Higher real K-theories and Topological Automorphic Forms

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Overview:

Interested in:

"v_n-periodic cohomology theories"

e.g. v_1-periodicity = Bott periodicity, Im J global

local

$n=1 \begin{cases} \text{good} & KU & KU_p \simeq E_1 \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1)_p \\ \text{better} & KO \simeq KU^{hC_2} & KO_p \simeq E_1^{hC_2} \end{cases}$

"Better"

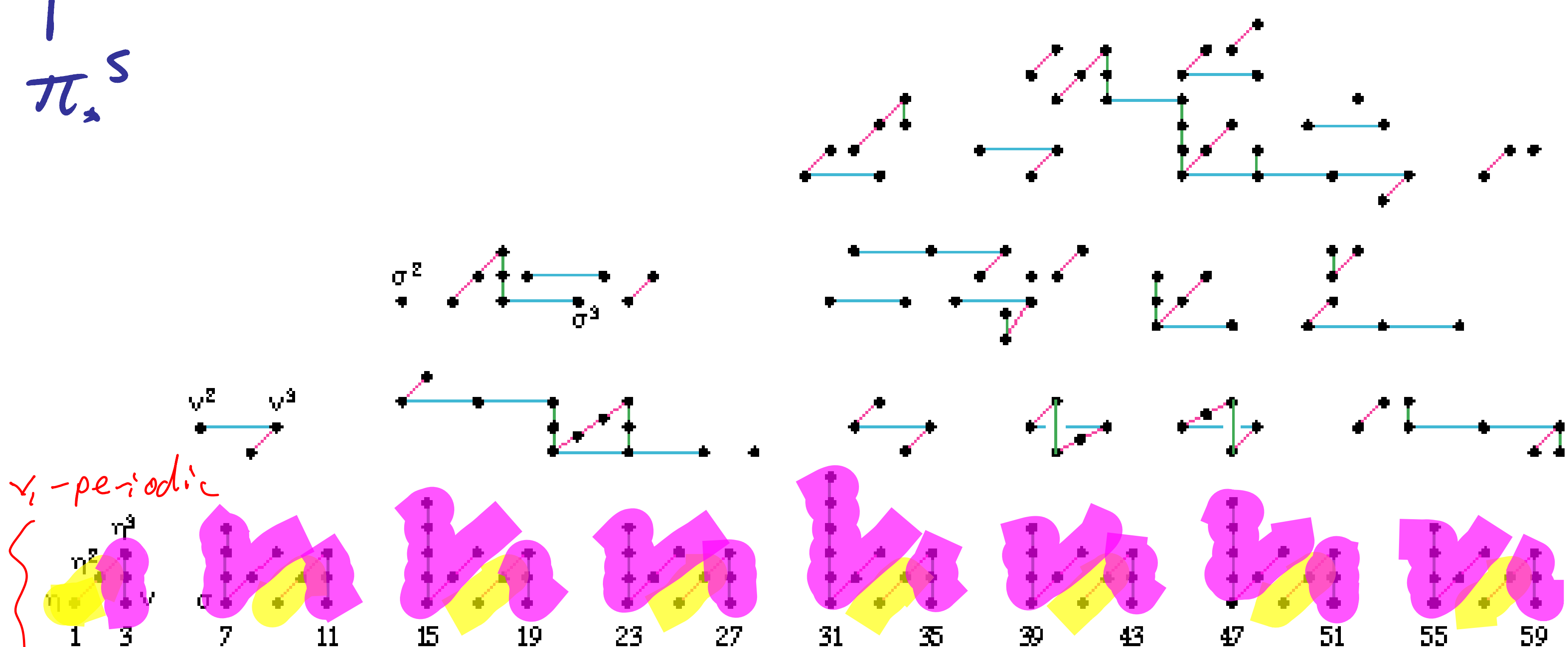
= Im J

$$\pi_* KO = \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \cdots$$

$\uparrow$

π_*^s

Stable Homotopy Groups of Spheres at the prime 2

