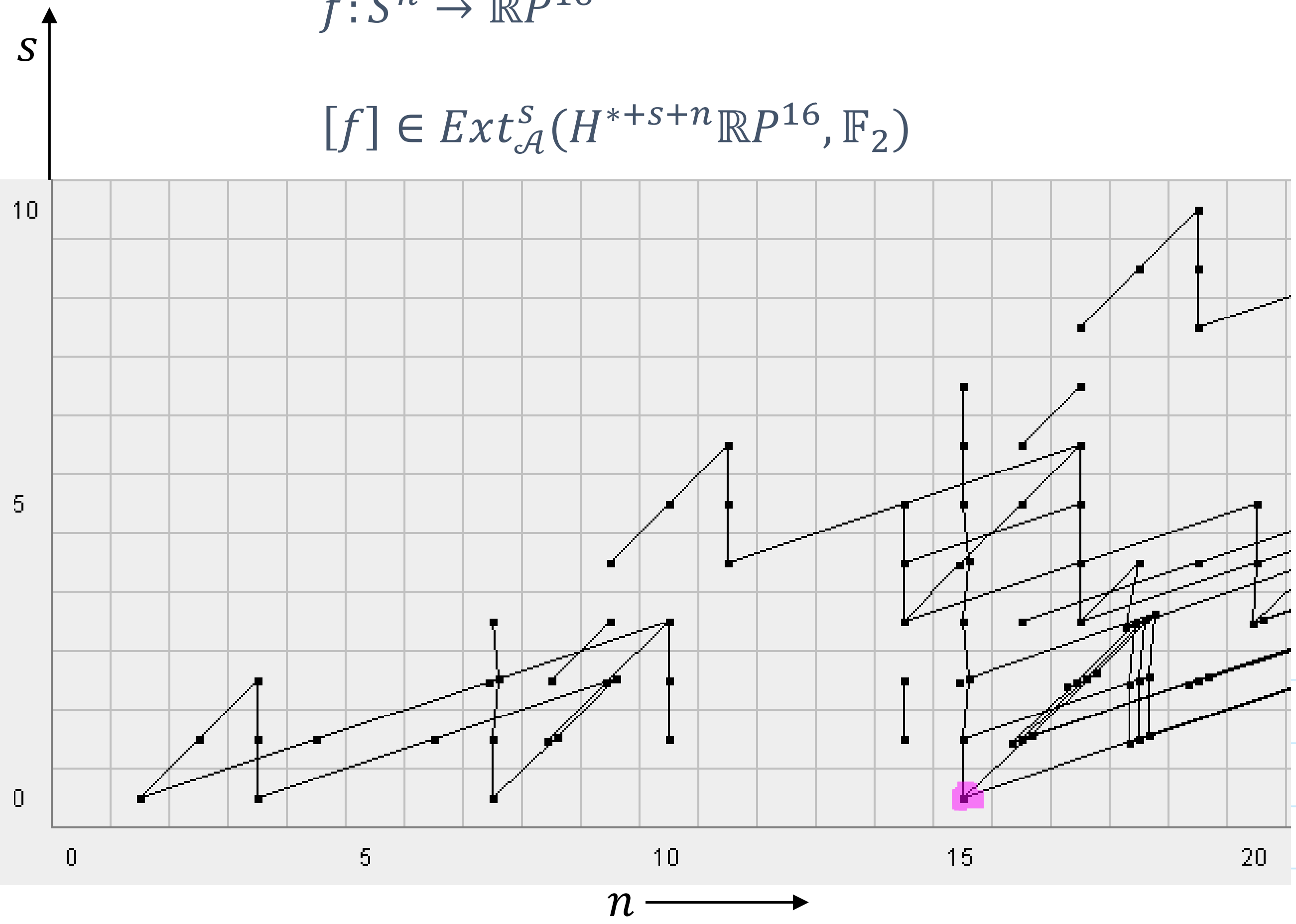
  For some signals
  $$x \in Ext_{\mathcal{A}}^{s}(H^{*+s}X, H^{*}Y)$$
  $x \neq [f]$ for any f

Adams differentials - "Noise cancellation"

Turns out you can use physically impossible signals to cancel noise!

$$f: S^n \to \mathbb{R}P^{16}$$

$$[f] \in Ext_{\mathcal{A}}^{s}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$$



Physically impossible event: there is no topological map

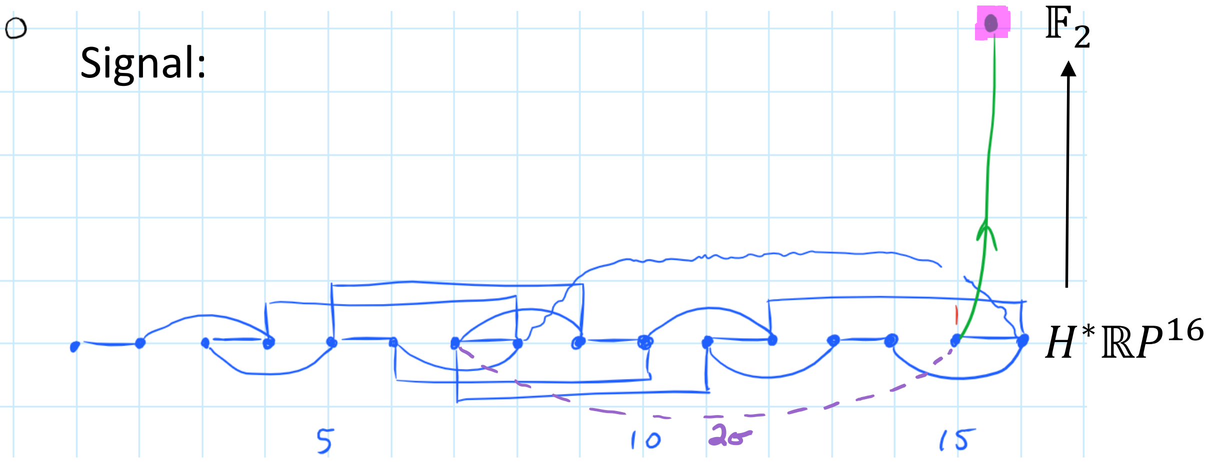
$$S^{15} \xrightarrow[\iota_{15}]{} \mathbb{R}P^{16}$$

Issue: 15-cell attaches to 7-cell with attaching map

$$2\iota_7 \circ \sigma: S^{14} \to S^7$$

invisible to Steenrod operations!

Signal:

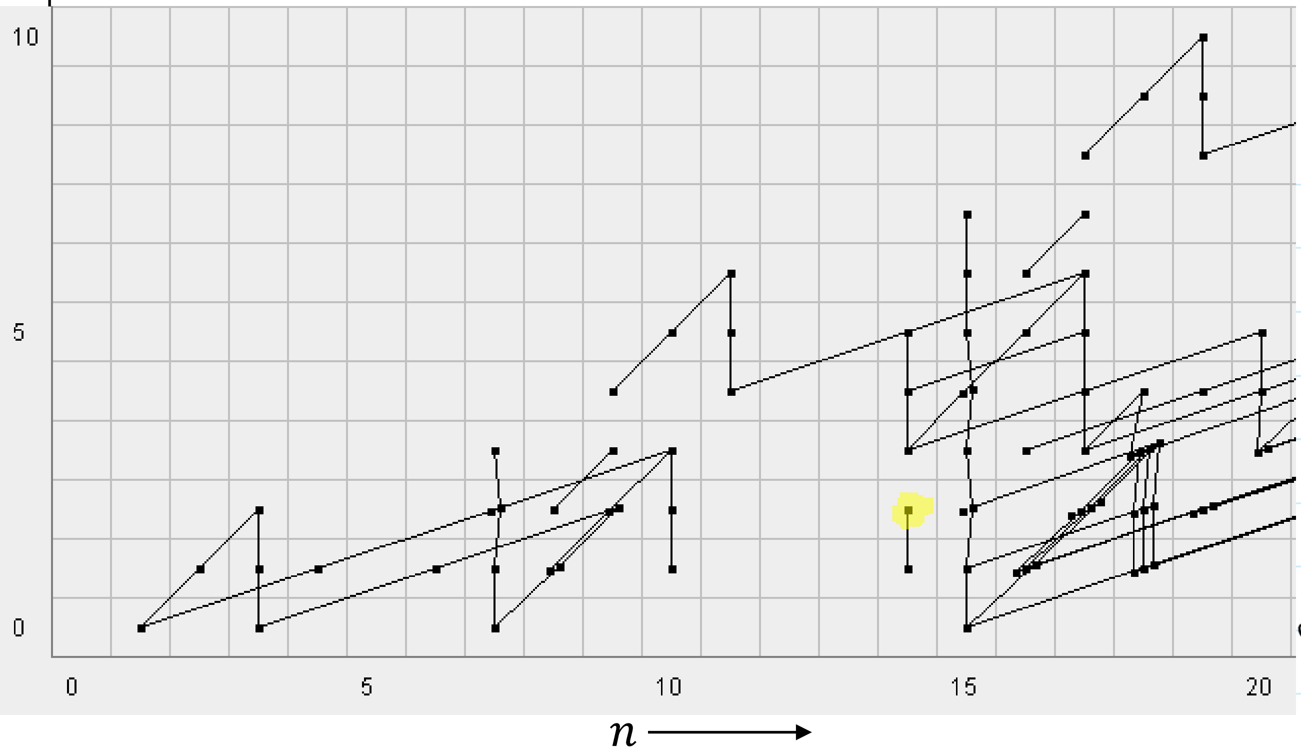


Adams differentials - “Noise cancellation”

Turns out you can use physically impossible signals to cancel noise!

