Morava E-theory of the Goodwillie tower

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Goal: Compute $v_n^{-1}\pi_* S^q$ q odd

(unstable v_n-periodic homotopy gps of spheres)

[n=1 Mahowald $p=2$

Mahowald-Thompson $p>2$]

Goodwillie tower of Id

$$\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
D_3(Id)(S^2) & \longrightarrow & P_3(Id)(S^2) \\
 & & \downarrow \\
D_2(Id)(S^2) & \longrightarrow & P_2(Id)(S^2) \\
 & & \downarrow \\
D_1(Id)(S^2) & = & P_1(Id)(S^2)
\end{array}$$

[Goodwillie tower

$Id : Top_* \to Top_*$]

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- $\mathbb{D}_i(Id)(X) = \partial_i(Id) \wedge_{h\Sigma_i} X^{\wedge i}$

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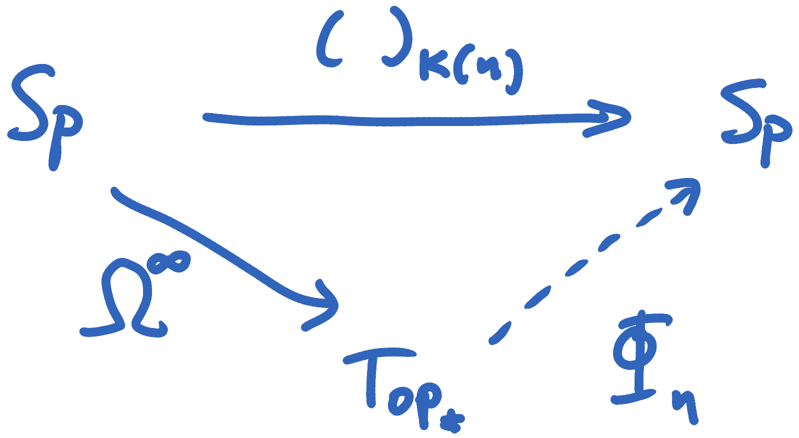
- $\mathbb{D}_i(Id)(X) = \partial_i(Id) \wedge_{h\Sigma_i} X^{\wedge i}$
- $\varprojlim P_i(Id)(S^2) \simeq S^2$

[Goodwillie tower
$Id : Top_* \to Top_*$]

Bousfield-Kuhn functor



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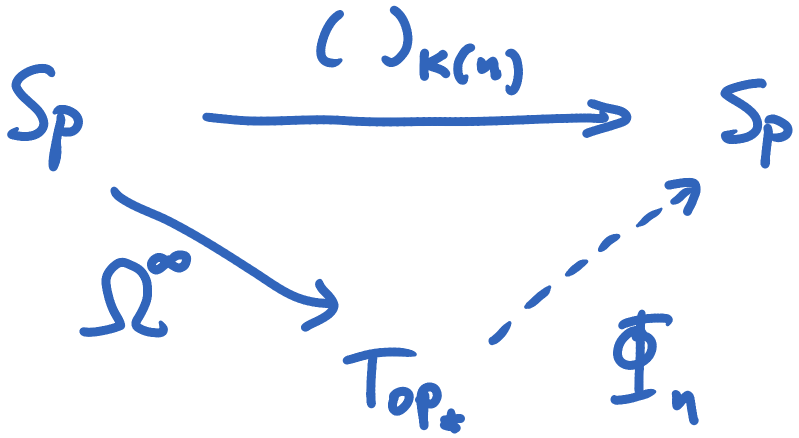
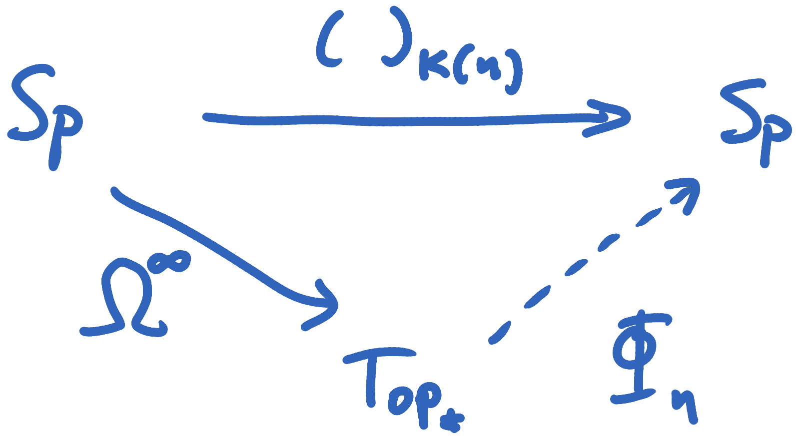