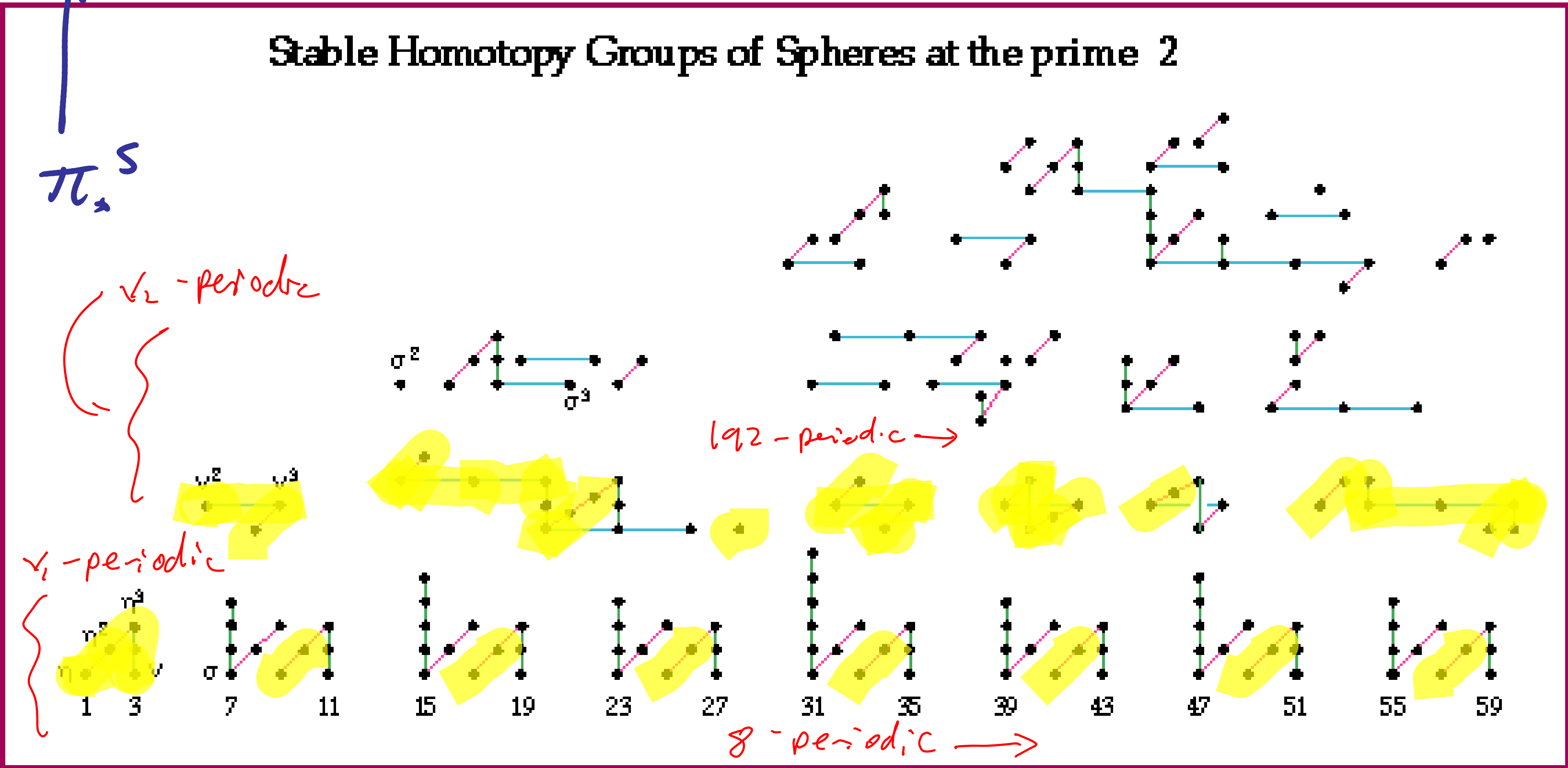


v_1-periodic

		global	local
$n=1$	good	KU	$KU_p \simeq E_1 \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1)_p$
	better	$KO \simeq KU^{hC_2}$	$KO_p \simeq E_1^{hC_2}$
$n=2$	good	Ell_C C = elliptic curve	$E(2)$, $E_2 \simeq \bigvee \Sigma^{(-)} E(2) \otimes \mathbb{Z}_{p^2}$
	better	TMF	$p=2,3$ $TMF_{K(2)} \simeq E_2^{hG} =: EO_2$ $\lvert G \rvert = \begin{cases} 48 & p=2 \\ 24 & p=3 \end{cases}$

$\pi_* TMF$

π_*^s



Stable Homotopy Groups of Spheres at the prime 2
ν_2-periodic
ν_1-periodic
192-periodic
8-periodic
σ^2
σ^3
ν^2
ν^3
η
η^2
η^3
ν
σ
1
3
7
11
15
19
23
27
31
35
39
43
47
51
55
59