$f: S^n \to \mathbb{R}P^{16}$

$[f] \in Ext^s_{\mathcal{A}}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



Noise: the event below gives a non-zero signal

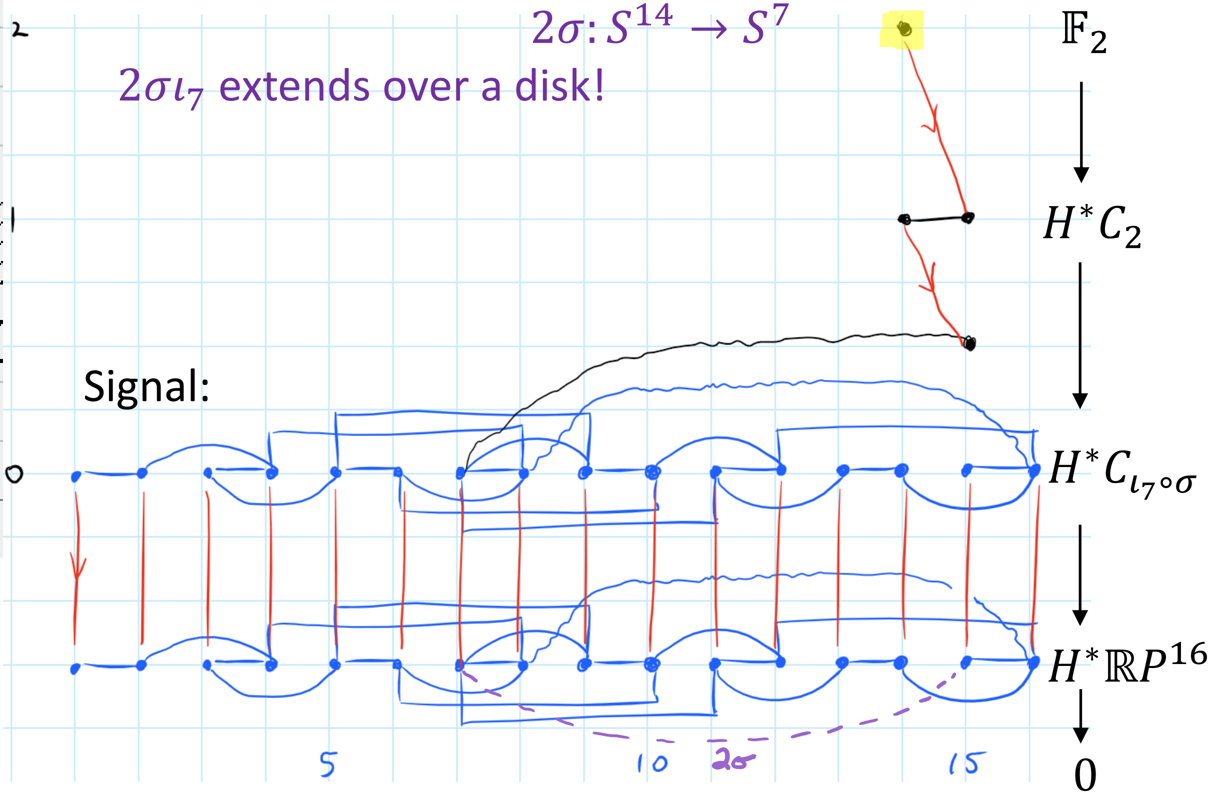
$$S^{14} \underset{2}{\to} S^{14} \xrightarrow[\iota_7 \circ \sigma]{} \mathbb{R}P^{16}$$

Issue: the element $2\sigma\iota_7 \in \pi^{st}_{14}(\mathbb{R}^{16})$ is null homotopic- since the 15-cell attaches to 7-cell with attaching map

$$2\sigma: S^{14} \to S^7$$

$2\sigma\iota_7$ extends over a disk!

Signal:

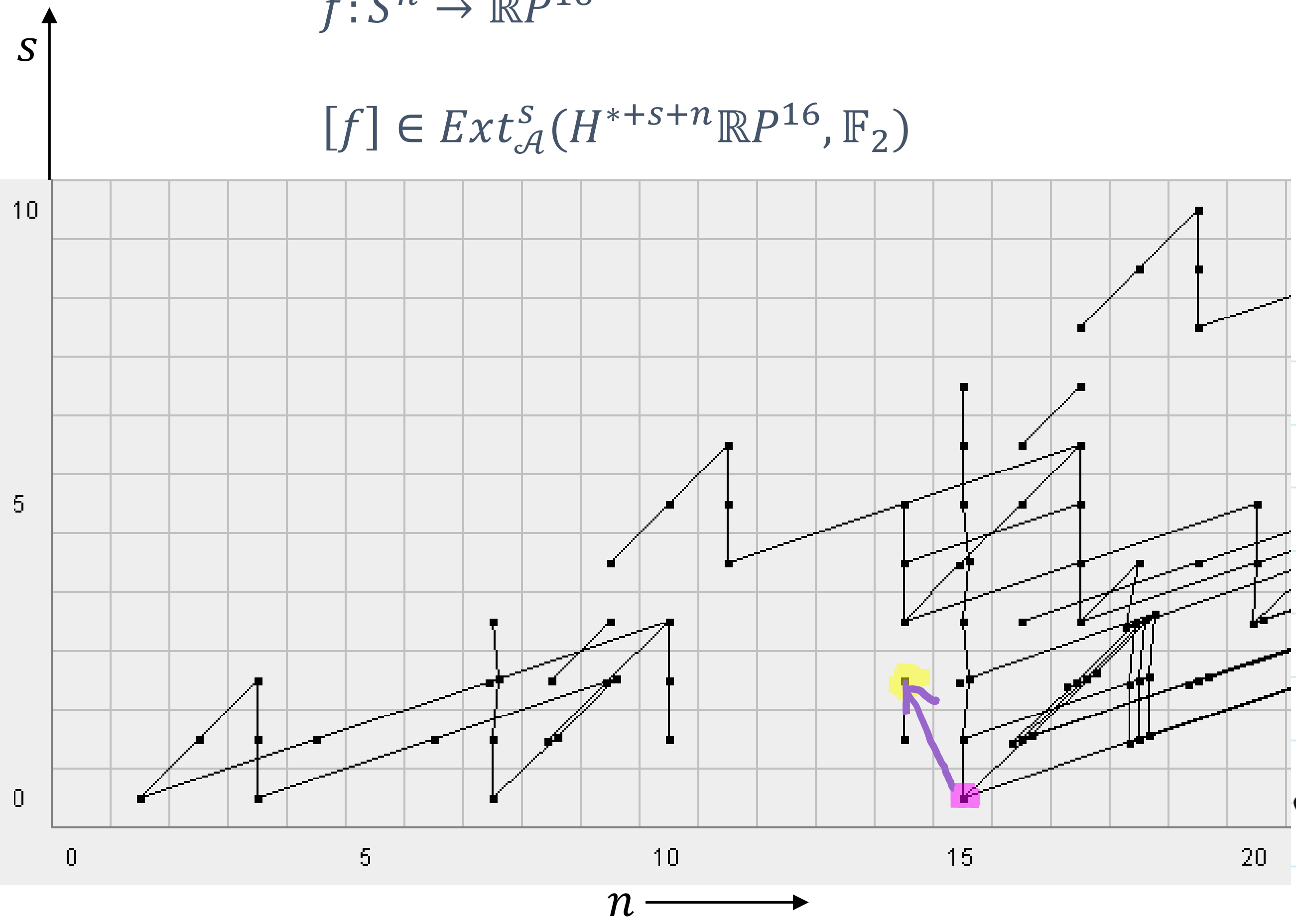


Adams differentials - "Noise cancellation"

Turns out you can use physically impossible signals to cancel noise!

$f: S^n \to \mathbb{R}P^{16}$

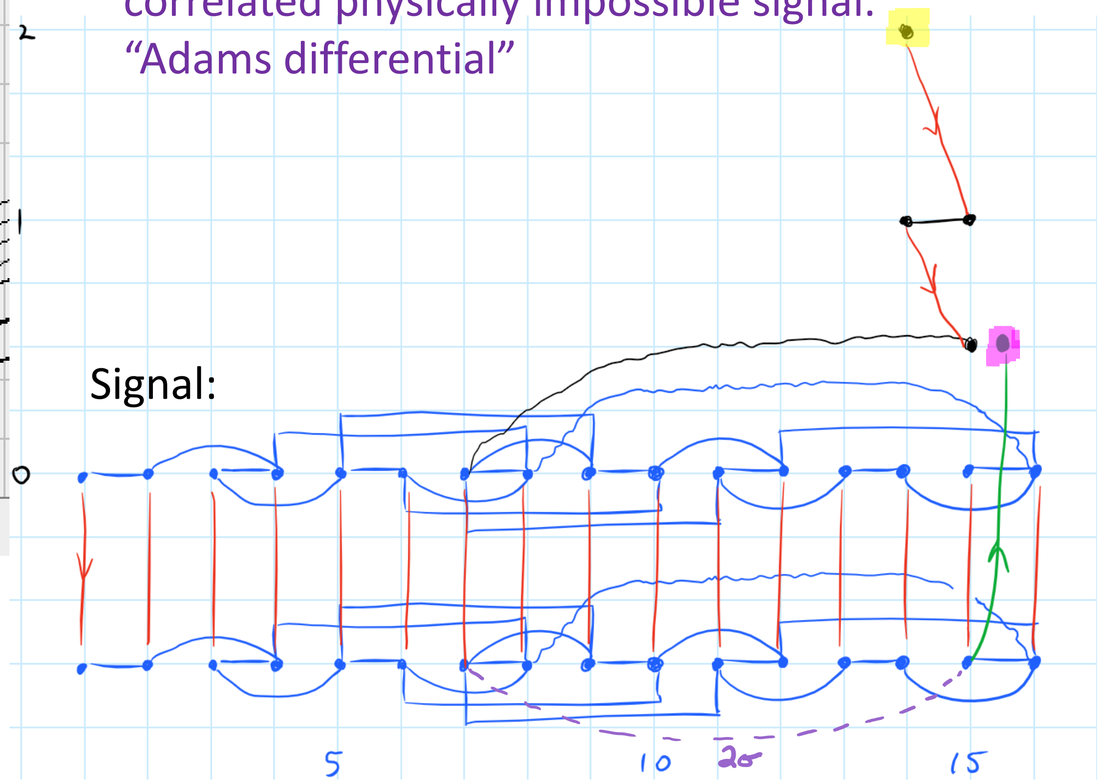
$[f] \in Ext^s_{\mathcal{A}}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$



The "invisible" attaching map $2\sigma: S^{14} \to S^7$ simultaneously creates a physically impossible signal a noise signal.

Noise cancellation – cancel the noise with the correlated physically impossible signal.
"Adams differential"

Signal:



Adams differentials - “Noise cancellation”