- Φ_n preserves fiber sequences

Bousfield-Kuhn functor



- Φ_n preserves fiber sequences
- $\pi_* \Phi_n(X) = v_n^{-1} \pi_* X$

v_n -periodic Goodwillie tower

Arone-Mahowald

- $\mathbb{D}_n(Id)(S^q)_p \simeq *$ unless $n = p^k$

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Consequence [Kuhn]

$$\Phi_n(S^q) \xrightarrow{\simeq} \Phi_n(P_{p^n}(Id)(S^q))$$

v_n -periodic Goodwillie tower

$$\Phi_n(S^q)$$

$$\|$$

$$\mathbb{D}_{p^n}(Id)(S^q)_{K(n)} \to \Phi_n(P_{p^n}(Id)(S^q))$$

$$\downarrow$$

$$\mathbb{D}_{p^{n-1}}(Id)(S^q)_{K(n)} \to \Phi_n(P_{p^{n-1}}(Id)(S^q))$$

$$\downarrow$$

$$\vdots$$

$$\downarrow$$

$$\mathbb{D}_1(Id)(S^q)_{K(n)} \to \Phi_n(P_1(Id)(S^q))$$

v_n -periodic Goodwillie tower

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Goodwillie SS

$$\pi_* \mathbb{D}_{p^*}(\mathrm{Id})(S^2)_{K(n)}$$

$$\Downarrow$$

$$v_n^{-1}\pi_* S^2$$

v_n -periodic Goodwillie tower

$$
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 & & \parallel \\
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E-homology Goodwillie SS

$$E_*^\wedge \mathbb{D}_{p^k}(\mathrm{Id})(S^q) \Rightarrow E_*^\wedge \Phi_n(S^q)$$

[$E = E_n$ = n^{th} Morava E-thy]

v_n -periodic Goodwillie tower

Recover $v_n^{-1}\pi_*(S^q) = \pi_*\Phi(S^q)$

via "homotopy fixed point SS"

$$H_c^*(\mathbb{G}_n; E_*^\wedge\Phi(S^q)) \Rightarrow \pi_*\Phi(S^q)$$

[$\mathbb{G}_n$ = n^{th} extended Morava stabilizer gp]

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$$\Downarrow$$

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n=1 : Exactly Mahowald-Thompson method $v_1^{-1}\pi_* S^q$

Morava E-theory Dyer-Lashof Algebra

$$[QE^*(-)]^2 : Top_* \longrightarrow Ab$$

$$X \longmapsto [QE^*(X)]^2$$

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e.g.

$n=1 \qquad \Delta^{0} = \mathbb{Z}_p[\theta]$

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Δ^2 = algebra of additive natural endomorphisms

e.g. $n=2$, $p=2$

$$\Delta^0 \cong \frac{\pi_0 E\langle Q_0, Q_1, Q_2\rangle}{\begin{array}{l} Q_1Q_0 = 2Q_2Q_1 - 2Q_0Q_2 \\ Q_2Q_0 = Q_0Q_1 + aQ_0Q_2 - 2Q_1Q_2 \\ Q_0a = a^2Q_0 - 2aQ_1 + 6Q_2 \\ Q_1a = 3Q_0 + aQ_2 \\ Q_2a = -aQ_0 + 3Q_1 \end{array}}$$