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n arbitrary	good	"Automorphic spectra"	$E(n)$, $E_n \simeq \bigvee \Sigma^{(-)} E(n) \otimes \mathbb{Z}_{p^n}$
	better	TAF (Top'l automorphic forms)	$EO_n := E_n^{h(\text{certain finite sg})}$

$$KO_{K(1)} \simeq E_1^{hC_2} = EO_1 \qquad p=2$$

$$TMF_{K(2)} \simeq EO_2 \qquad p=2,3$$

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$$TMF_{K(2)} \simeq EO_2 \qquad p = 2, 3$$

Q: What is the relationship between

$TAF_{K(n)}$ and EO_n ?

Background on EO_n

$$\pi_0 E_n = \mathbb{Z}_{p^n}[[u_1, \ldots, u_{n-1}]]$$

$$\Big|$$

$$\mathbb{F}_{p^n}$$

$\tilde{H}_n$ = Lubin-Tate universal deformation

H_n = Honda ht n formal gp

$$\text{Formal gp}(E_n) = \tilde{H}_n$$

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$\tilde{H}_n$ = Lubin-Tate universal deformation

H_n = Honda ht n formal gp

$$\mathbb{S}_n = \mathrm{Aut}(H_n) \curvearrowright \pi_0 E_n \curvearrowleft \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

Morava Stab. Gp

Background on EO_n

Hopkins-Miller:

$$\mathbb{G}_n \circlearrowright E_n$$

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e.g. $n=1$

$$EO_1 = \begin{cases} KO_2 & p=2 \quad G = C_2 \\ E(1)_p & p \text{ odd} \quad G = C_{p-1} \end{cases}$$

Background on EO_n

Hopkins-Miller: $\quad G \leq \mathbb{S}_n$ maximal finite

$$\mathbb{S}_n \rtimes Gal \circlearrowright E_n \qquad EO_n := E_n^{hG}$$

e.g. $n=1$ $\quad EO_1 = \begin{cases} KO_2 & p=2 & G=C_2 \\ E(1)_p & p \text{ odd} & G=C_{p-1} \end{cases}$

Issue: If $(p-1) \mid n > 1$, maximal finites not unique!

"Many EO_n's"

Thm (T. Hewett)

Write $n = (p-1)p^{r-1}s \qquad p \nmid s$

Then $\exists!$ (up to iso) $G_{n,\alpha} \overset{\text{max finite}}{\hookrightarrow} S_n \qquad \begin{array}{ll} 0 \le \alpha \le r & p \text{ odd} \\ 1 \le \alpha \le r & p = 2 \end{array}$

s.t. $G_{n,\alpha}$ has elts w/ maximal p-order p^α

Background on TAF (B-Lawson using Lurie)

n — alg gp — moduli space — coh thy

Background on TAF

n	alg gp	moduli space	coh thy
1	GL_1	$* = \{\mathbb{G}_m\}$	KO

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n	$U = U(1, n-1)$	Sh = shimura variety	TAF_u

only exists at 50% primes

Background on TAF

Sh is a moduli space:

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Initial data:

$$\begin{array}{cc} F & u\bar{u} \\ \big|\,{\scriptstyle 2} & \big| \\ \mathbb{Q} & p \end{array}$$

V = F v.s. dim n

$\langle -,- \rangle$ = alternating herm. form on V

sign = $(1, n-1)$

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Satisfying:

- $\underset{n\text{-dim}}{\hat{A}} \cong \underset{1\text{-dim}}{\hat{A}_u} \oplus \underset{(n-1)\text{-dim}}{\hat{A}_{\bar{u}}}$
- $\langle -,- \rangle_\lambda \cong \langle -,- \rangle$

Thm: (B-Lawson)

$$TAF_{K(n)} \simeq \left(\bigvee_{(A,i,\lambda) \in Sh^{[n]}(\overline{\mathbb{F}}_p)} \bar{E}_n^{\,hAut(A,i,\lambda)} \right)^{hGal_{\mathbb{F}_p}}$$

ht $\hat{A}_u = n$

$$Aut(A,i,\lambda) \underset{\text{finite}}{<} \mathbb{S}_n$$

$$\bar{E}_n := E_n \otimes_{\mathbb{Z}_{p^n}} \mathbb{Z}_p^{nr}$$

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Note that

$\# | Sh^{[n]}(\mathbb{F}_p) | < \infty$

but typically $\neq 1$ $\left(\begin{array}{c} \text{class} \\ \# \end{array} \right)$

Question **Reduces** **to**:

Given $G_{n,\alpha} < S_n$

does there exist $(F, V, \langle -, - \rangle)$ s.t.

