Exotic spheres and topological modular forms

Mark Behrens (MIT)
(joint with Mike Hill, Mike Hopkins, and Mark Mahowald)

Fantastic survey of the subject:

Milnor, “Differential topology: 46 years later” (Notices of the AMS, June/July 2011)

http://www.ams.org/notices/201106/

Poincaré Conjecture

Q: Is every homotopy n-sphere homeomorphic to an n-sphere?

A: Yes!

- $n = 2$: easy.
- $n \geq 5$: (Smale, 1961) h-cobordism theorem
- $n = 4$: (Freedman, 1982)
- $n = 3$: (Perelman, 2003)

Smooth Poincaré Conjecture

Q: Is every homotopy n-sphere diffeomorphic to an n-sphere?

A: Depends on n.

- $n = 2$: True - easy.
- $n = 7$: (Milnor, 1956) False – produced a smooth manifold which was homeomorphic but not diffeomorphic to S^7! [exotic sphere]
- $n \geq 5$: (Kervaire-Milnor, 1963) – `often' false. (true for $n = 5,6$).
- $n = 3$: (Perelman, 2003) True.
- $n = 4$: Unknown.

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  Goal for this talk
- $n = 3$: (Perelman, 2003) True.
- $n = 4$: Unknown.

Main Question

For which n do there exist exotic n-spheres?

Kervaire-Milnor

$\Theta_n \coloneqq$ {oriented smooth homotopy n-spheres}/h-cobordism

(note: if $n \neq 4$, h-cobordant $\Leftrightarrow$ oriented diffeomorphic)

For $n \not\equiv 2(4)$:

$$0 \to \Theta_n^{bp} \to \Theta_n \to \frac{\pi_n^s}{Im\, J} \to 0$$

Kervaire-Milnor

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Θ_n^{bp} = subgroup of those which bound a parallelizable manifold

π_n^s = stable homotopy groups of spheres

$= \pi_{n+k}(S^k)$ for $k \gg 0$

$J: \pi_n(SO) \to \pi_n^s$ is the J-homomorphism.

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We'll get back to these

Kervaire-Milnor

Θ_n := {smooth homotopy *n*-spheres}/h-cobordism

For $n \not\equiv 2(4)$:

$\pi_n^s = \Omega_n^{fr}$

$$0 \to \Theta_n^{bp} \to \Theta_n \to \frac{\pi_n^s}{Im\,J} \to 0$$

framed surgery

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For $n \equiv 2(4)$:

$$0 \to \Theta_n^{bp} \to \Theta_n \to \frac{\pi_n^s}{Im\, J} \to \mathbb{Z}/_2 \to \Theta_{n-1}^{bp} \to 0$$

$$[M] \mapsto \Phi_K(M)$$

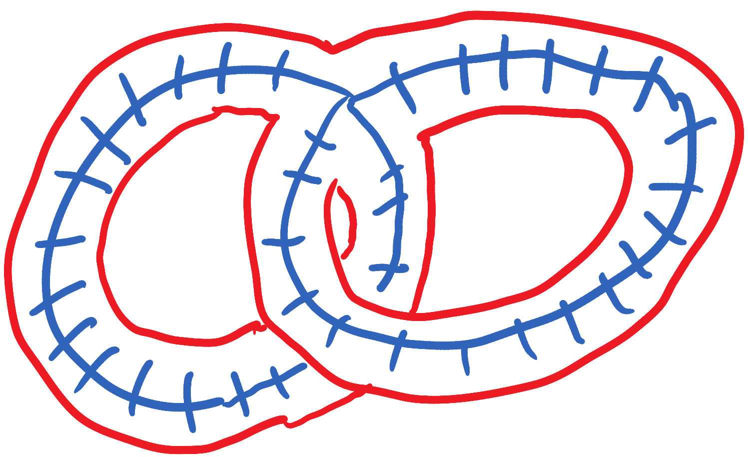
Kervaire Invariant

Θ_n^{bp}

- Trivial for *n* even
- Cyclic for *n* odd

$$\Theta_n^{bp}$$

- Trivial for *n* even
- Cyclic for *n* odd
  - Generated by boundary of an explicit parallelizable manifold given by plumbing construction



Θ_n^{bp}

- Trivial for *n* even
- Cyclic for *n* odd:

$$|\Theta_n^{bp}| = \begin{cases} 2^{2k}(2^{2k+1}-1)num\left(\frac{4B_{k+1}}{k+1}\right), n = 4k+3 \\ \mathbb{Z}/_2, \quad n \equiv 1(4), \exists M^{n+1}\ with\ \Phi_K = 1 \\ 0, \quad n \equiv 1(4), \nexists M^{n+1}\ with\ \Phi_K = 1 \end{cases}$$

Upshot: *n* even ⇒ bp gives no exotic spheres

$n \equiv 3\ (4)$ ⇒ bp gives exotic spheres ($n \geq 7$)

$n \equiv 1\ (4)$ ⇒ bp gives exotic sphere only if there are no M^{n+1} with $\Phi_K = 1$

J-homomorphism

$$J\colon \pi_n SO \to \pi_n^s \cong \Omega_n^{fr}$$

Given $\alpha\colon S^n \to SO$, apply it pointwise to the standard stable framing of S^n to obtain a non-standard stable framing of S^n.

Homotopy spheres are stably parallelizable, but not uniquely so – only get a well defined map

$$\Theta_n \to \frac{\pi_n^s}{Im\, J}$$

J-homomorphism

$$J: \pi_n SO \to \pi_n^s \cong \Omega_n^{fr}$$

$$\begin{array}{lccccccccccccccccc} & & \| & & & & & & & & & & & & & & & \\ & (\mathbb{Z}/2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & \mathbb{Z}/2 & \cdots) \\ n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \cdots \end{array}$$

$\mathbb{Z}/2$'s map in nontrivially

Adams, Mahowald

$$|\mathrm{Im}\, J|_{4k-1} = \mathrm{denom}\left(\frac{B_k}{4k}\right)$$

π_*^s

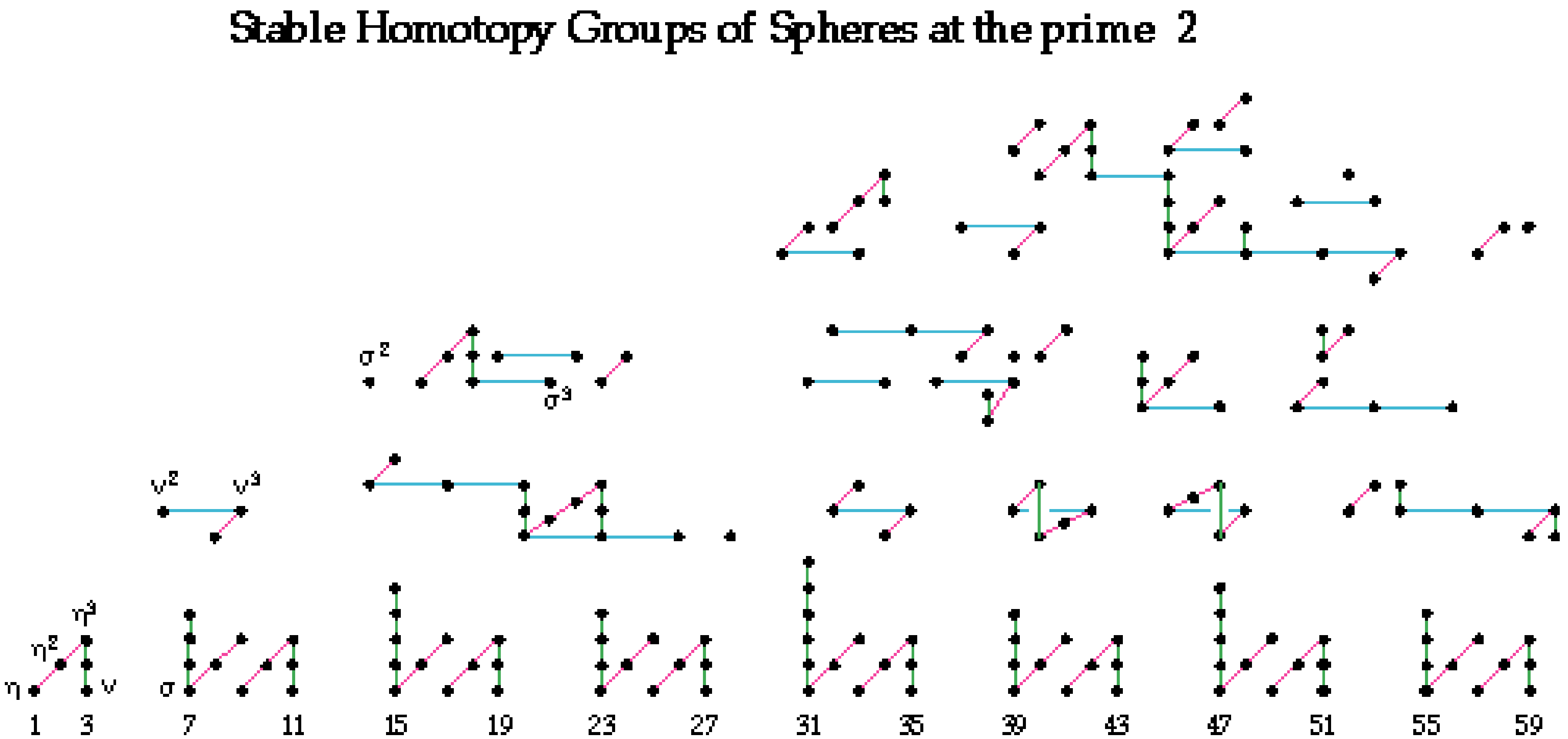
Stable homotopy groups:

$$\pi_n^s := \lim_{k\to\infty} \pi_{n+k}(S^k) \quad \text{(finite abelian groups for } n > 0\text{)}$$

Primary decomposition:

$$\pi_n^s = \bigoplus_{p\ prime} (\pi_n^s)_{(p)}$$

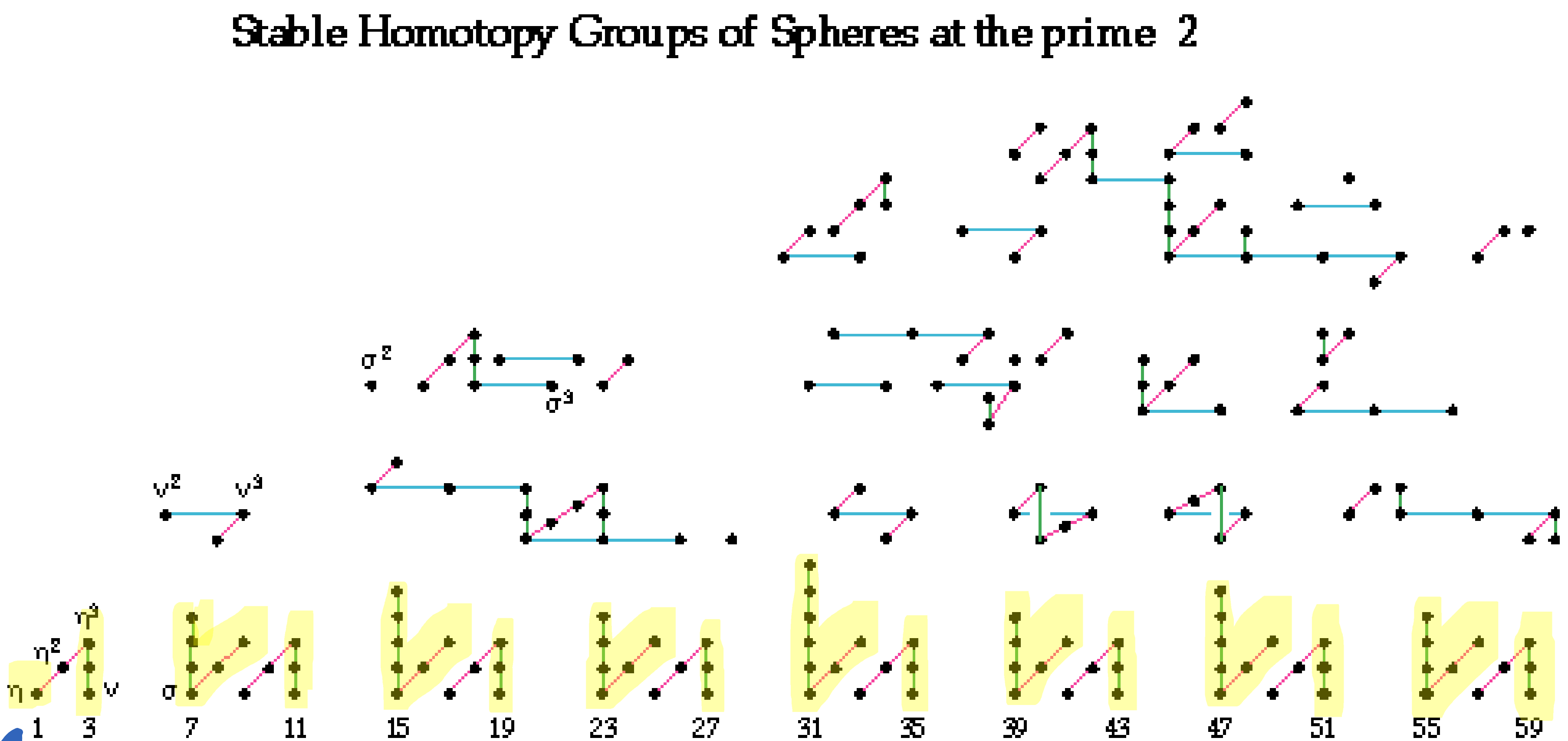
e.g.: $\pi_3^s = \mathbb{Z}_{24} = \mathbb{Z}_8 \oplus \mathbb{Z}_3$