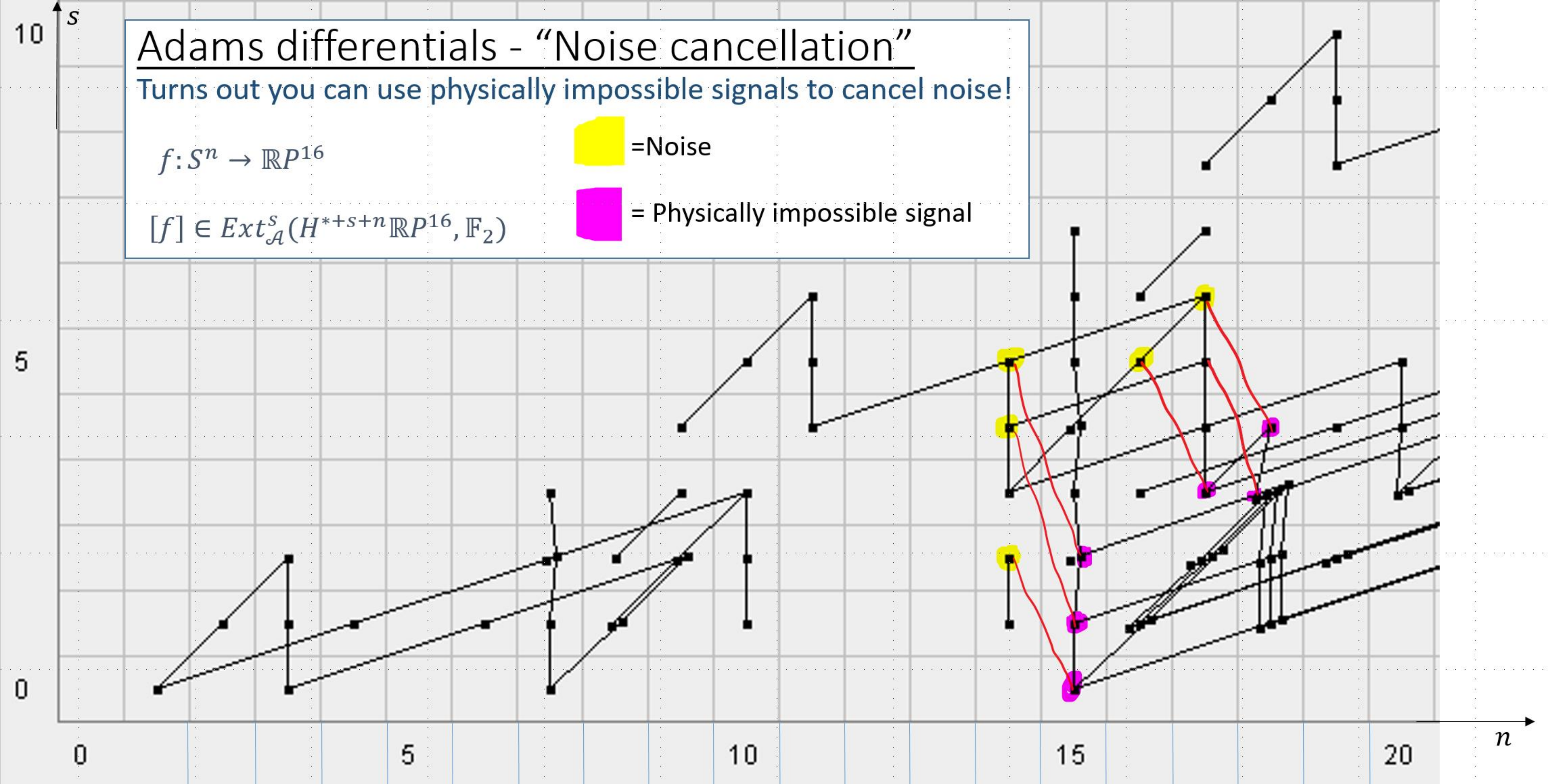
Turns out you can use physically impossible signals to cancel noise!

$f: S^n \to \mathbb{R}P^{16}$

$[f] \in Ext^s_{\mathcal{A}}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$

=Noise

= Physically impossible signal



Adams differentials – “noise cancellation”

Turns out, there are differentials (Adams spectral sequence)

$$d_r : Ext^s_{\mathcal{A}}(H^{*+s+n}X, \mathbb{F}_2) \to Ext^{s+r}_{\mathcal{A}}(H^{*+s+r+n-1}X, \mathbb{F}_2)$$

$$d_r(impossible\ signal) = noise$$

Theorem (Adams)

$$H^*(Ext^*_{\mathcal{A}}(H^*X, \mathbb{F}_2), \{d_r\}) \cong \pi^{st}_*(X)^\wedge_2$$ [2-completion = “2-torsion”]

[Isomorphism of sets]

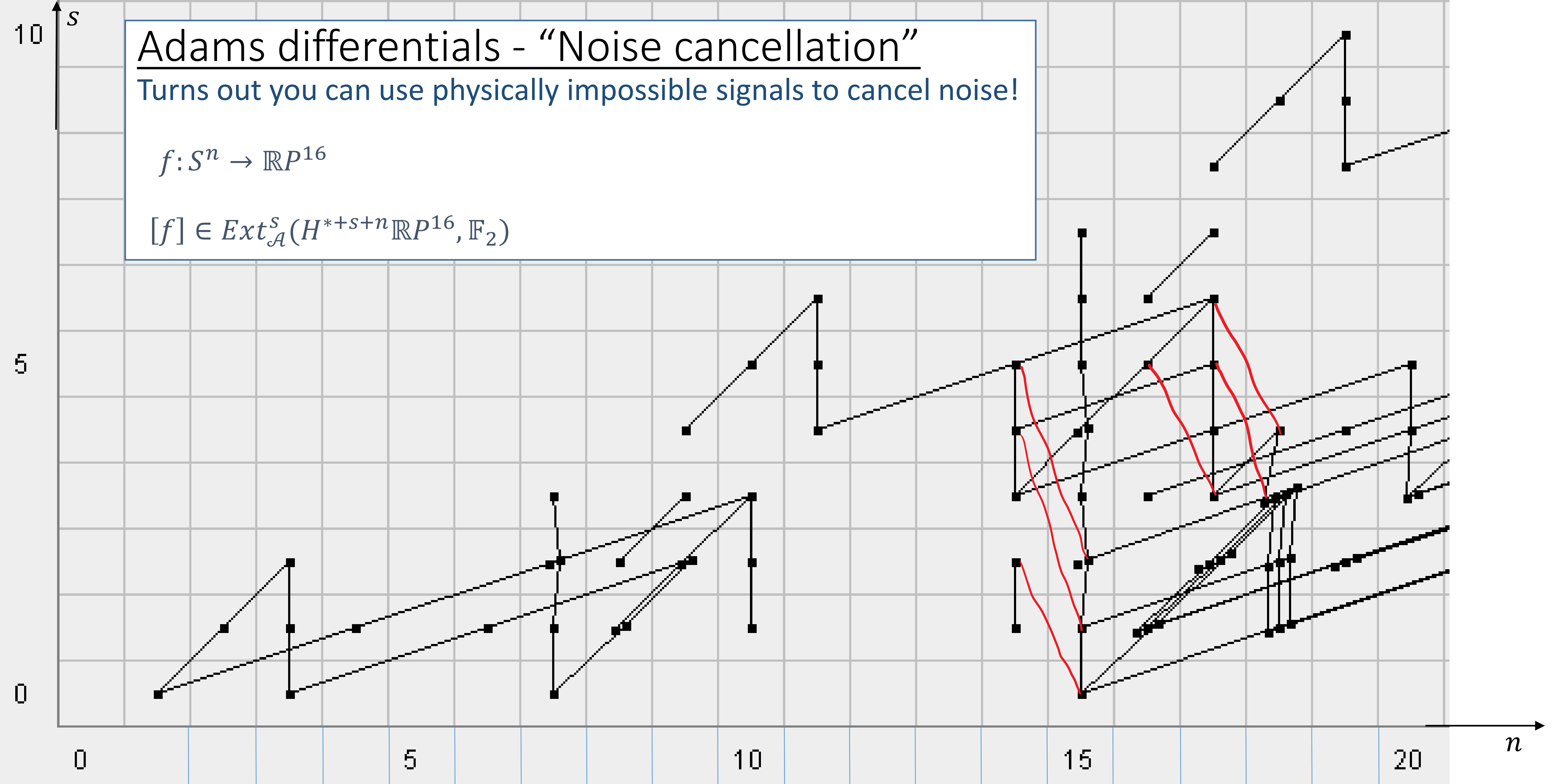
(version for $H^*(-; \mathbb{F}_p) \Rightarrow$ p-completion)

Adams differentials - "Noise cancellation"

Turns out you can use physically impossible signals to cancel noise!

$f: S^n \to \mathbb{R}P^{16}$

$[f] \in Ext^s_{\mathcal{A}}(H^{*+s+n}\mathbb{R}P^{16}, \mathbb{F}_2)$

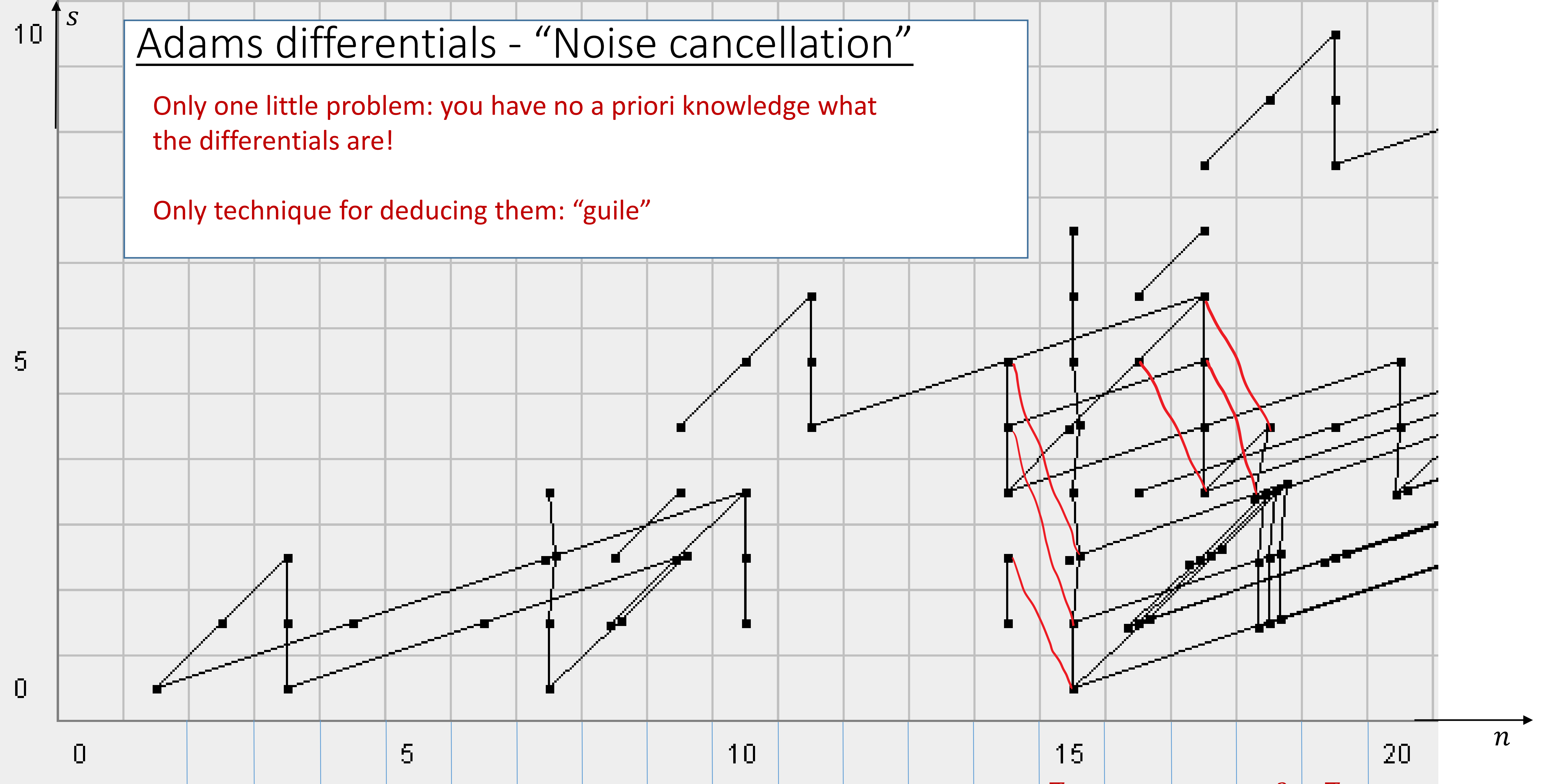


$\pi_*^{st}(\mathbb{R}P^{16})$: $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_8$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_{16} \times \mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_8 \times \mathbb{Z}_2$ | $\mathbb{Z}_8$ | 0 | 0 | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{32} \times \mathbb{Z}_2^2$ | $\mathbb{Z}_2^5$ | $\mathbb{Z}_2^7$ | $\mathbb{Z}_8^3 \times \mathbb{Z}_2$ | $\mathbb{Z}_8 \times \mathbb{Z}_2^4$ | $\mathbb{Z}_8 \times \mathbb{Z}_2$