$\exists\ (A, i, \lambda) \in Sh^{[n]}(\bar{\mathbb{F}}_p)$, $\mathrm{Aut}(A, i, \lambda) = G_{n,\alpha}$?

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A: $\boxed{\text{Main Thm}}$ (B-Hopkins)

$p \leq 7$, $n = (p-1)p^{r-1}$, $\alpha = r$ $\Longrightarrow$ yes!

p odd, not in above situation $\Longrightarrow$ No!

$p = 2$, not in above situation $\Longrightarrow$?

Consolation Prize

Thm (B-Hopkins)

for all $n = (p-1)p^{r-1}$ $\exists$ $(A, i, \lambda) \in Sh^{[n]}(\bar{\mathbb{F}}_p)$

s.t. $\mathrm{Aut}(A, i, \lambda)$ contains an elt of order p^r

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i.e.

"$TAF_{K(n)}$ sees as much as EO_n,

just not as efficiently"

Consequences for orientation theory

$\hat{A}: \quad MSpin \longrightarrow KO \qquad (ABS)$

$W: \quad MString \longrightarrow TMF \qquad (AHR)$

Consequences for orientation theory

characteristic series

$\hat{A}$: $MSpin \longrightarrow KO$ (ABS) Bernoulli #'s

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Eisenstein series

Q: Does there exist

$$MO\langle N\rangle \longrightarrow TAF_{U(1,n-1)}$$

Eisenstein series on $U(1,n-1)$

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Q: Does there exist

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Eisenstein series on $U(1, n-1)$

A: If $(p-1) \mid n$, $n \geq 2$ then NO!

(using work of Hovey, showed $E_n^{hC_p}$ does not admit such an orientation)

New Question:

For which top'l gps

$$G \longrightarrow O$$

does there exist

$$MG \longrightarrow TAF$$

(Connective covers won't work!)

Method of Proof of Main thm ...

Local + Global Class fld thy

Division Algebras:

K = local/global fld

$$\mathrm{Br}(K) \cong H^2(\mathrm{Gal}(\bar{K}/K); \bar{K}^\times)$$

$$\begin{array}{ccc} \Psi & & \Psi \\ B & \longleftrightarrow & \mathrm{Inv}_B \end{array}$$

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Explicit construction: (Serre 1950)

$$\begin{tikzcd} x \in H^2(\mathrm{Gal}(\bar{K}/K); \bar{K}^\times) \\ x \in H^2(\mathrm{Gal}(M/K); M^\times) \arrow[u, mapsto] \arrow[u, hook] \end{tikzcd}$$

$$\begin{array}{c} M \\ | \\ K \end{array}$$

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M
$|$
K

$x \sim$ central extension

$$M^\times \to E \to Gal(M/K)$$

$$B_x = \mathbb{Z}[E] \otimes_{\mathbb{Z}[M^\times]} M$$

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Conversely: $M \hookrightarrow B$ $\;\dim/_K = n^2$

M

$| \, n$

K

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M

$|n$

K

Get $M^\times \longrightarrow E \longrightarrow Gal(M/K)$

$$E := N_{B^\times}(M^\times)$$

More Explicitly: Assume $\mathrm{Gal}(M/K) = C_n = \langle \sigma \rangle$

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K local $\Rightarrow$ $H^2(\mathrm{Gal}(M/K); M^\times) \cong \mathbb{Z}/n \subset \mathbb{Q}/\mathbb{Z}$

$$\mathbb{Q}/\mathbb{Z} \cong \mathrm{Br}(K)$$

More Explicitly: Assume $\mathrm{Gal}(M/K) = C_n = \langle \sigma \rangle$

$\underline{K \text{ local}} \implies H^2(\mathrm{Gal}(M/K); M^\times) \cong \mathbb{Z}/n \subset \mathbb{Q}/\mathbb{Z}$

$$\mathbb{Q}/\mathbb{Z} \cong \mathrm{Br}(K)$$

$$M^\times \longrightarrow E \longrightarrow \mathrm{Gal}(M/K)$$

$$\varphi$$

$$\tilde{\sigma} \longmapsto \sigma$$

More Explicitly: Assume $\mathrm{Gal}(M/K) = C_n = \langle \sigma \rangle$

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$$\begin{array}{ccccc}
M^\times & \longrightarrow & E & \longrightarrow & \mathrm{Gal}(M/K) \\
\rotatebox{90}{$\subset$} & & & & \rotatebox{90}{$\in$} \\
K^\times & & \tilde{\sigma} & \longmapsto & \sigma \\
\rotatebox{90}{$\in$} & & & & \\
a & \longmapsto & \tilde{\sigma}^n & &
\end{array}$$