Computation: Mahowald-Tangora-Kochman
Picture: A. Hatcher

- Each dot represents a factor of 2, vertical lines indicate additive extensions

  e.g.: $(\pi_3^S)_{(2)} = \mathbb{Z}_8, \qquad (\pi_8^S)_{(2)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$
- Vertical arrangement of dots is arbitrary, but meant to suggest patterns



Im J

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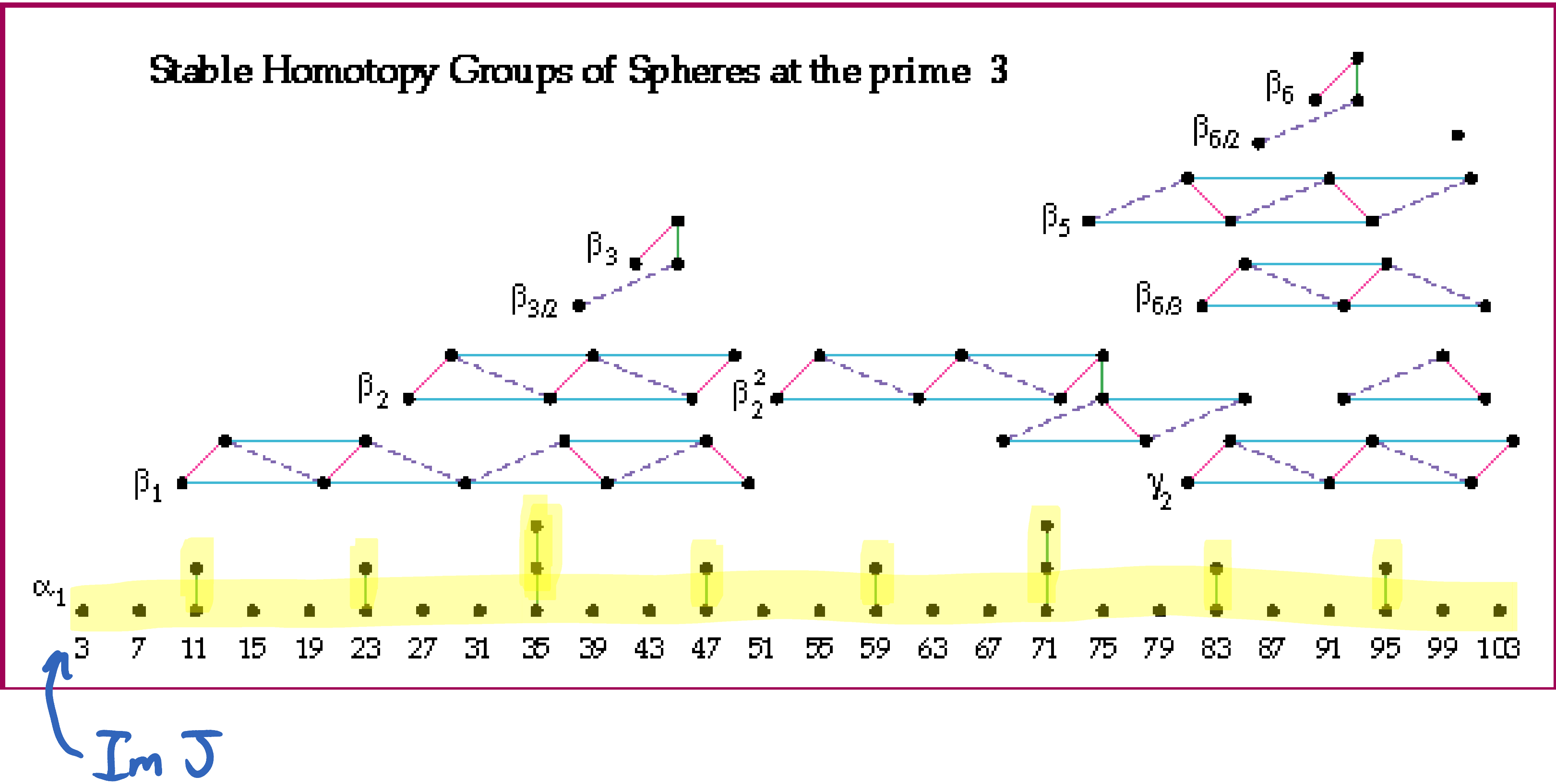
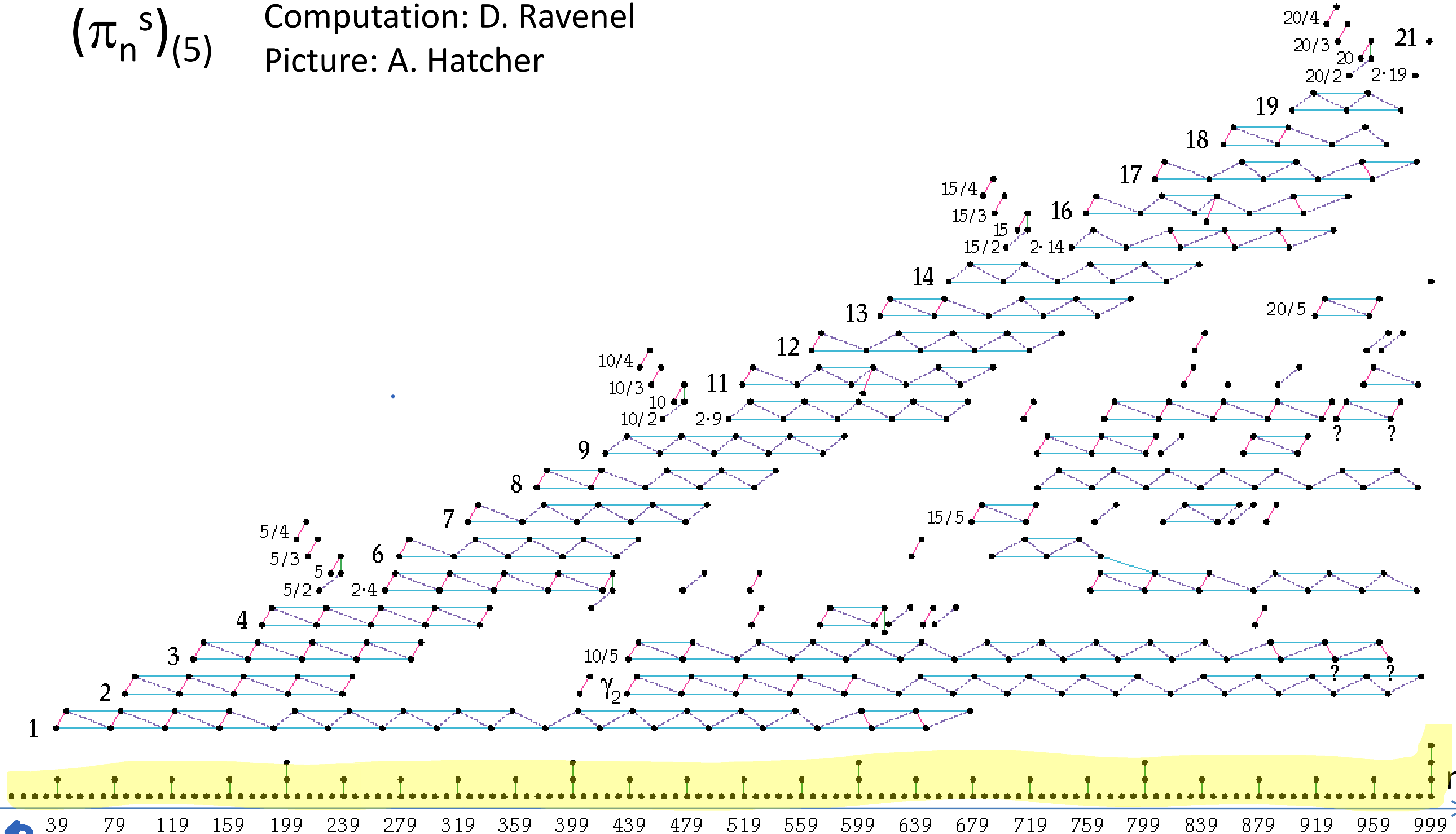
Computation: Nakamura -Tangora
Picture: A. Hatcher



$(\pi_n^s)_{(5)}$

Computation: D. Ravenel
Picture: A. Hatcher



Im J

Adams spectral sequence

$$Ext_A^{s,t}(\mathbb{Z}/p\,,\mathbb{Z}/p) \Rightarrow (\pi^s_{t-s})_p$$

[p=2]

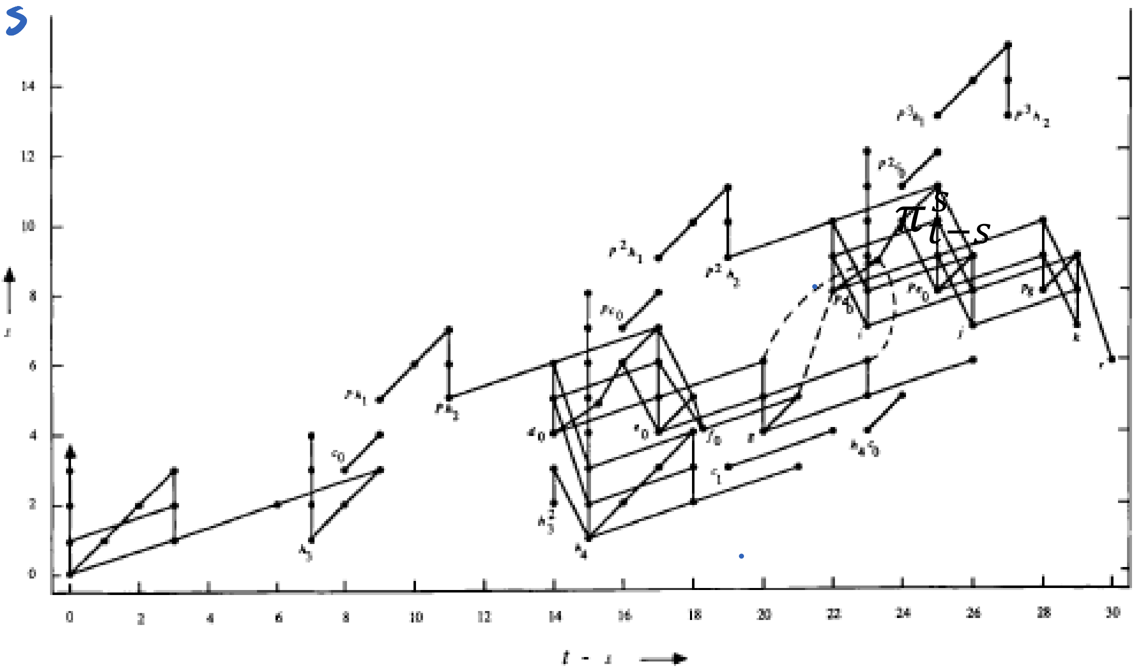
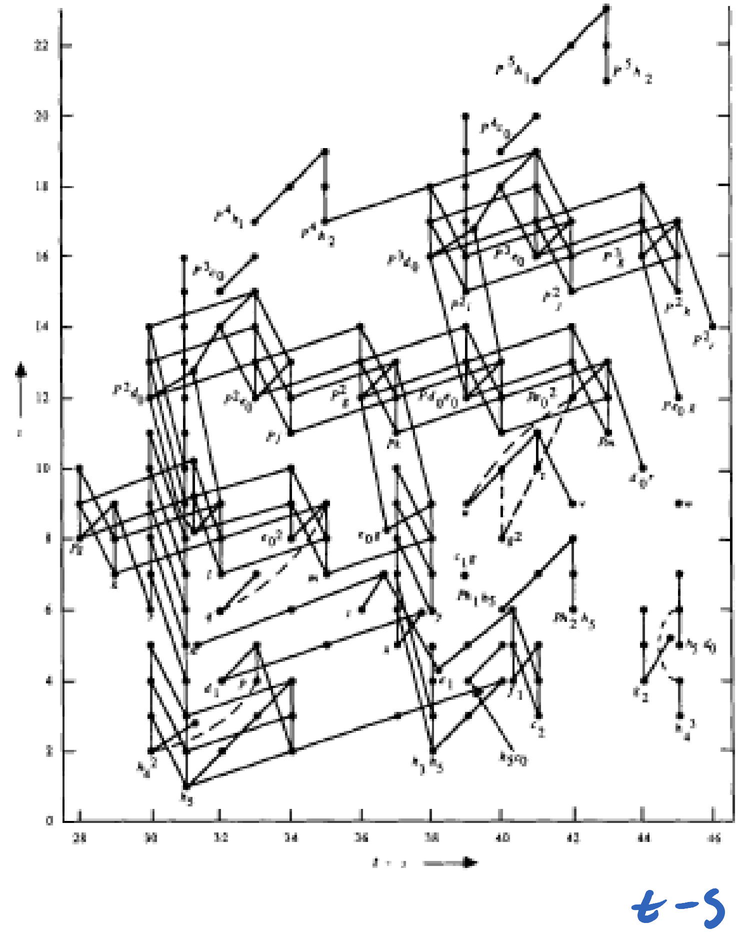


Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.



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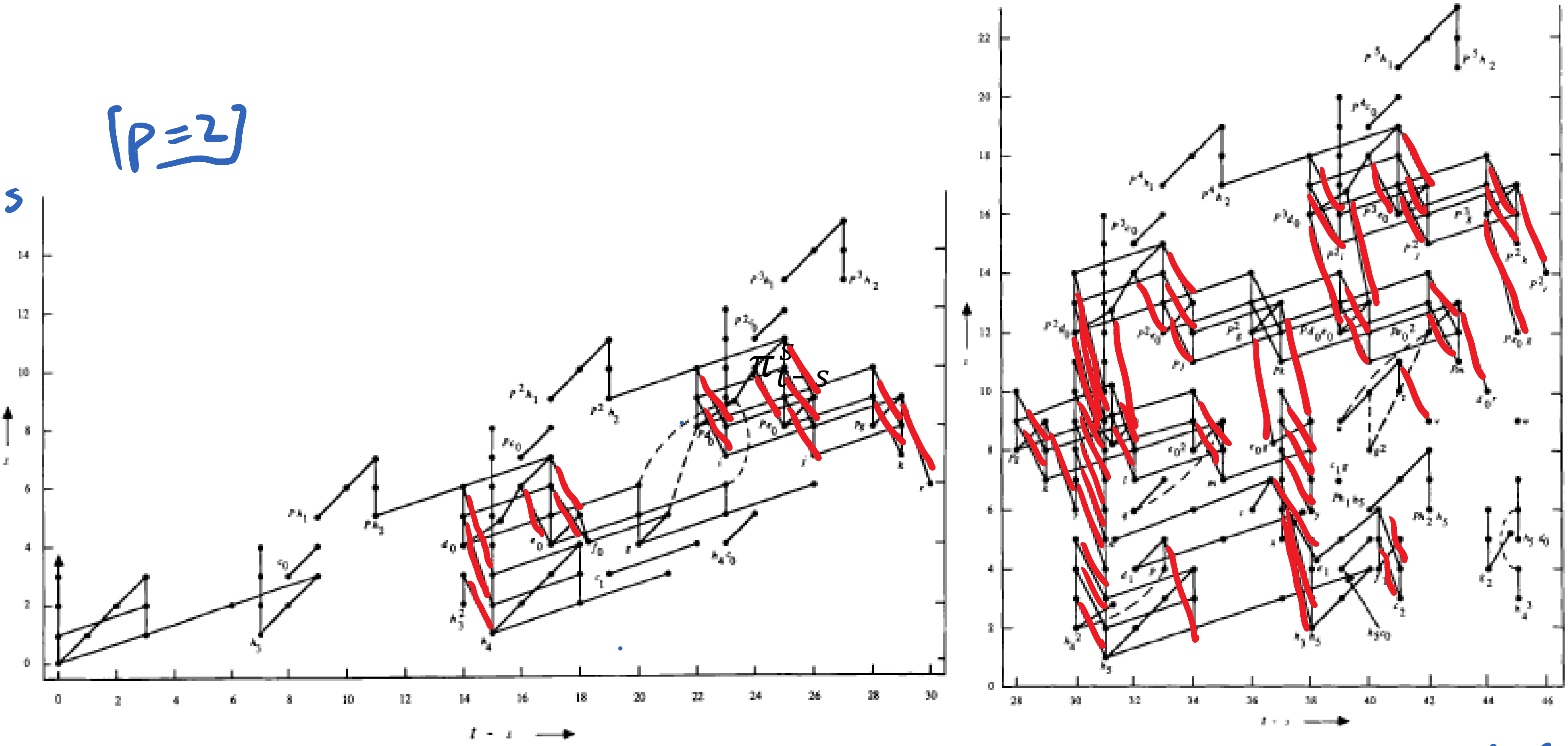

Figure A3.1a The Adams spectral sequence for $p=2$, $t-s \le 29$.

- Many differentials
- d_r differentials go back by 1 and up by r

Adams spectral sequence

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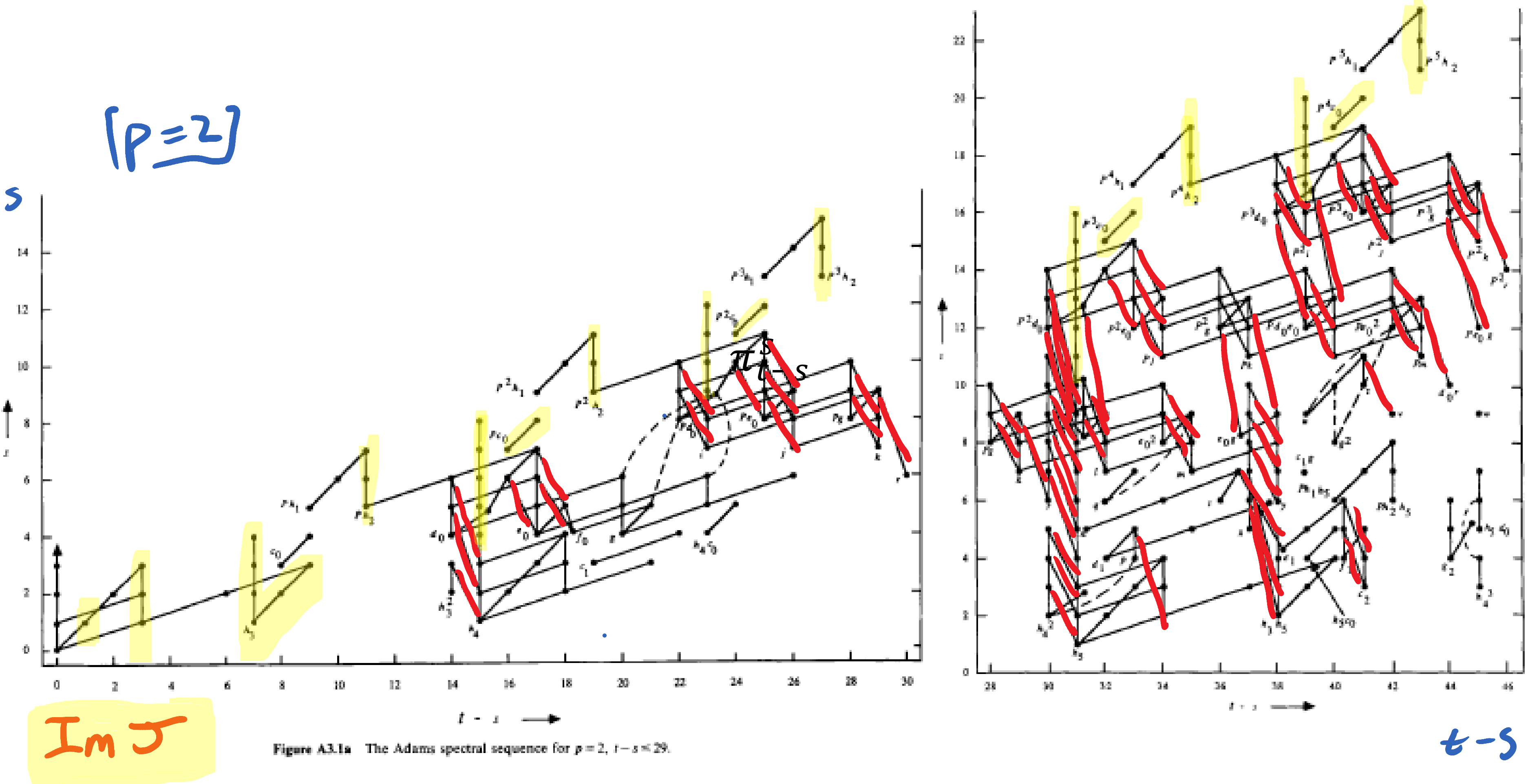

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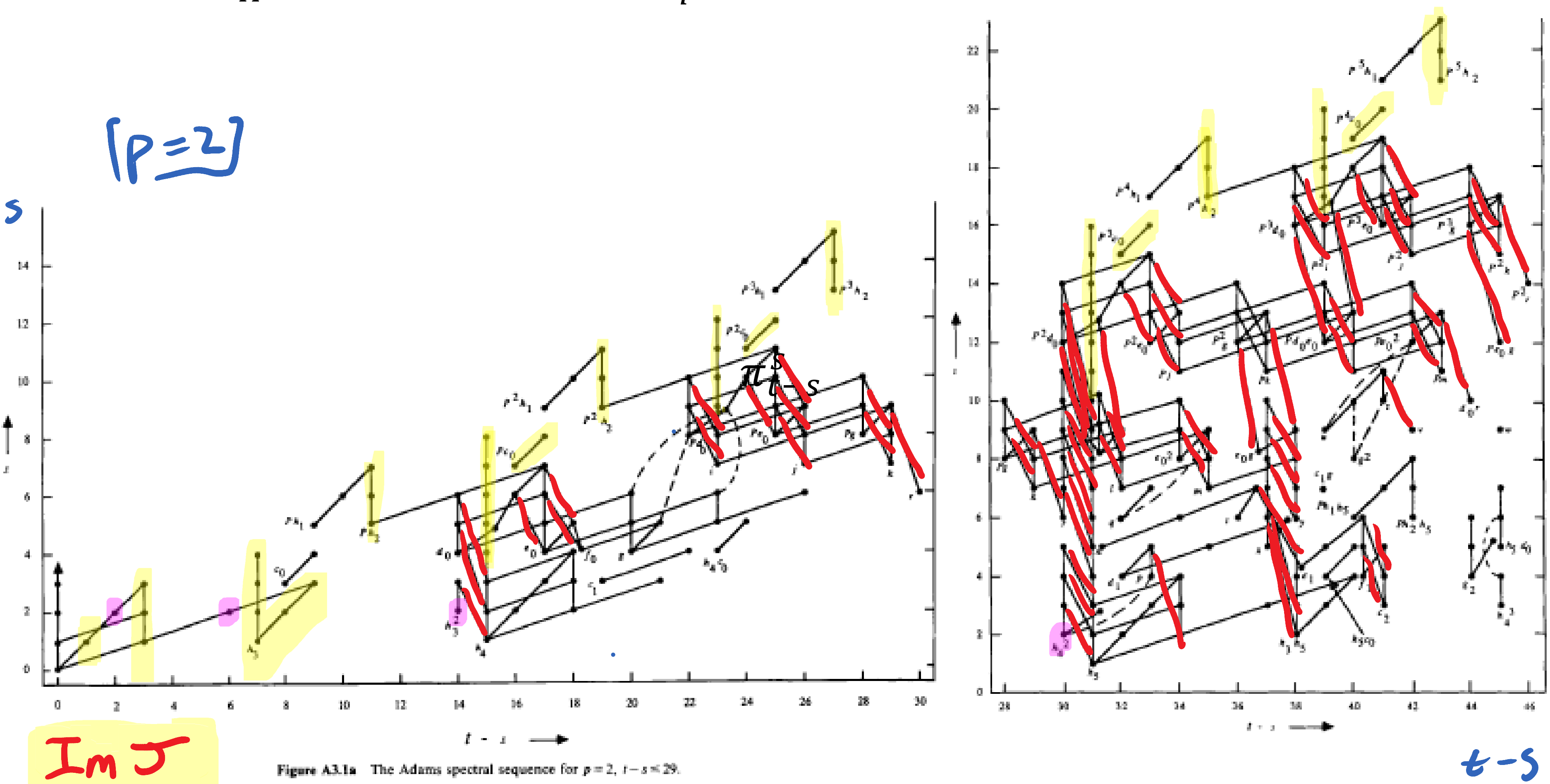