Adams differentials - “Noise cancellation”

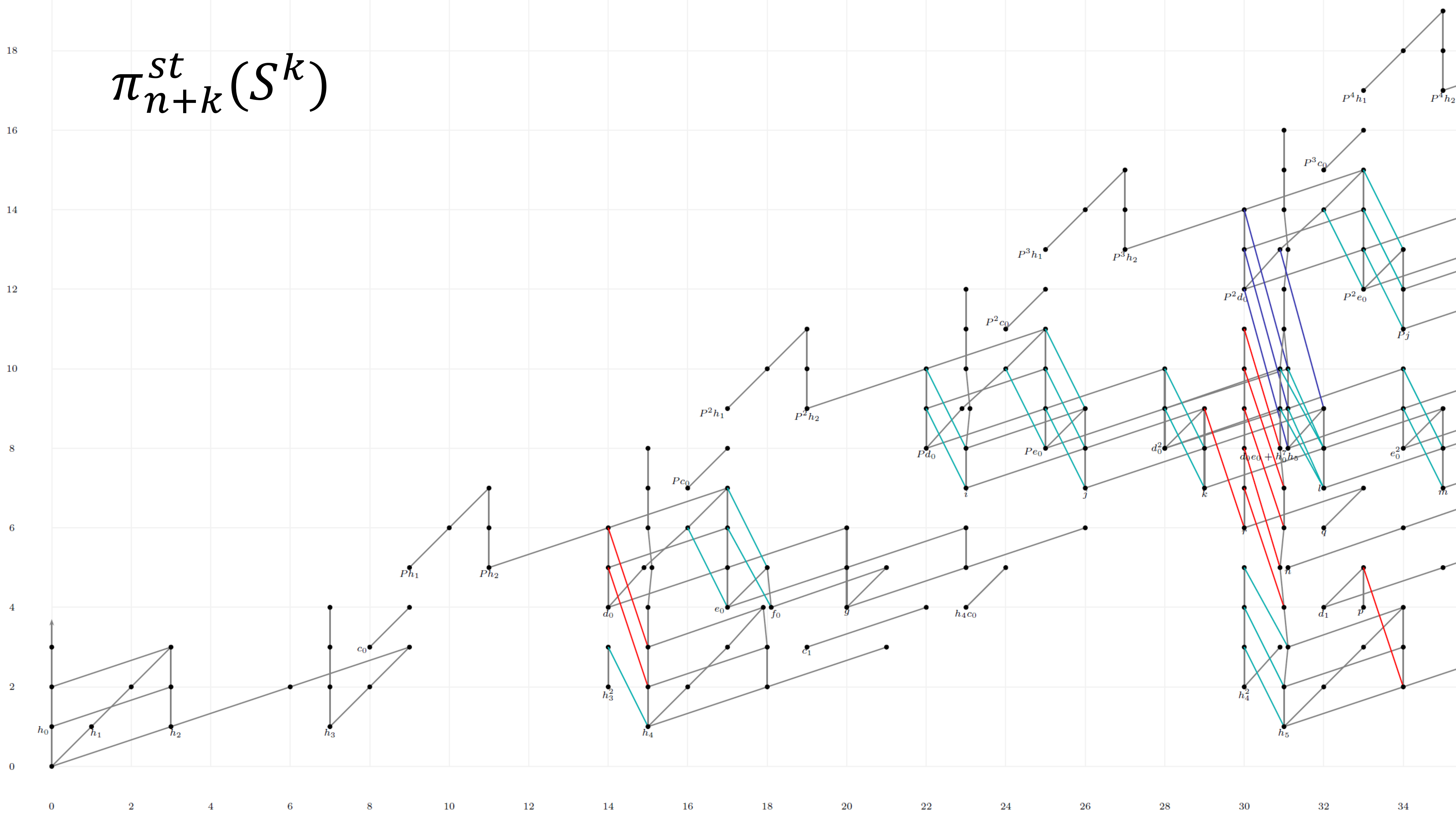
Only one little problem: you have no a priori knowledge what the differentials are!

Only technique for deducing them: “guile”

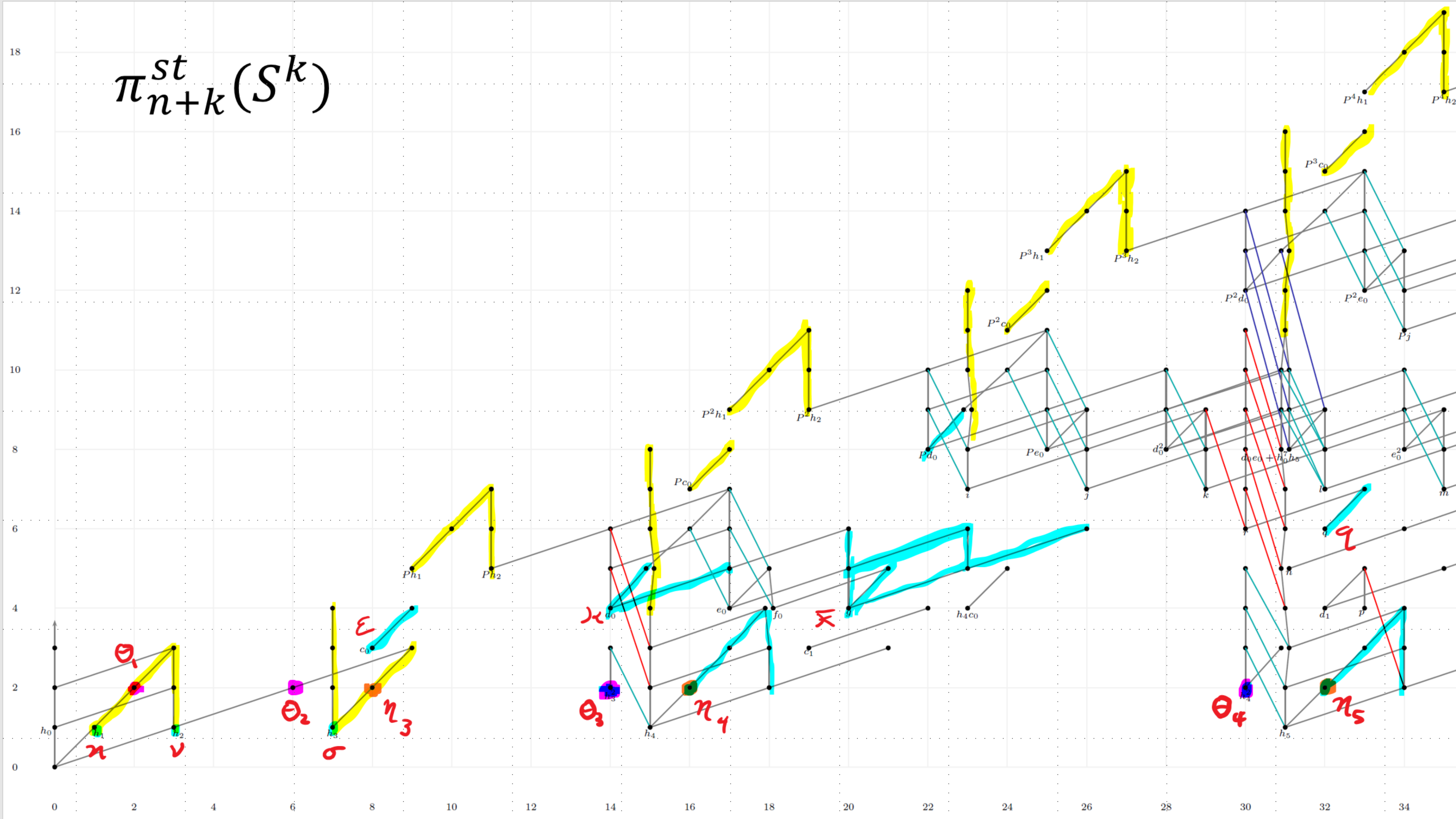


$\pi_*^{st}(\mathbb{R}P^{16})$:																			
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16} \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_8$	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_{32} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^7$	$\mathbb{Z}_8^3 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$

$\pi^{st}_{n+k}(S^k)$



h_0 h_1 h_2 h_3 c_0 Ph_1 Ph_2 h_3^2 h_4 d_0 e_0 f_0 Pc_0 c_1 g P^2h_1 P^2h_2 h_4c_0 Pd_0 P^2c_0 Pe_0 i j P^3h_1 P^3h_2 d_0^2 k h_4^2 r $d_0e_0+h_0^7h_5$ P^2d_0 h_5 n l d_1 q p P^3c_0 P^2e_0 e_0^2 Pj m P^4h_1 P^4h_2
0 2 4 6 8 10 12 14 16 18
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34