More Explicitly: Assume $\mathrm{Gal}(M/K) = C_n = \langle \sigma \rangle$

$$\underline{K \text{ local}} \implies H^2(\mathrm{Gal}(M/K); M^\times) \cong \mathbb{Z}/n \subset \mathbb{Q}/\mathbb{Z} \cong \mathrm{Br}(K) \ni \mathrm{Inv}\, B = k/n$$

$$\begin{array}{ccccc} M^\times & \longrightarrow & E & \longrightarrow & \mathrm{Gal}(M/K) \\ \cup & & & & \in \\ K^\times & & \tilde{\sigma} & \longmapsto & \sigma \\ \in & & & & \\ a & \longmapsto & \tilde{\sigma}^n & & \end{array}$$

$$a \in K^\times \xrightarrow[\mathrm{Art}_K]{\cong} \mathrm{Gal}(K^{ab}/K) \twoheadrightarrow \mathrm{Gal}(M/K), \qquad a \longmapsto \sigma^k$$

More Explicitly: Assume $\mathrm{Gal}(M/K) = C_n = \langle \sigma \rangle$

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Notation $B = B_{[a]}$

Presentation of B_a

$a \in K^\times$, $\begin{array}{c} M \\ \mid \\ K \end{array} \; C_n = \langle \sigma \rangle$

$$B_a = M\langle s \rangle \big/ \left(s^n = a, \; sx = x^\sigma s, \; x \in M \right)$$

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e.g.

$$M = \mathbb{Q}_{p^n}$$

$$\begin{array}{c} \mathbb{Q}_{p^n} \\ | \; \mathrm{Gal} = \langle \sigma \rangle \\ \mathbb{Q}_p \end{array}$$

$$\mathbb{Q}_p^\times \xrightarrow{\mathrm{Art}} \mathrm{Gal}\left(\mathbb{Q}_{p^n}/\mathbb{Q}_p\right)$$

$$p \longmapsto \sigma$$

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$\quad \big| \; \mathrm{Gal} = \langle \sigma \rangle$
$\mathbb{Q}_p$

$$\mathbb{Q}_p^\times \xrightarrow{\mathrm{Art}} \mathrm{Gal}\left(\mathbb{Q}_{p^n} / \mathbb{Q}_p \right)$$

$$p \longmapsto \sigma$$

$$B_{[p]} = D_{1/n} \cong \mathbb{Q}_{p^n}\langle S \rangle \big/ \left(S^n = p, \; Sx = x^\sigma S \right)$$

Global version of this theory

K = global fld

$$\begin{array}{ccc} Br(K) & \hookrightarrow & \bigoplus_v Br(K_v) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ D & \longmapsto & (D_v)_v \end{array}$$

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$a \in K^\times$, $\quad \begin{array}{c} M \\ | \\ K \end{array} \; C_n = \langle \sigma \rangle \Longrightarrow$ Get algebra $B_{[a]}$

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Get algebra

$a \in K^\times$, $\quad \begin{matrix} M \\ | \; C_n = \langle \sigma \rangle \\ K \end{matrix} \quad \Longrightarrow \quad B_{[a]}$

$$\prod{}' K_v^\times \cong \mathbb{I}_K \xrightarrow[\text{Art}]{} Gal(M/K) \cong \mathbb{Z}/n \subset \mathbb{Q}/\mathbb{Z}$$

$$Inv_v(B_{[a]}) = Art(a_v)$$

Easy Construction of Hewett's $G_{n,r} < \mathbb{S}_n$

$$D_{1/n} = \mathrm{End}(H_n) \otimes \mathbb{Q}$$

$n = (p-1)p^{r-1}$

p odd

$$G_{n,r} \hookrightarrow \mathbb{S}_n \longrightarrow D_{1/n}$$

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$$\begin{tikzcd}
\mathbb{Q}_p(\zeta_{p^r}) \arrow[d, dash, "n"'] \\
\mathbb{Q}_p
\end{tikzcd}$$