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= Kervaire Invariant 1

Kervaire Invariant

$$\Phi_K: \pi_n^s \to \mathbb{Z}/2$$

Browder:

$$(\Phi_K \neq 0) \Rightarrow \left(n = 2^k - 2\right)$$

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Computation in ASS: $\Phi_K \neq 0$ for

$$n \in \{2, 6, 14, 30, 62\}$$

Barratt-Jones-Mahowald '84

Kervaire Invariant

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Hill-Hopkins-Ravenel:

$\Phi_K = 0$ for all $n \geq 254$

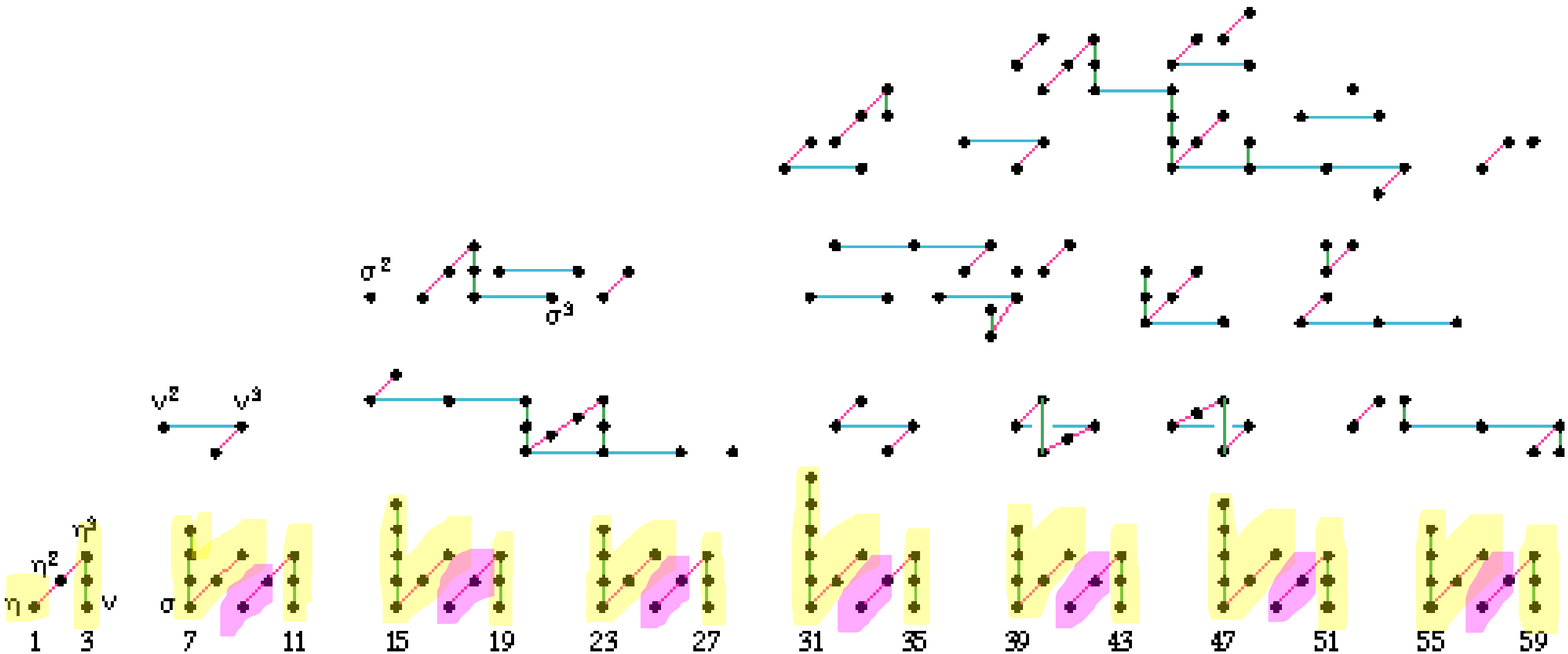
(Note: the case of $n = 126$ is still open)

Summary: Exotic spheres

$\Theta_n \neq 0$ if:

- $\Theta_n^{bp} \neq 0$:
  - $n \equiv 3\ (4)$ and $n \geq 7$
  - $n \equiv 1\ (4)$ and $n \notin \{1,5,13,29,61,125?\}$ [Kervaire]
- Remains to check: is $\frac{\pi_n^S}{Im\ J} \neq 0$ for
  - n even
  - $n \in \{1,5,13,29,61,125?\}$

Stable Homotopy Groups of Spheres at the prime 2



= Im J

= 8-fold periodic $\Rightarrow \dfrac{\pi_n^s}{\mathrm{Im} J} \neq 0$ for $n = 8k+2$

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  - $n \equiv 3\ (4)$ and $n \geq 7$
  - $n \equiv 1\ (4)$ and $n \notin \{1,5,13,29,61,125?\}$
- $\frac{\pi_n^s}{Im\ J} \neq 0$ for $n \equiv 2\ (8)$
- Remains to check: is $\frac{\pi_n^s}{Im\ J} \neq 0$ for
  - $n \equiv 0\ (4)$ or $n \equiv -2\ (8)$
  - $n \in \{1,5,13,29,61,125?\}$

Low dimensional computations

- Limitation: only know $(\pi_n^s)_2$ for $n \leq 63$
- $\left(\frac{\pi_n^s}{Im\ J}\right)_p = 0$ in this range for $p \geq 7$.

Low dimensional computations

Non-trivial elements in $Coker\ J$:

$$n \equiv 0\ (4)$$

Stem	p = 2	p = 3	p = 5
4	0	0	0
8	ε	0	0
12	0	0	0
16	η4	0	0
20	κbar	β1^2	0
24	h4 ε η	0	0
28	ε κbar	0	0
32	q	0	0
36	t	β2 β1	0
40	κbar^2	β1^4	0
44	g2	0	0
48	e0 r	0	0
52	κbar q	β2^2	0
56	κbar t	0	0
60	κbar^3	0	0

Low dimensional computations

Non-trivial elements in $Coker\ J$:
$n \equiv -2\ (8)$

■ = Kervaire inv 1

Stem	p = 2	p = 3	p = 5
6	ν^2	0	0
14	k	0	0
22	ε k	0	0
30	θ4	β1^3	0
38	y	β3/2	β1
46	w η	β2 β1^2	0
54	ν2^8 ν^2	0	0
62	h5 n	β2^2 β1	0

Low dimensional computations

Non-trivial elements in $Coker\ J$:

$n \in \{1,5,13,29,61\}$ [where $\Theta_n^{bp} = 0$ because of Kervaire classes]

Stem	p = 2	p = 3	p = 5
1	0	0	0
5	0	0	0
13	0	$\beta 1\ \alpha 1$	0
29	0	$\beta 2\ \alpha 1$	0
61	0	$\beta 4\ \alpha 1$	0

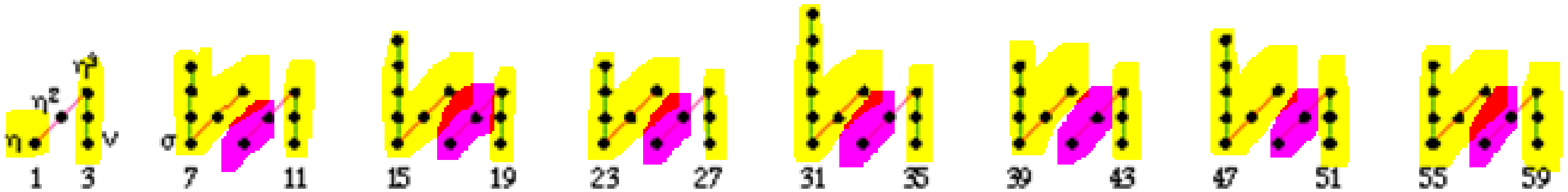
Low dimensional computations

Conclusion

For $n \leq 63$, the only n for which $\Theta_n = 0$ are:

$$1,2,3,4,5,6,12,61$$

Beyond low dimensions…



<u>Strategy</u>: try to demonstrate Coker J is non-zero in certain dimensions by producing infinite periodic families such as the one above.

Need to study <u>periodicity</u> in π_*^s

Periodicity in π_*^s

Work in stable cat:

Mod p^{i_0} Moore Spectrum:

$$S^0 \xrightarrow[p^{i_0}]{} S^0 \longrightarrow M(p^{i_0})$$

Periodicity in π_*^s

Work in stable cat:

Mod p^{i_0} Moore spectrum:

$$S^0 \xrightarrow{p^{i_0}} S^0 \longrightarrow M(p^{i_0})$$

Adams:

$M(p^{i_0})$ has "v_1 - self map"

$$\Sigma^{2i_1(p-1)} M(p^{i_0}) \xrightarrow{v_1^{i_1}} M(p^{i_0}) \qquad i_1 \gg 0$$

[not nilpotent]

Periodicity in π_*^s

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Devinatz-Hopkins-Smith Generalized Adam's result

$$\exists \quad \Sigma^{2i_2(p^2-1)} M(p^{i_0}, v_1^{i_1}) \xrightarrow{v_2^{i_2}} M(p^{i_0}, v_1^{i_1})$$

Not nilpotent "v_2-self-map"

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$$\Sigma^{N_2} M(p^{i_0}, v_1^{i_1}) \xrightarrow{v_2^{i_2}} M(p^{i_0}, v_1^{i_1}) \to M(p^{i_0}, v_1^{i_1}, v_2^{i_2})$$

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$$\Sigma^{N_3} M(p^{i_0}, v_1^{i_1}, v_2^{i_2}) \xrightarrow{v_3^{i_3}} M(p^{i_0}, v_1^{i_1}, v_2^{i_2}) \to M(p^{i_0}, v_1^{i_1}, v_2^{i_2}, v_3^{i_3})$$

$N_3 = 2i_3(p^3-1)$, Not Nilpotent

$|v_k| = 2(p^k-1)$

Periodicity in π_*^S

$$M^0_{(i_0, \ldots, i_k)} := \Sigma^{-\dim} M(p^{i_0}, v_1^{i_1}, \ldots, v_k^{i_k})$$

$$\alpha \in (\pi_n^S)_{(p)}$$

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$$\begin{array}{c} \pi_n^S \\ \big\downarrow p^{i_0} \\ \pi_n^S \end{array}$$

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$$\begin{tikzcd}
\pi_n^S \arrow[d, "p^{i_0}"] & \pi_n^S M^0_{i_0} \ni \alpha_0 \arrow[l] \\
\pi_n^S &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & & \alpha_1 \arrow[d, phantom, "\curvearrowright"] & \\
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\end{tikzcd}$$

Periodicity in π_*^S

α lifts to $M^0_{i_0,\dots,i_{k-1}}$, no further

$\Rightarrow \alpha$ is v_k-periodic

$$
\begin{tikzcd}
 & & \alpha_1 \arrow[d, mapsto] & \\
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\pi_n^S & \pi_{n+N_1}^S M^0_{i_0} & \pi_{n+N_2}^S M^0_{i_0,i_1} &
\end{tikzcd}
$$