$\pi^{st}_{n+k}(S^k)$
η
ν
σ
ε
κ
κ̄
q
θ1
θ2
θ3
θ4
η3
η4
η5
h_0
h_1
h_2
h_3
h_4
h_5
c_0
c_1
d_0
e_0
f_0
g
h_4c_0
d_1
p
Ph_1
Ph_2
Pc_0
Pd_0
Pe_0
P^2h_1
P^2h_2
P^2c_0
P^2d_0
P^2e_0
P^3h_1
P^3h_2
P^3c_0
P^4h_1
P^4h_2
Pj
d_0^2
e_0^2
i
j
k
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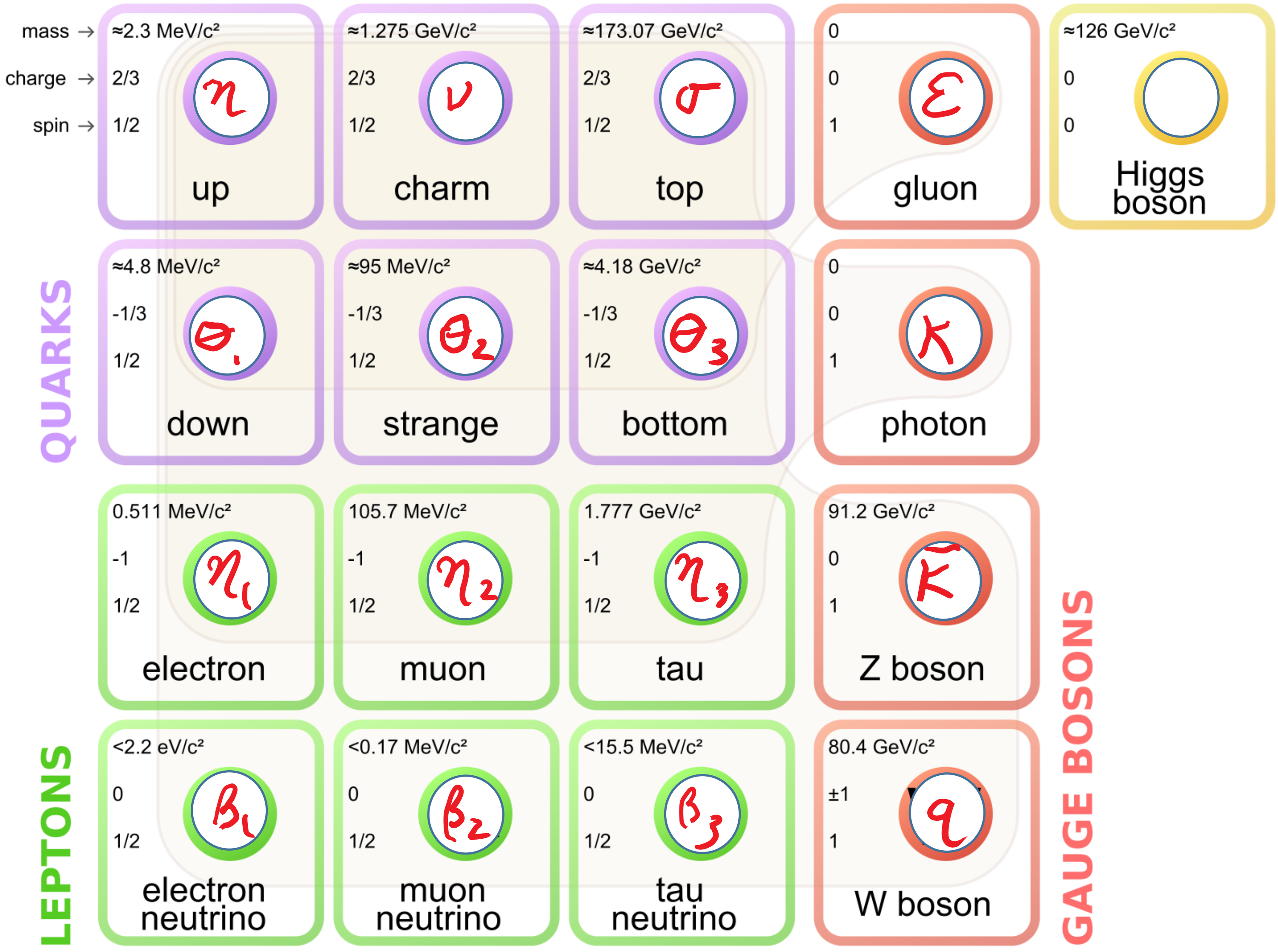
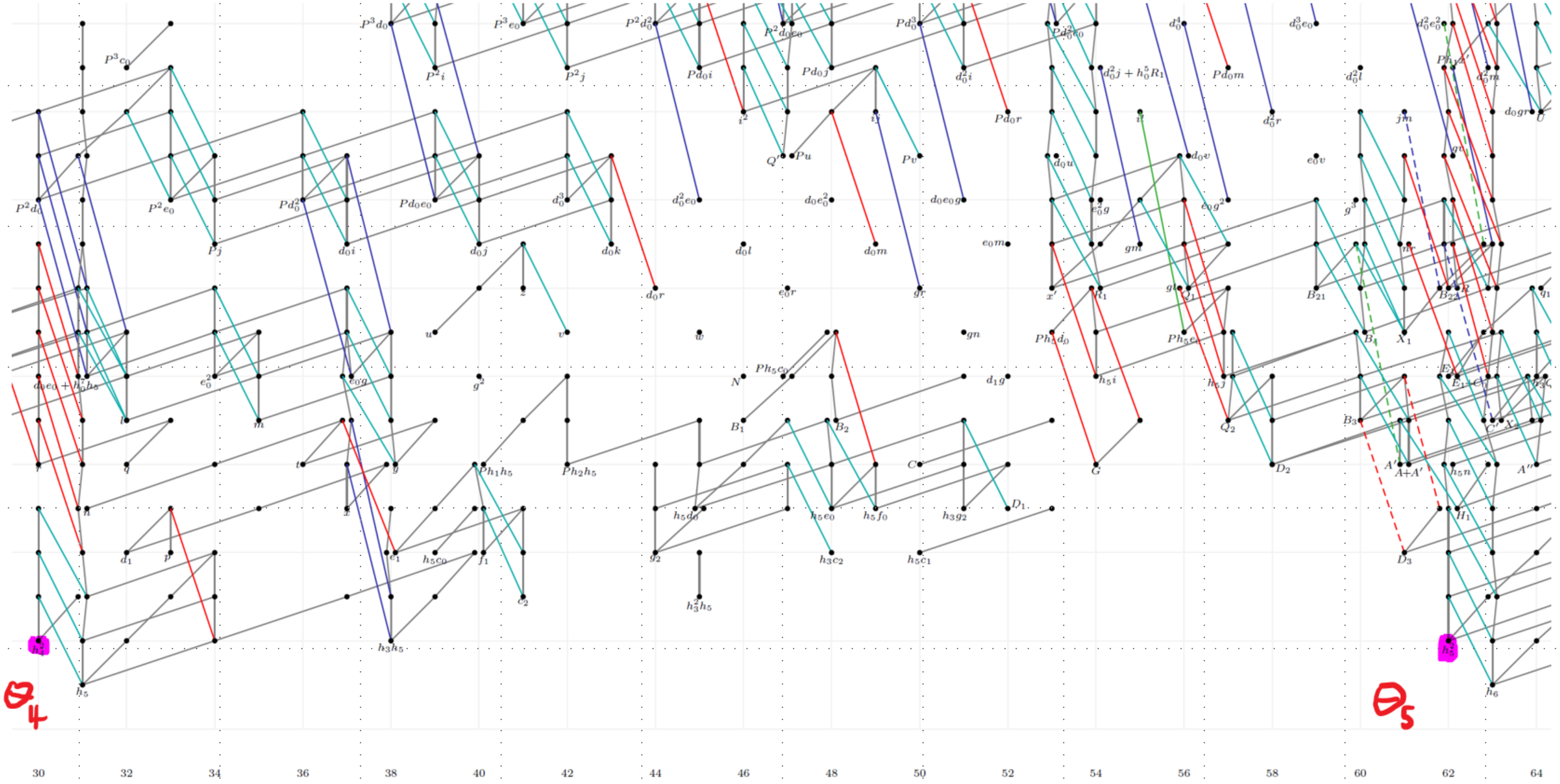
r

Elementary particles of homotopy theory:

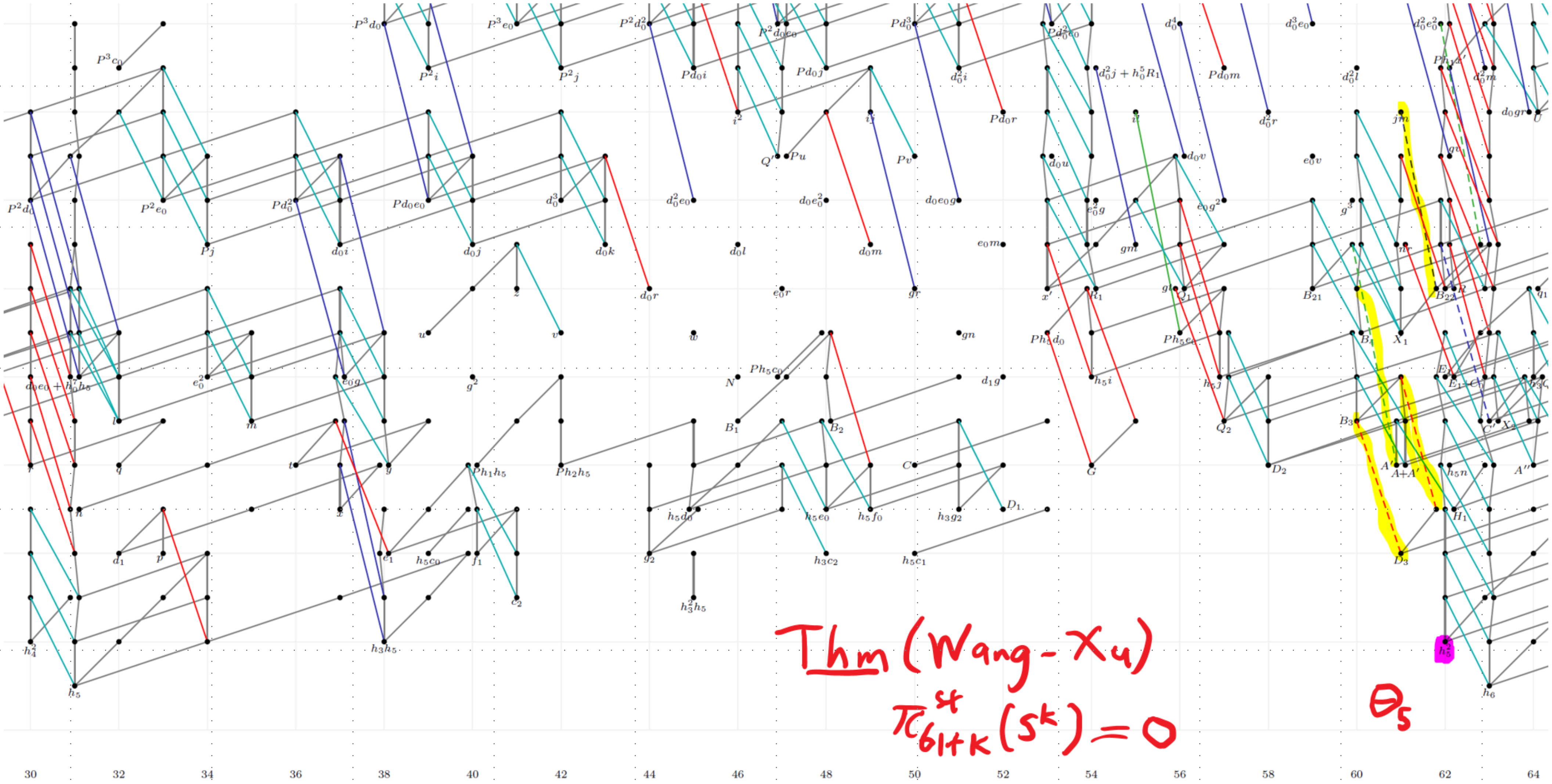


[Wikipedia commons]