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$$\begin{tikzcd}
\mathbb{Q}_p(\zeta_{p^r}) \arrow[dd, dash, "(p-1)p^{r-1} = n"'] \arrow[dr, dash, "p-1"] & \\
& L \\
\mathbb{Q}_p \arrow[ur, dash, "p^{r-1}"'] &
\end{tikzcd}$$

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& L \arrow[dl, dash, "p^{r-1}"] \\
\mathbb{Q}_p &
\end{tikzcd}$$

$D' \subset D_{1/n}$

$\shortparallel$

centralizer of L

$$\mathrm{Inv}(D') = \frac{1}{p-1}$$

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 & L \arrow[dl, dash, "p^{r-1}"] \\
\mathbb{Q}_p &
\end{tikzcd}$$

$D' \subset D_{1/n}$

$D' =$ centralizer of L

$\mathrm{Inv}(D') = \frac{1}{p-1}$

$$\begin{tikzcd}
M^\times \arrow[r] & E \arrow[r, two heads] & \mathrm{Gal}(M/\mathbb{Q}_p) \\
M^\times \arrow[u, equal] \arrow[r] & E' \arrow[u, hook] \arrow[r, two heads] & \mathrm{Gal}(L/\mathbb{Q}_p) \arrow[u, hook]
\end{tikzcd}$$

Easy Construction of Hewett's $G_{n,r} < \mathbb{S}_n$

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$n = (p-1)p^{r-1}$

p odd

$$G_{n,r} \hookrightarrow \mathbb{S}_n \longrightarrow D_{1/n}$$

$$\begin{tikzcd}
\mathbb{Q}_p(\zeta_{p^r}) := M \arrow[d, dash, "p-1"] \arrow[dd, dash, bend right, "(p-1)p^{r-1} = n"'] \\
L \arrow[d, dash, "p^{r-1}"] \\
\mathbb{Q}_p
\end{tikzcd}$$

$D' \subset D_{1/n}$

$D' \cong$ centralizer of L

$\mathrm{Inv}(D') = \frac{1}{p-1}$

$$\begin{tikzcd}
M^\times \arrow[r] \arrow[d, equal] & E \arrow[r] & \mathrm{Gal}(M/\mathbb{Q}_p) \\
M^\times \ni \omega \arrow[r] & E' \ni z^{p-1} \arrow[u, hook] \arrow[r] & \mathrm{Gal}(M/L) \arrow[u, hook] \arrow[d, equal] \\
C_{(p-1)p^r} \arrow[r] \arrow[u, hook] & G_{n,r} \arrow[r] \arrow[u, hook] & C_{p-1}
\end{tikzcd}$$

$\omega \mapsto z^{p-1}$

$G_{n,r} = \langle z, \zeta \rangle$

$\langle \omega, \zeta \rangle = C_{(p-1)p^r}$

$\omega \in \mathbb{Z}_p$

$\omega^{p-1} = 1$

To get main thm: Globalize

$(A, i, \lambda) \in Sh^{[n]}$

$\mathrm{End}(A, i) \otimes \mathbb{Q} = D/F$

$\mathrm{Inv}_u(D) = 1/n$

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$\mathrm{Aut}(A, i, \lambda) \hookrightarrow \{x \in D^\times \mid x x^\dagger = 1\}$

Idea: case $p = 5, 7$

$$\mathbb{Q}(\omega) =: F \qquad \omega^{p-1} = 1$$

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[excludes $p > 7$]

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$$D' \subset D$$

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$$\begin{tikzcd}
M^\times \arrow[r] \arrow[d, equal] & E \arrow[r] & \mathrm{Gal}(M/F) \\
M^\times \ni \omega \arrow[r] & E' \ni z^{p-1} \arrow[r] \arrow[u, hook] & \mathrm{Gal}(M/L) \arrow[u, hook] \\
\langle \omega, \zeta \rangle = C_{(p-1)p^r} \arrow[r] \arrow[u, hook] & G_{n,r} = \langle z, \zeta \rangle \arrow[r] \arrow[u, hook] & C_{p-1} \arrow[u, equal]
\end{tikzcd}$$

$$\omega \mapsto z^{p-1}$$

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Need to construct an involution $(\dagger)$

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Then: use classification: $\{\langle -,- \rangle \text{ on } V\}$

$$\updownarrow$$

$\{\text{involutions on } D\}$