Periodicity in π_*^S

α lifts to $M^0_{i_0,\ldots,i_{k-1}}$, no farther

$\Rightarrow$ α is v_k-periodic

α generates an infinite family in π_*^S:

$$\pi_n^S M^0_{i_0\cdots i_{k-1}} \xrightarrow{v_k^{si_k}} \pi^S_{n+sN_k} M^0_{i_0\cdots i_{k-1}} \to \pi^S_{n+N_k}$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \\ \alpha_{k-1} & \longmapsto & v_k^{si_k}\alpha \end{array}$$

$$\text{period} = 2i_k(p^k-1) \qquad \forall\, s \in \mathbb{N}$$

Periodicity in π_*^s

E.g. "Greek letter elts"

$$\begin{tikzcd} S^{2j(p-1)-1} \arrow[r, hook] \arrow[rrr, bend right, "\alpha_{j/i}"'] & \Sigma^{2j(p-1)} M_i^0 \arrow[r, "v_1^j"] & M_i^0 \arrow[r] & S^0 \end{tikzcd}$$

Periodicity in π_*^s

E.g. "Greek letter elts"

$$\begin{tikzcd} S^{2j(p-1)-1} \arrow[r, hook] \arrow[rrr, bend right, "\alpha_{j/i}"'] & \Sigma^{2j(p-1)} M_i^0 \arrow[r, "v_1^j"] & M_i^0 \arrow[r] & S^0 \end{tikzcd}$$

$$\begin{tikzcd} S^{2k(p^2-1)-2j(p-1)-2} \arrow[r, hook] \arrow[rrr, bend right, "\beta_{k/j,i}"'] & \Sigma^{2k(p^2-1)} M_{i,j}^0 \arrow[r, "v_2^k"] & M_{i,j}^0 \arrow[r] & S^0 \end{tikzcd}$$

Periodicity in π_*^s

E.g. "Greek letter elts"

$$\begin{tikzcd} S^{2j(p-1)-1} \arrow[r, hook] \arrow[rrr, bend right, "\alpha_{j/i}"'] & \Sigma^{2j(p-1)} M_i^0 \arrow[r, "v_1^j"] & M_i^0 \arrow[r] & S^0 \end{tikzcd}$$

$$\begin{tikzcd} S^{2k(p^2-1)-2j(p-1)-2} \arrow[r, hook] \arrow[rrr, bend right, "\beta_{k/j,i}"'] & \Sigma^{2k(p^2-1)} M_{i,j}^0 \arrow[r, "v_2^k"] & M_{i,j}^0 \arrow[r] & S^0 \end{tikzcd}$$

$\gamma_{\ell/k,j,i}$

v_3-periodic

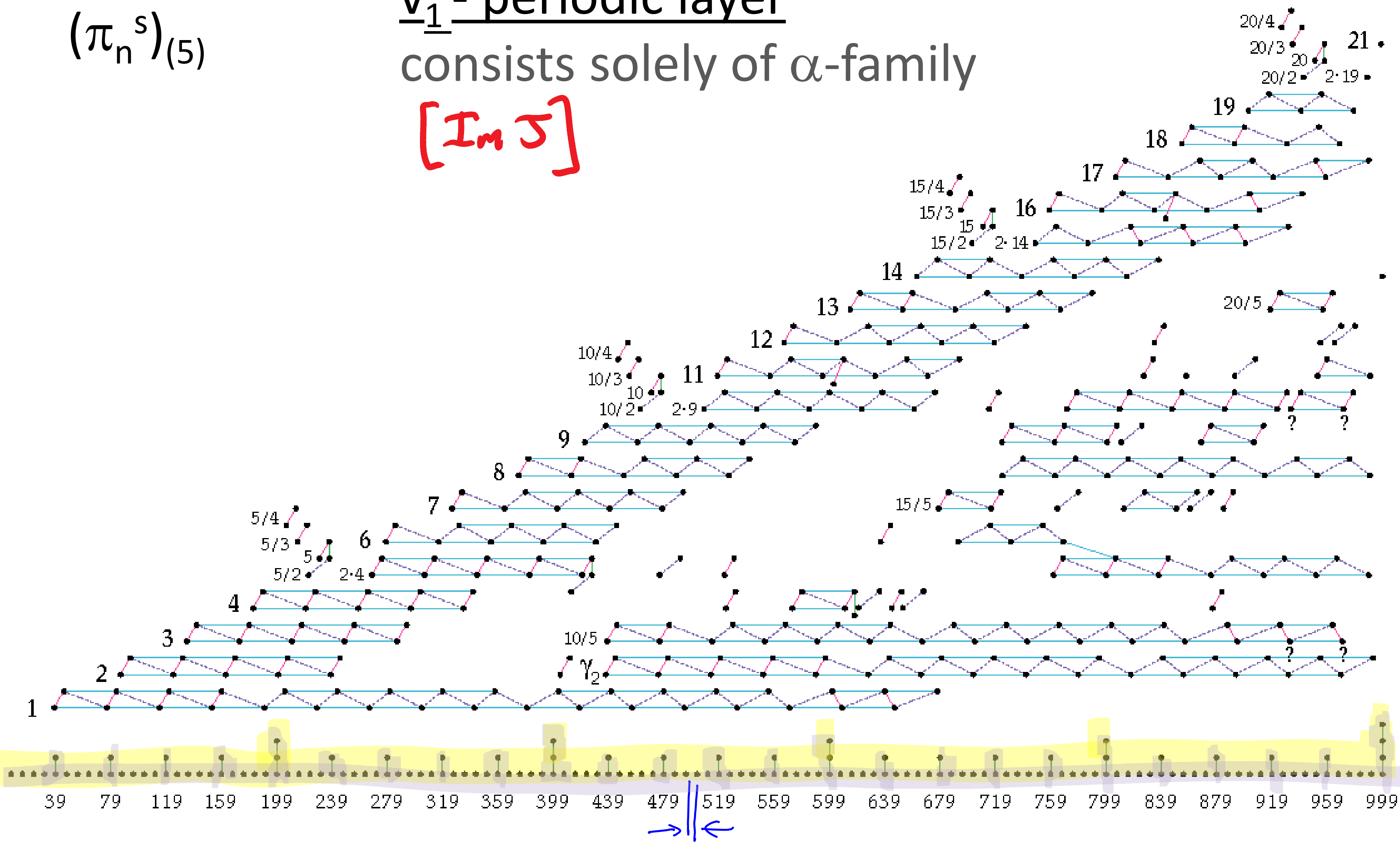
$\vdots$

$(\pi_n^s)_{(5)}$

v_1 - periodic layer

consists solely of α-family

[Im J]