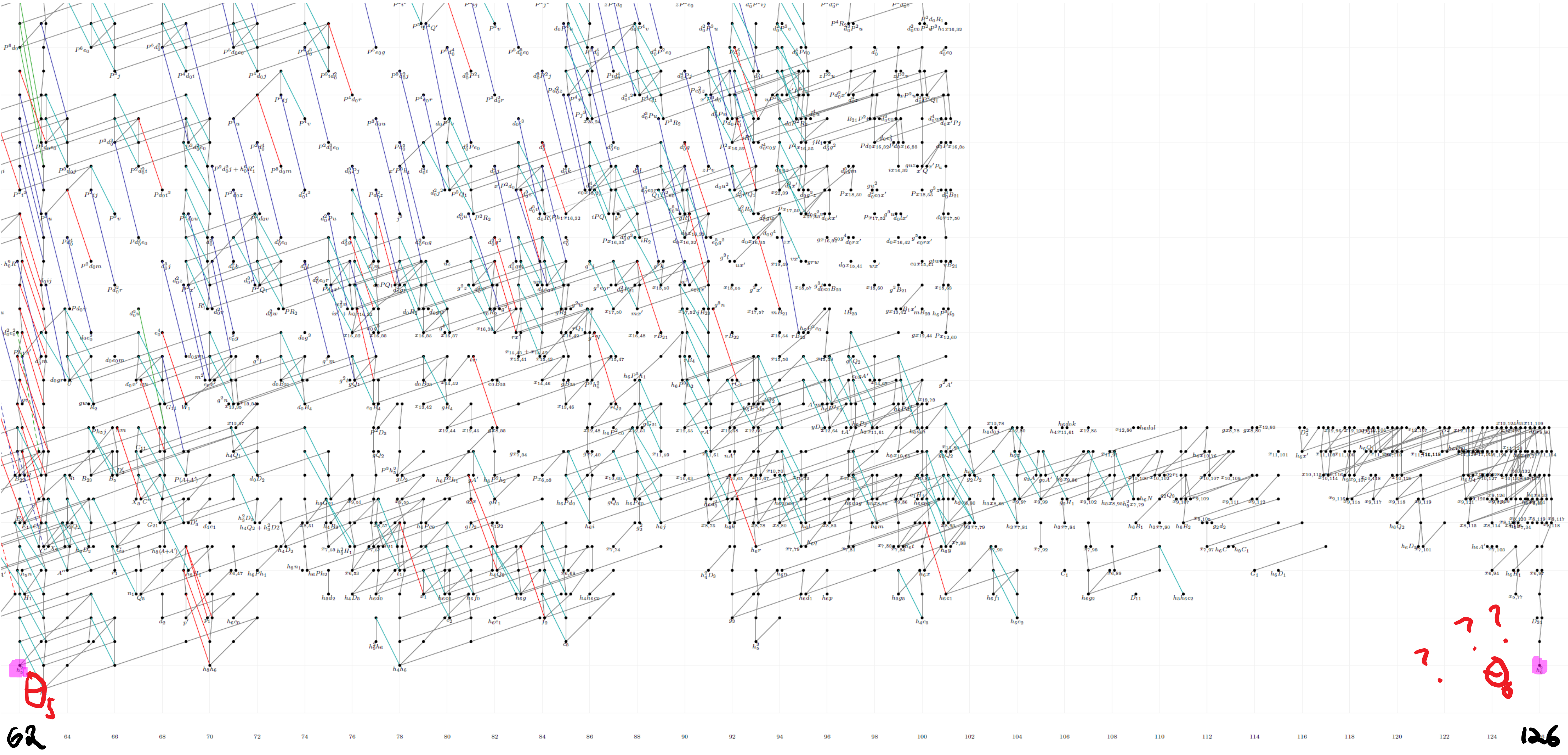
Higher energies



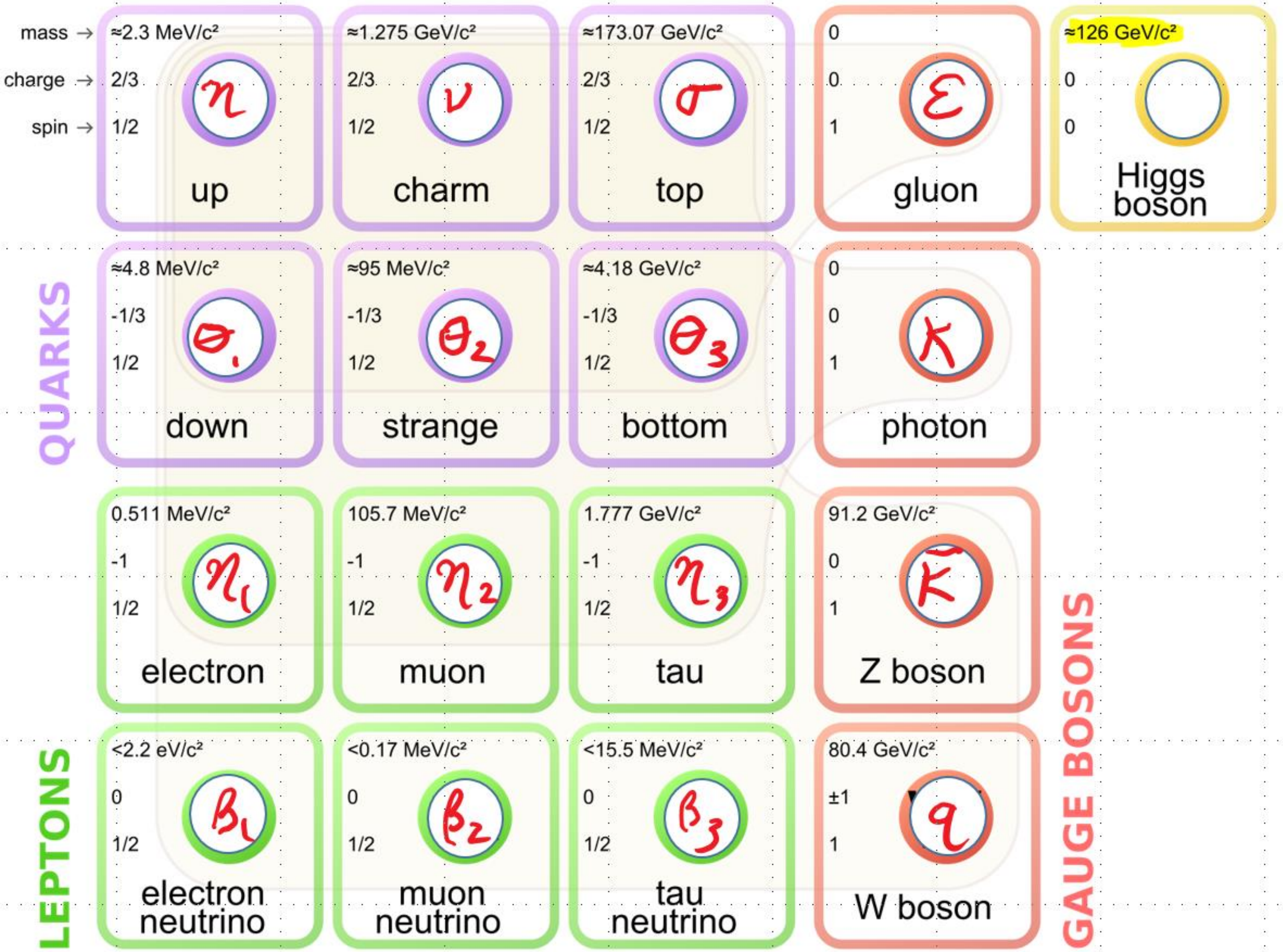
Higher energies



Higher energies

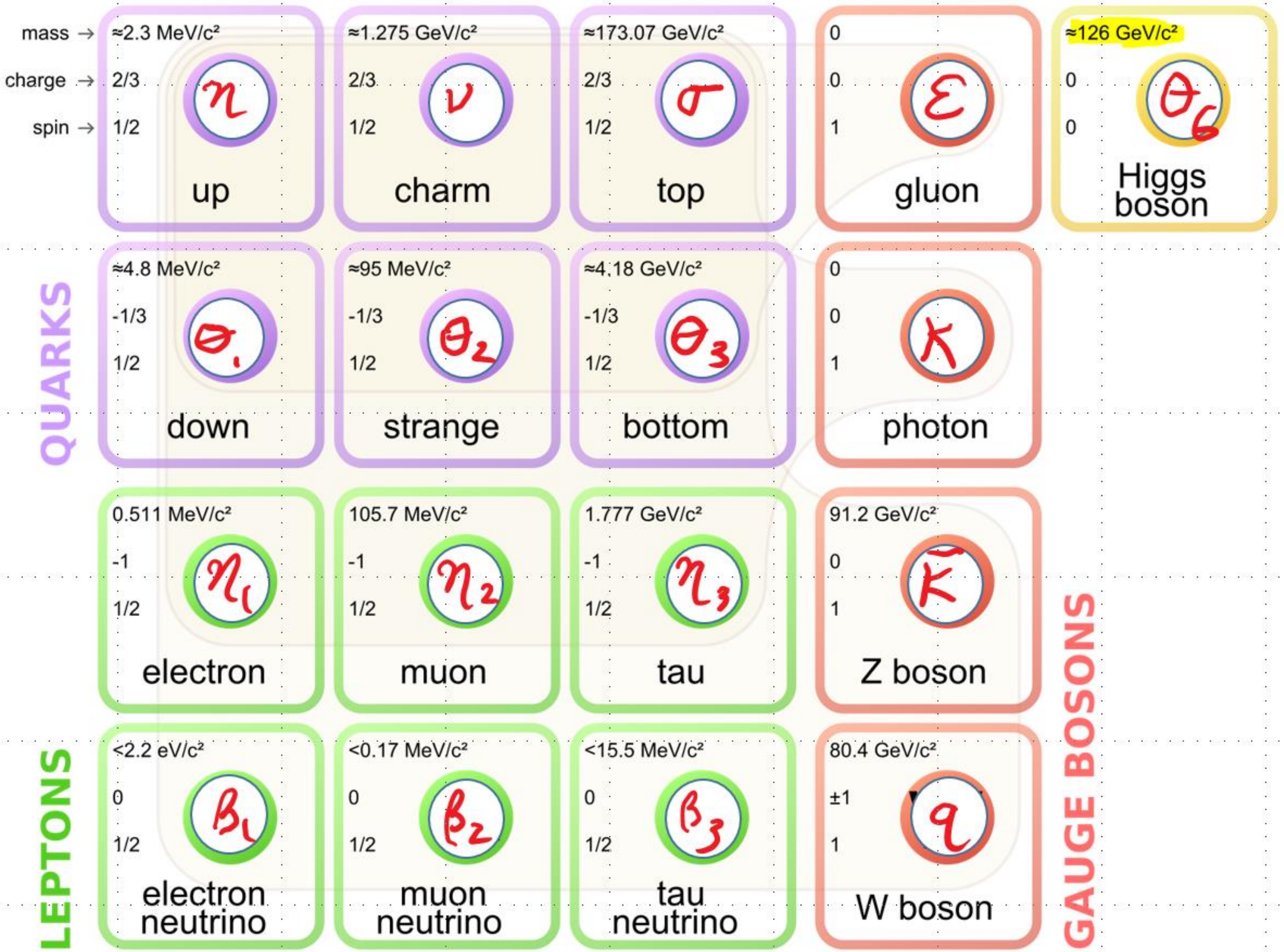


Elementary particles of homotopy theory:



[Wikipedia commons]