(Rezk)

$\pi_0 E = \hat{\mathbb{Z}}_2[\![a]\!]$

Morava E-theory Dyer-Lashof Algebra

Thm (Rezk)

Δ^q is Koszul

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$$\Delta^q = \bigoplus_{k \geq 0} \Delta^q[k]$$

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$$\text{Koszul} \iff H_*(B_\bullet(\Delta^q)[k]) = \begin{cases} C[k]_2, & * = k \\ 0, & * \neq k \end{cases}$$

Morava E-theory Dyer-Lashof Algebra

Thm (Rezk)

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$M = \Delta^q$ module: Koszul resolution

$C[0]_q \otimes_{E_0} M \leftarrow C[1]_q \otimes_{E_0} M \leftarrow \cdots$ $\quad H_s = \mathrm{Tor}_{\Delta^q}(E_0, M)$

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Morava E-theory of the layers

Thm

$$E^{\wedge}_{q}\left(\Sigma^{k}\mathbb{D}_{p^k}(Id)(S^q)\right) \cong C[k]_q^{\vee}$$

← E_0-linear dual

Morava E-theory of the tower

Thm

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"Pre-thm" The d_1 diff'l in the E_*-GSS

$$C[k]_q^\vee \cong E_q^\wedge(\Sigma^k \mathbb{D}_{p^k}(Id)(S^q)) \to E_q^\wedge(\Sigma^{k+1} \mathbb{D}_{p^{k+1}}(Id)(S^q)) \cong C[k+1]_q^\vee$$

is dual to the Koszul resolution for $\tilde{E}^q(S^q)$

Proof of the "Pre-thm" – TAQ

apply Φ_n to $X \longrightarrow QX$ gives $\eta_n : \Phi_n(X) \longrightarrow (\Sigma^\infty X)_{K(n)}$

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Dualize this to get "comparison map"

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Goodwillie tower hard to understand

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Goodwillie tower easy to understand "Kuhn Filtration"

Modular Isogeny Complex

Γ = formal gp for Morava E-thy

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$$\mathrm{Sub}_{p^{k_1},\ldots,p^{k_s}}(\Gamma) = \{H_1 < \cdots < H_s < \Gamma \mid |H_i/H_{i-1}| = p^{k_i}\}$$

$$= \mathrm{Spf}(S_{p^{k_1},\ldots,p^{k_s}})$$

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$$= \mathrm{Spf}(S_{p^{k_1},\ldots,p^{k_s}})$$

$$\mathcal{K}^\bullet_{p^k} : \quad 0 \to S_{p^k} \to \prod_{k_1+k_2=k} S_{p^{k_1},p^{k_2}} \to \cdots \to S_{\underbrace{p,\ldots,p}_{k}}$$

$$d = \sum (-1)^i u_i \qquad u_i : (H_1 < \cdots < H_s) \mapsto (H_1 < \cdots < \hat{H}_i < \cdots < H_s)$$

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Thm

$$H^*(\mathcal{K}^\bullet_{p^k}) = \begin{cases} C[k]_1^\vee, & * = k \\ 0, & o/w \end{cases}$$

Modular Isogeny Complex

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$$H^*(\mathcal{K}^\bullet_{p^k}) = \begin{cases} \mathbb{C}[k]^\vee_1, & * = k \\ 0, & o/w \end{cases}$$

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Koszul Res for $\tilde{\hat{E}}'(S')$

$$\begin{array}{ccc}
(H_2/H_1 < \cdots < H_k/H_1 < I/H_1) & \longleftarrow\!\shortmid & (H_1 < \cdots < H_k < I) \\
\int_{\underbrace{p,\ldots,p}_{k-1}} & \longrightarrow & \int_{\underbrace{p,\ldots,p}_{k}} \\
\downarrow & & \downarrow \\
C[k-1]_1^\vee & \xrightarrow{\ d\ } & C[k]_1^\vee
\end{array}$$