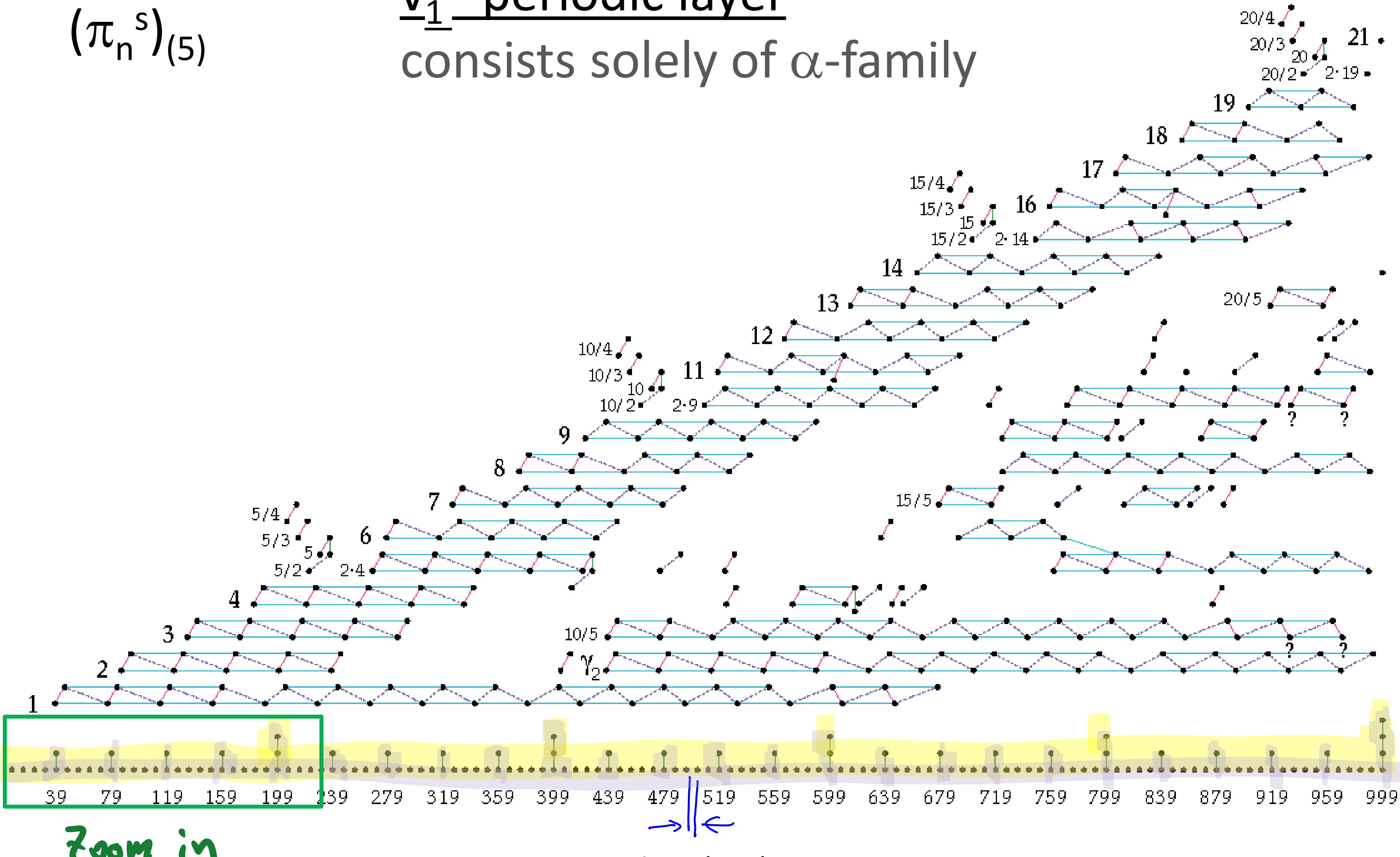


period = 2(p-1) = 8

v_1 - periodic layer

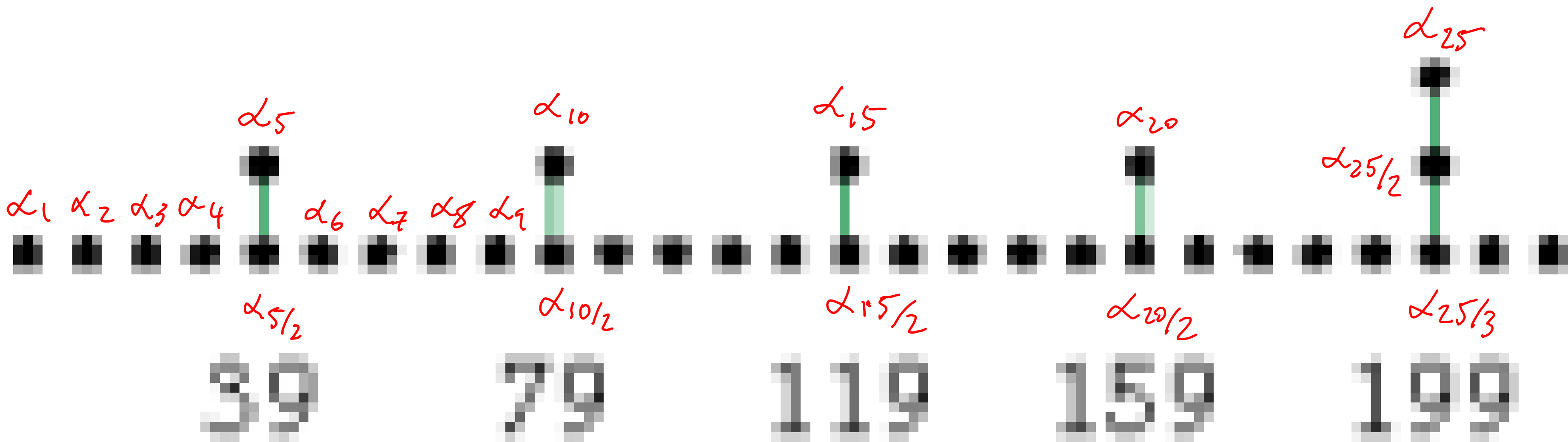
consists solely of α-family

$(\pi_n^s)_{(5)}$



period = 2(p-1) = 8

Greek letter notation: the α-family



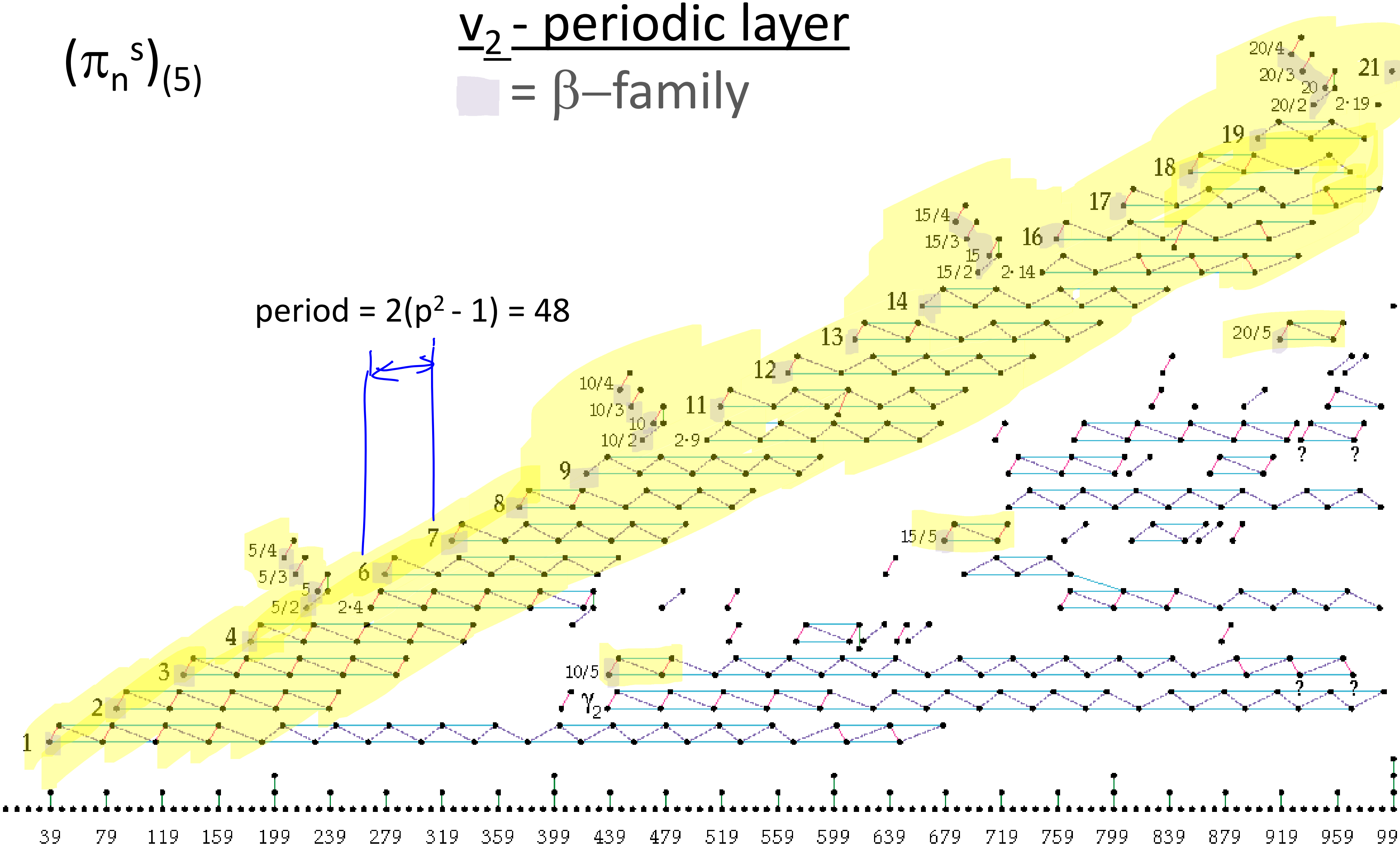
$\alpha_{i/j}$ is p^j-torsion

$(\alpha_i := \alpha_{i/1})$

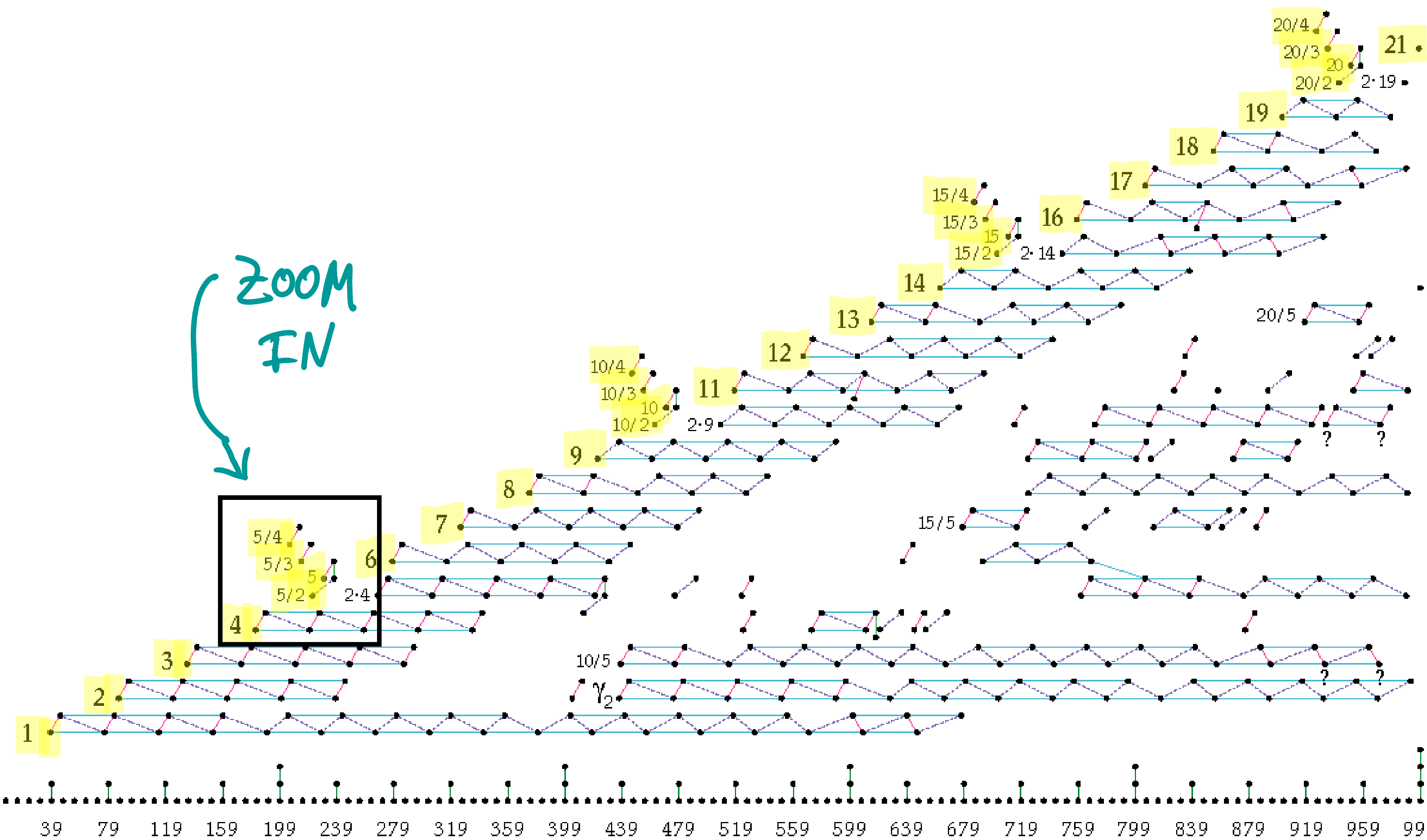
$(\pi_n^s)_{(5)}$

v_2 - periodic layer

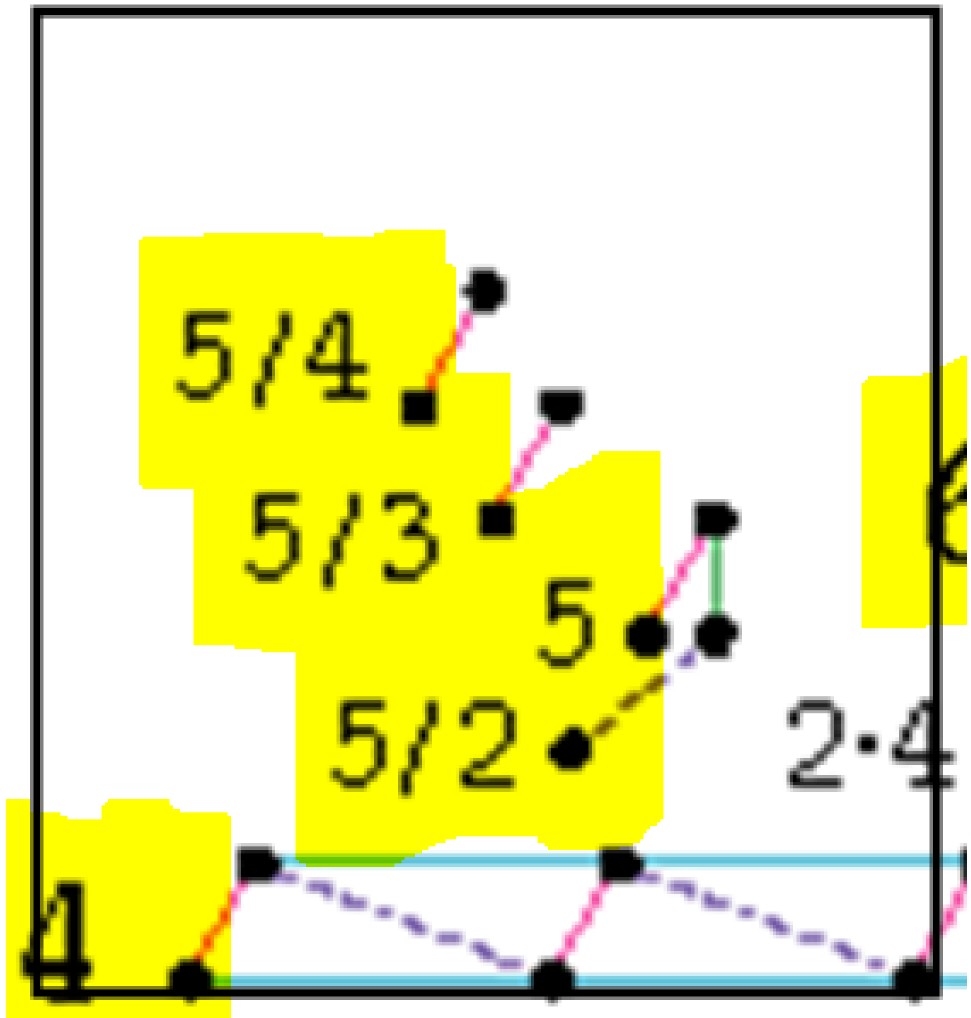
= β–family



$(\pi_n^s)_{(5)}$

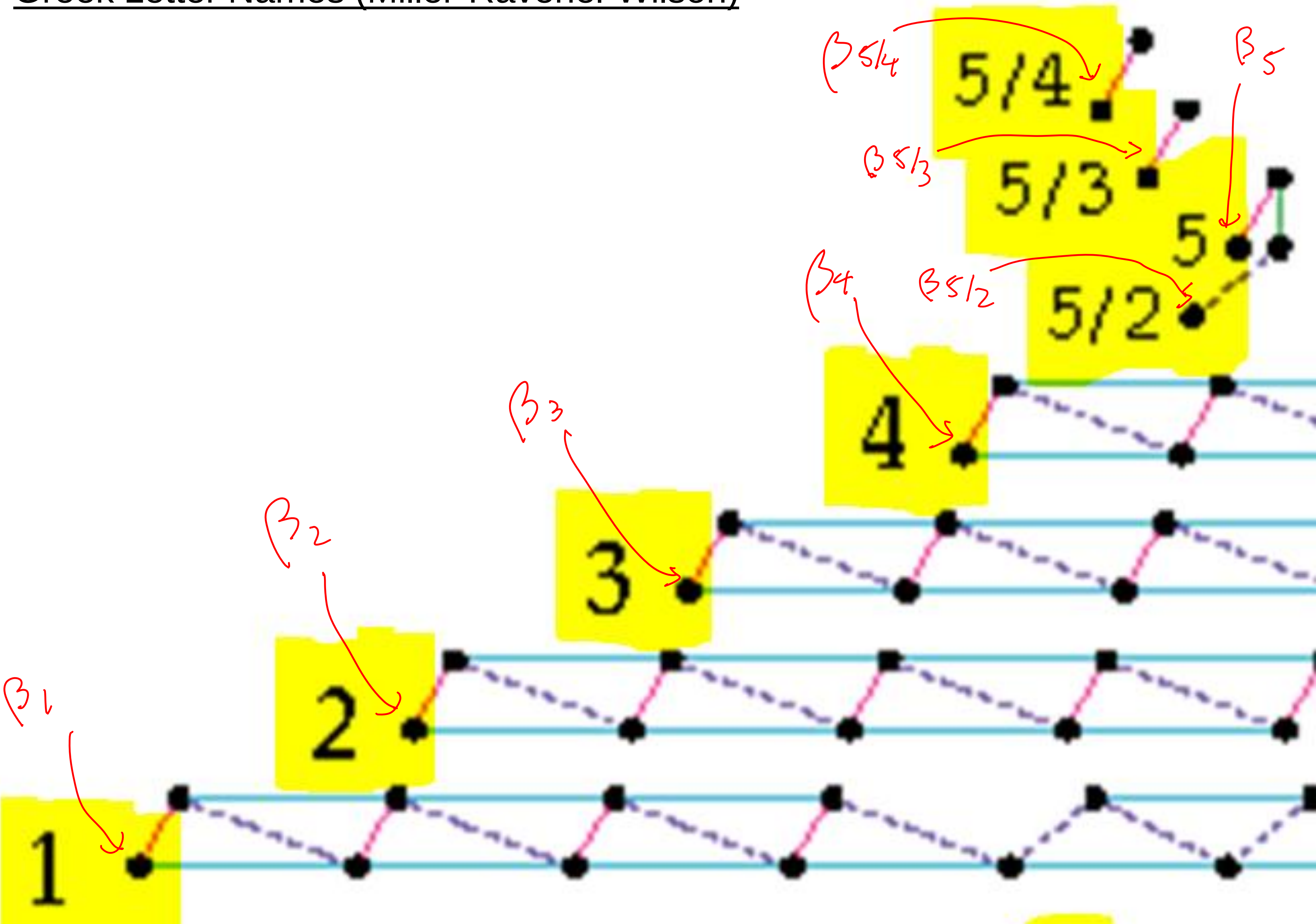

ZOOM IN
1
2
3
4
5/4
5/3
5
5/2
2·4
6
7
8
9
10/4
10/3
10
10/2
2·9
11
12
13
14
15/4
15/3
15
15/2
2·14
16
17
18
19
20/4
20/3
20
20/2
2·19
21