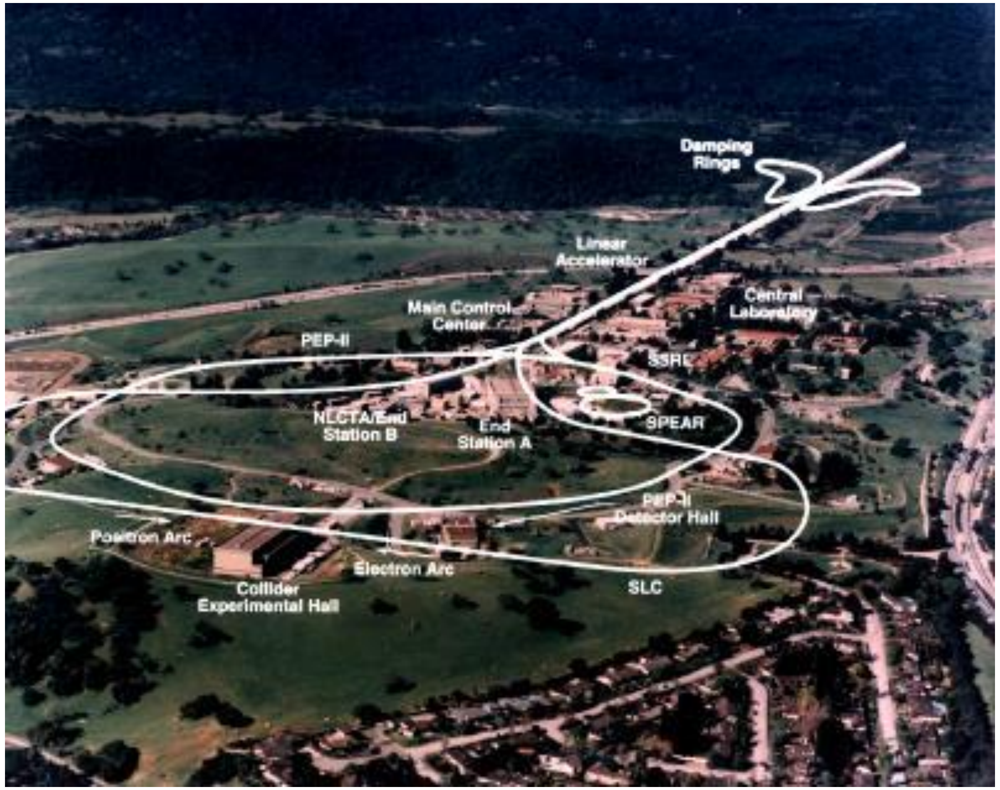
Elementary particles of homotopy theory:

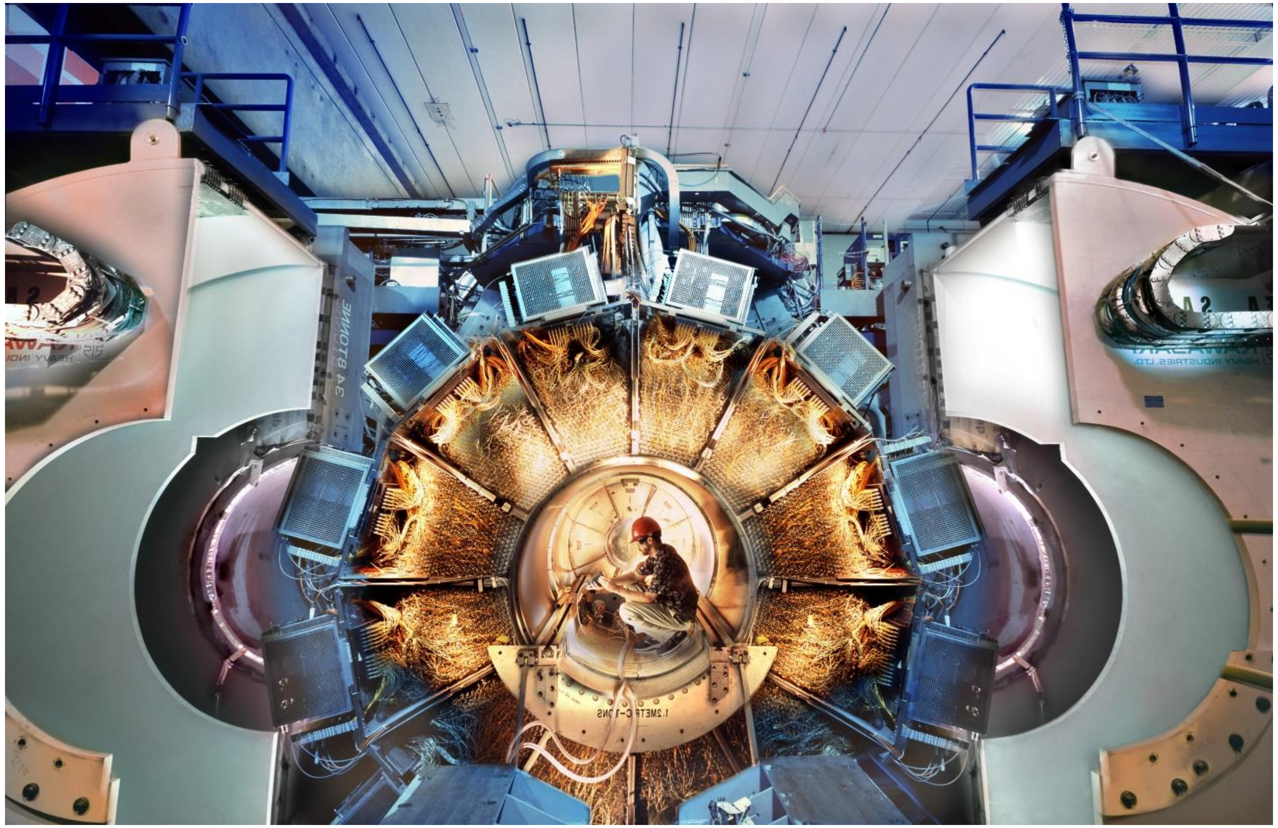


[Wikipedia commons]

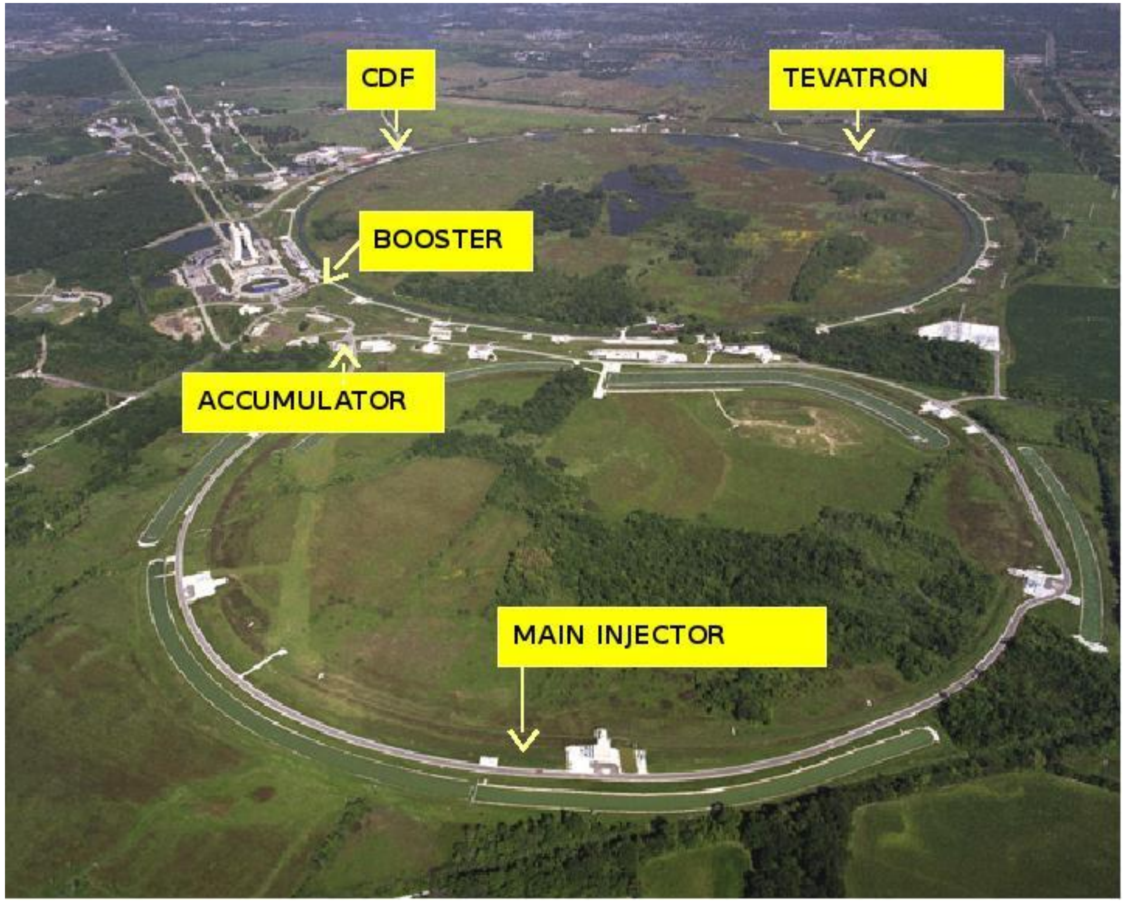
Higher energies require fancier detectors