20/5
15/5
10/5
γ_2
?
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

v_1-torsion in the v_2-family



“5/4” $\xrightarrow{v_1}$ “5/3” $\xrightarrow{v_1}$ “5/2” $\xrightarrow{v_1}$ “5” $\xrightarrow{v_1}$ 0

Greek Letter Names (Miller-Ravenel-Wilson)

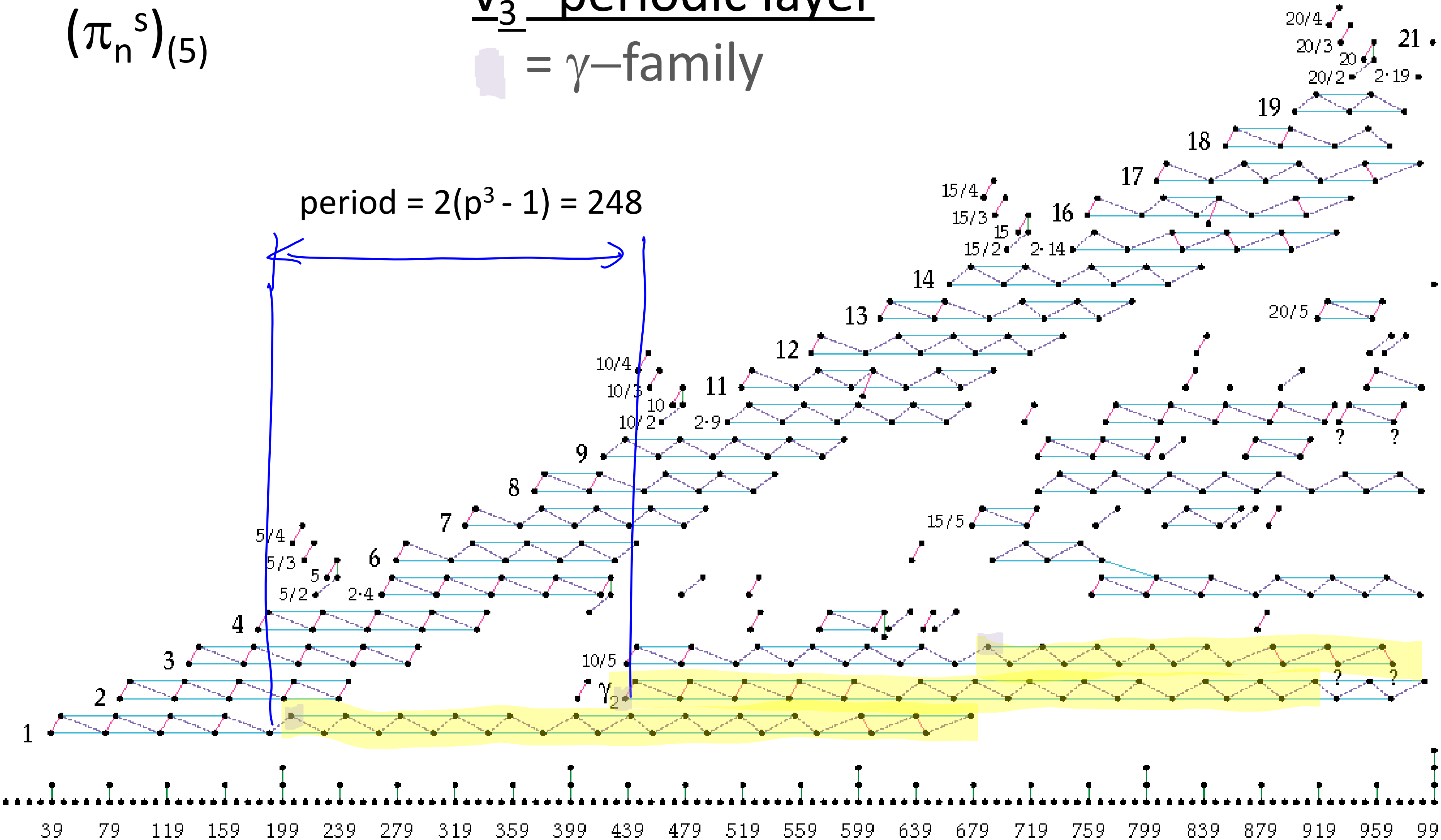


$(\pi_n^s)_{(5)}$

v_3 - periodic layer

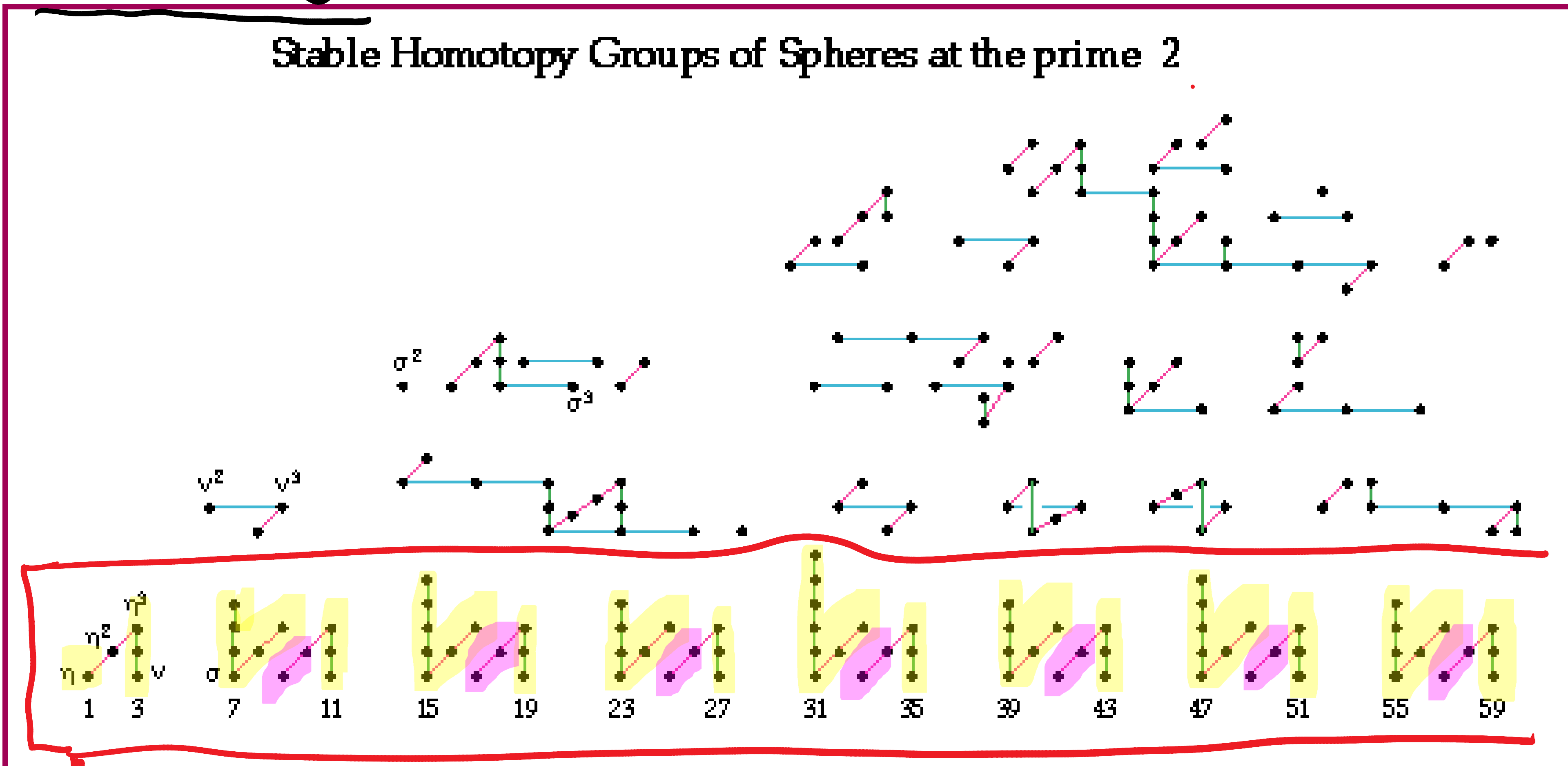
= γ–family

period = $2(p^3 - 1) = 248$



- v_1-periodicity – completely understood
- v_2-periodicity – know a lot for $p \geq 5$
  - Knowledge for $p = 2,3$ is subject of current research.
  - For Θ_n, we will see $p = 2$ dominates the discussion
- v_3-periodicity – know next to nothing!