SLAC

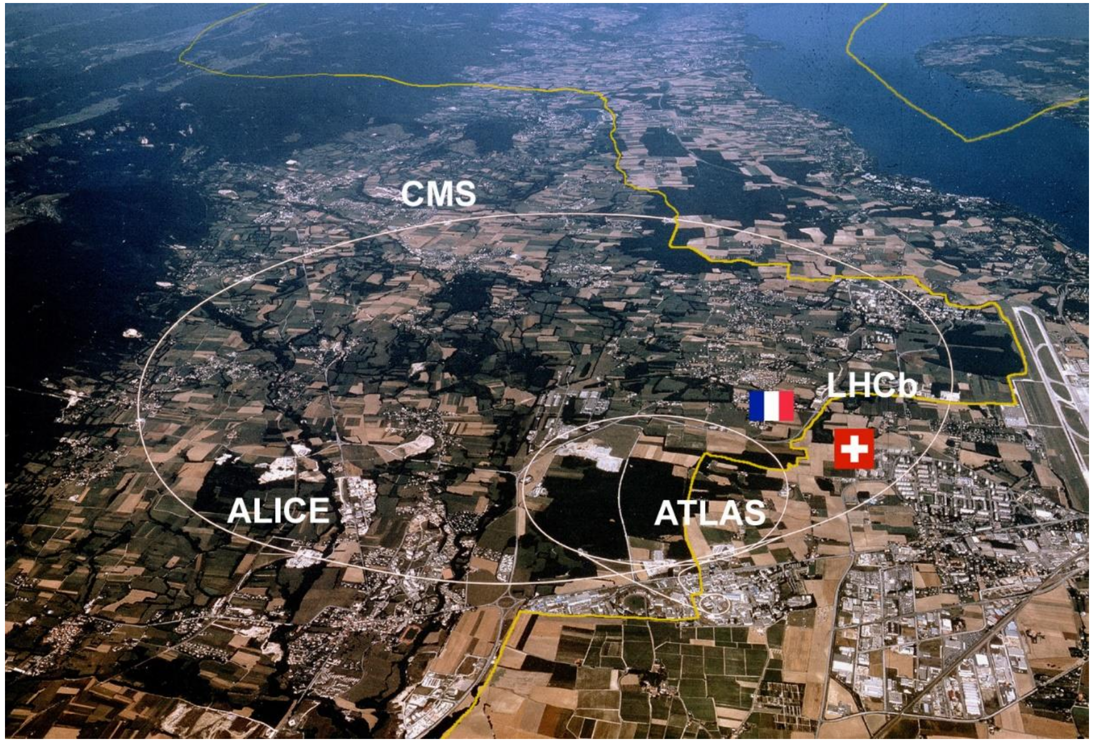


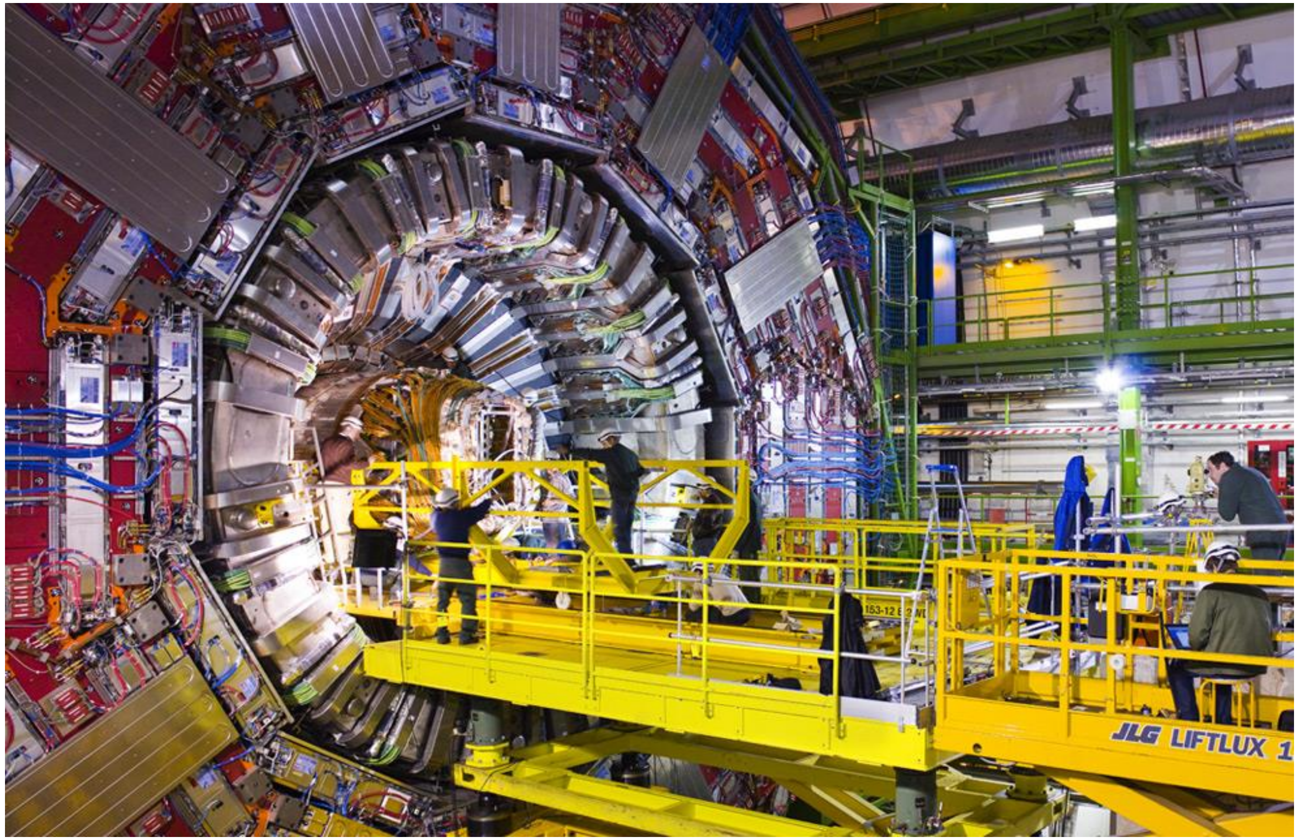


Tevatron



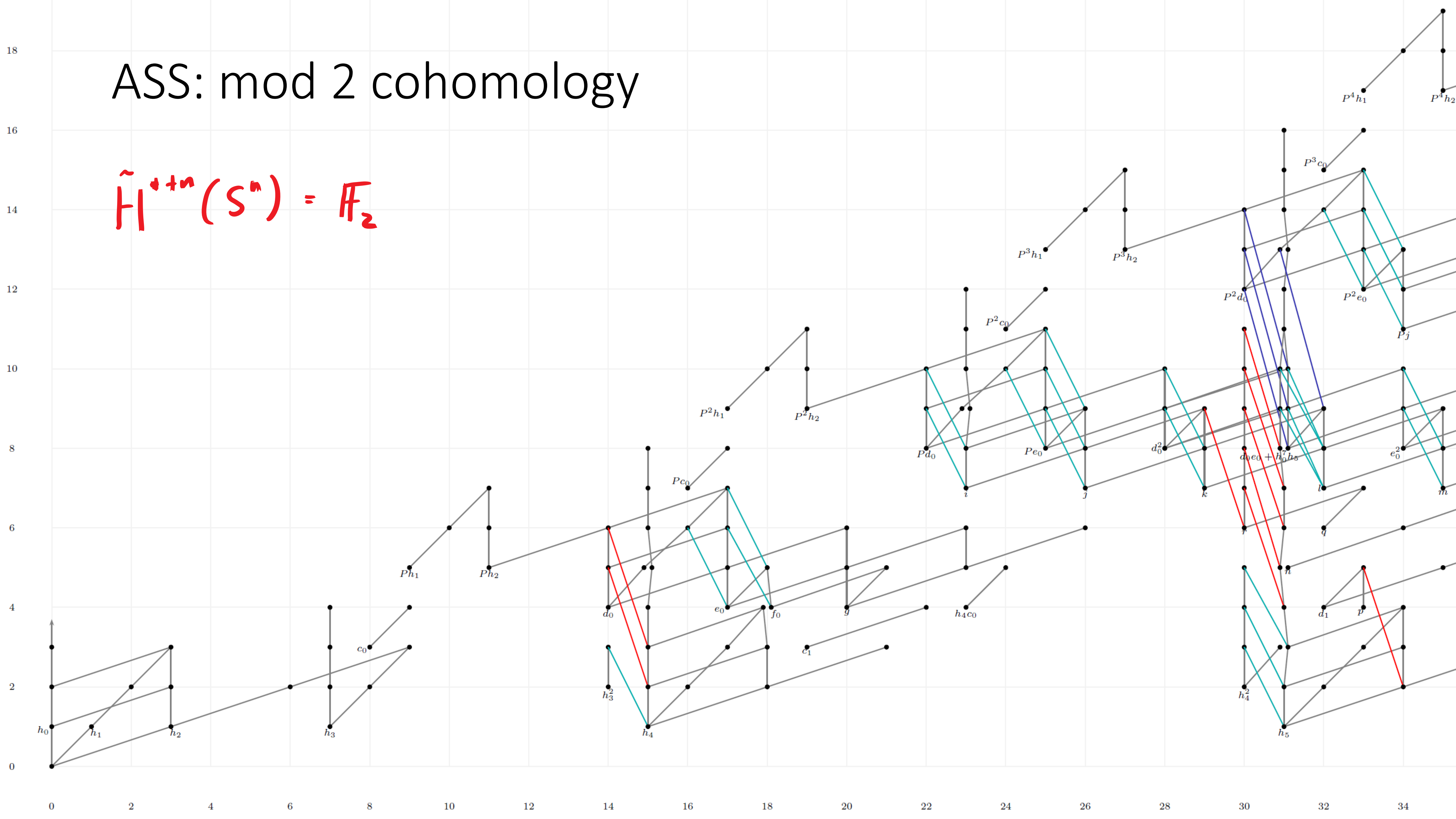
LHC





ASS: mod 2 cohomology

$\tilde{H}^{*+n}(S^n) = \mathbb{F}_2$



Real –K-theory (ko)

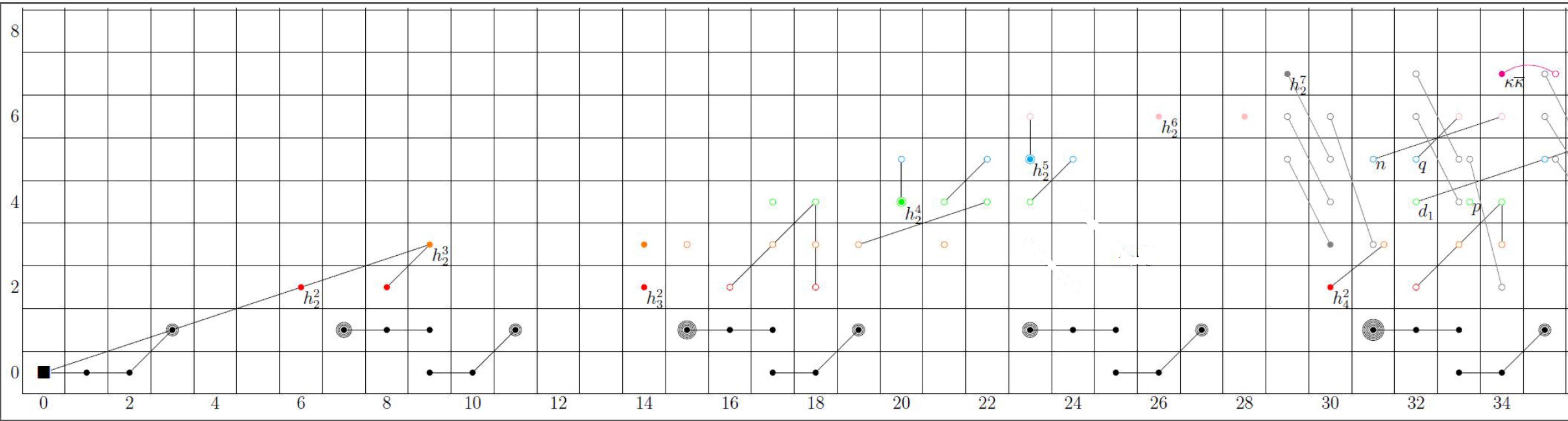
[Lellmann-Mahowald]
[Beaudry-B-Bhattacharya-Culver-Xu]

$ko^{*+n}(S^n) =$

$\mathbb{Z}$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ 0 $\mathbb{Z}$ ...

(8-periodic)

$Ext_{\mathcal{A}_{bo}}(ko^*(S^k), ko^*(S^{k+n}))$



Topological modular forms (tmf)

[Mahowald] – started thinking about the tmf-ASS
[B-Ormsby-Stojanoska-Stapleton] - $\mathcal{A}_{tmf}$
[Beaudry-B-Bhattacharya-Culver-Xu]
– computing $Ext_{\mathcal{A}_{tmf}}$ [work in progress]

$tmf^{*+n}(S^n)$:
192-periodic