Back to Θ_n:



v_1-periodic

(yellow) $= \mathrm{Im}\, J$

(pink) $\Rightarrow \Theta_n \neq 0$ for $n \equiv 2 \ (8)$

Exotic spheres from β-family

- $\beta_k = \beta_{k/1,1}$ exists for $p \geq 5$ and $k \geq 1$ [Smith-Toda]

  $\Theta_n \neq 0$ for $n \equiv -2(p-1) - 2 \; mod \; 2(p^2-1)$

$$\Sigma^{2(p^2-1)} M^0_{1,1} \xrightarrow[v_2]{} M^0_{1,1}$$

Coker J

n = 0 mod 4

Stem	p = 2	p = 3	p = 5
4	0	0	0
8	ε	0	0
12	0	0	0
16	η4	0	0
20	κbar	β1^2	0
24	h4 ε η	0	0
28	ε κbar	0	0
32	q	0	0
36	t	β2 β1	0
40	κbar^2	β1^4	0
44	g2	0	0
48	e0 r	0	0
52	κbar q	β2^2	0
56	κbar t	0	0
60	kbar^3	0	0
64		0	0
68		<α1,β3/2,β2>	0
72		β2^2 β1^2	0
76		0	β1^2
80		0	0
84		β5 β1	0
88		0	0
92		β6/3 β1	0
96		0	0
100		β2 β5	0
104			0
108			0
112			0
116			0
120			0
124			β2 β1
128			0
132			0
136			0
140			0
144			0
148			0
152			β1^4
156			0
160			0

n = -2 mod 8 (including Kervaire Inv 1)

Stem	p = 2	p = 3	p = 5
6	ν^2	0	0
14	k	0	0
22	ε k	0	0
30	θ4	β1^3	0
38	y	β3/2	β1
46	w η	β2 β1^2	0
54	ν2^8 ν^2	0	0
62	h5 n	β2^2 β1	0
70		0	0
78		β2^3	0
86		β6/2	β2
94		β5	0
102		β6/3 β1^2	0
110			0
118			0
126			0
134			β3
142			0
150			0
158			0
166			0
174			0
182			β4
190			β1^5
198			0
206			β5/4
214			β5/3
222			β5/2
230			β5
238			β2 β1^4
246			0
254			0
262			0
270			0
278			β1
286			β3 β1^4
294			0
302			0
310			0
318			0

n = 2^k - 3 (where Θ_n^bp = 0 because of Kervaire class)

Stem	p = 2	p = 3	p = 5
1	0	0	0
5	0	0	0
13	0	β1 α1	0
29	0	β2 α1	0
61	0	β4 α1	0
125?			0

Cohomology theories

- Use homology/cohomology to study homotopy
- A *cohomology theory* is a contravariant functor

$$E: \{\text{Topological spaces}\} \longrightarrow \{\text{graded ab groups}\}$$
$$X \longrightarrow E^*(X)$$

- Homotopy invariant: $f \simeq g \Rightarrow E(f) = E(g)$
- Excision: $Z = X \cup Y$ (CW complexes)

$$\cdots \to E^*(Z) \to E^*(X) \oplus E^*(Y) \to E^*(X \cap Y) \to$$

Cohomology theories

- Use homology/cohomology to study homotopy
- A *cohomology theory* is a contravariant functor

$$E: \{\text{Topological spaces}\} \longrightarrow \{\text{graded ab groups}\}$$
$$X \longrightarrow E^*(X)$$

- Homotopy groups:

$$\pi_n(E) := E^{-n}(pt)$$

(Note, in the above, n may be negative)

Cohomology theories

- Example: singular cohomology
  - $E^n(X) = H^n(X)$
  - $\pi_n(H) = \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & \text{else.} \end{cases}$

- Example: Real K-theory
  - $KO^0(X) = KO(X) =$ Grothendieck group of $\mathbb{R}$-vector bundles over X.
  - $\pi_* KO = (\mathbb{Z}, \mathbb{Z}/2\,, \mathbb{Z}/2\,, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2\,, \mathbb{Z}/2\,, 0, \mathbb{Z}, 0, 0, 0 \ldots)$

Hurewicz Homomorphism

- A cohomology theory E is a (commutative) *ring theory* if its associated cohomology theory has "cup products"

$E^*(X)$ is a graded commutative ring

- Such cohomology theories have a *Hurewicz homomorphism*:

$$h_E \colon \pi_*^s \to \pi_* E$$

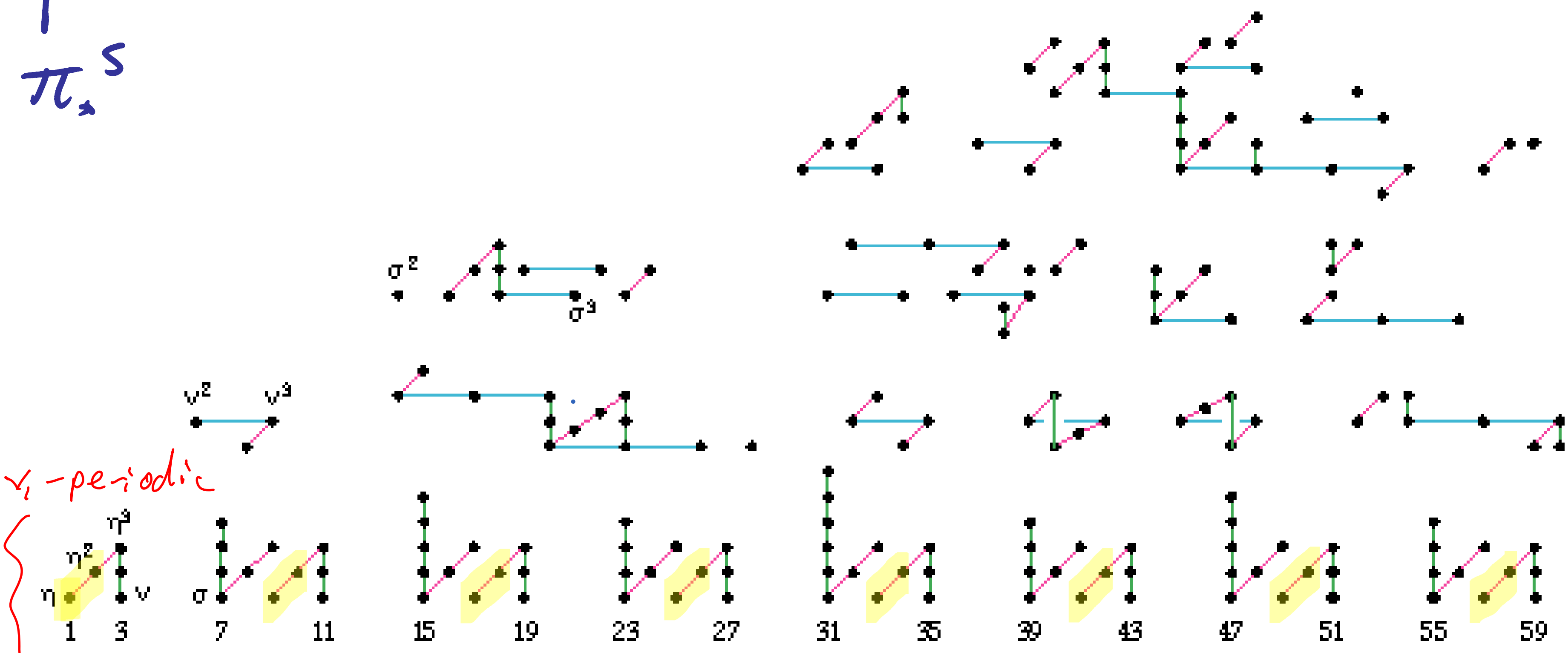
Example: H detects $\pi_0^s = \mathbb{Z}$.

Example: KO (real K-theory)

$$\pi_* KO = \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \ldots$$

$$\pi_*^s \xrightarrow{h_{KO}} \pi_* KO$$

Stable Homotopy Groups of Spheres at the prime 2



- To get more elements of Θ_n, need to start looking at v_2-periodic homotopy.

- Need a cohomology theory which sees a bunch of v_2-periodic classes in its Hurewicz homomorphism

- $tmf^*(X)$ - topological modular forms!

Topological Modular Forms

KO

- v_1-periodic
  - 8-periodic
- Multiplicative group
- Bernoulli numbers

TMF

- v_2-periodic
  - 576-periodic
- Elliptic curves
- Eisenstein series
  (modular forms)

192-periodic at $p = 2$

144-periodic at $p = 3$

Topological Modular Forms

- There is a descent spectral sequence:

$$H^s\left(\mathcal{M}_{ell}; \omega^{\otimes t}\right) \Rightarrow \pi_{2t-s}TMF$$

- Edge homomorphism:

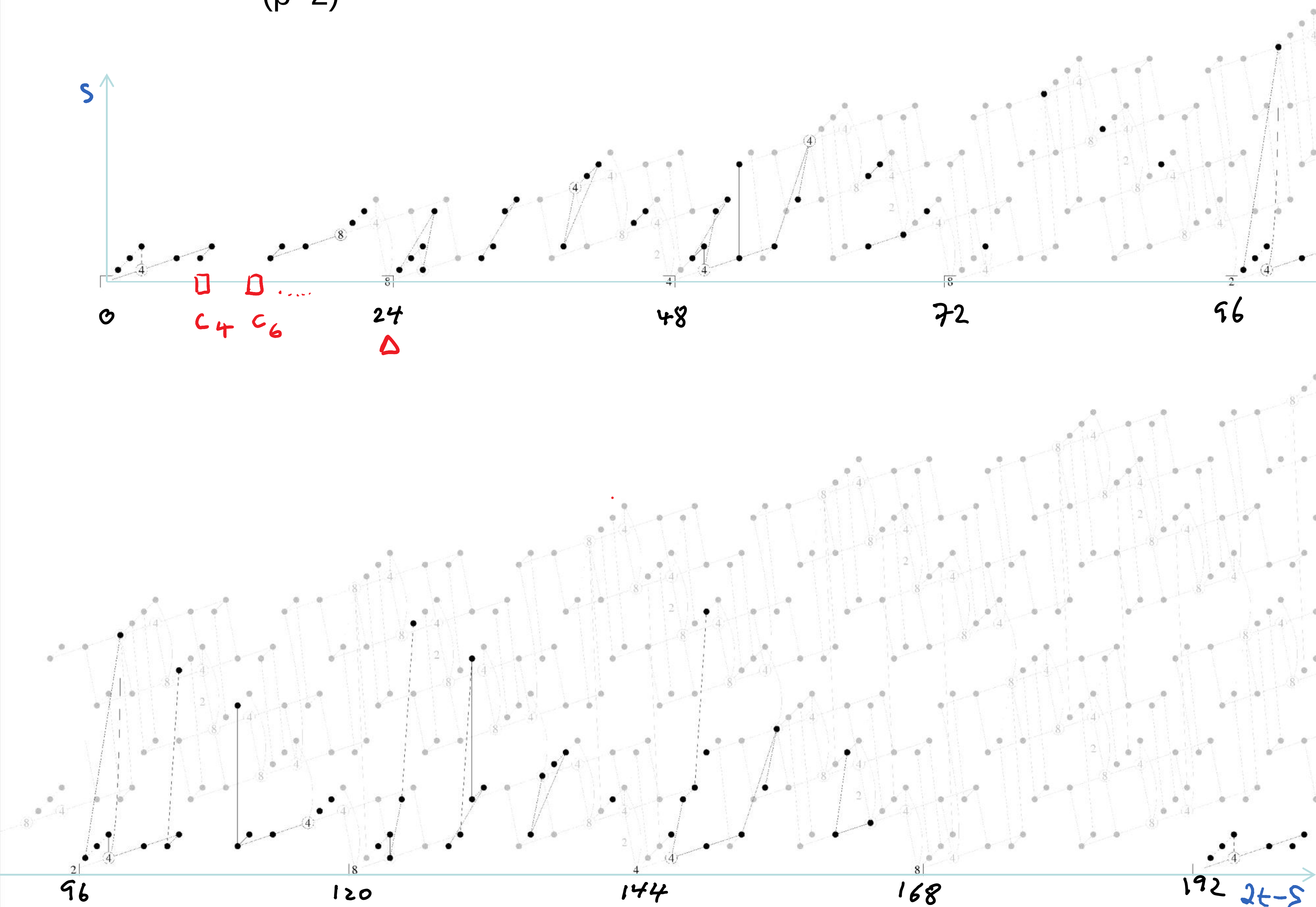
$\pi_{2k}TMF \rightarrow$ Ring of integral modular forms

(rationally this is an iso)

- $\pi_* TMF$ has a bunch of 2 and 3-torsion, and the descent spectral sequence is highly non-trivial at these primes.

The decent spectral sequence for TMF (p=2)

$$H^s(\mathcal{M}_{ell}; \omega^{\otimes t}) \Rightarrow \pi_{2t-s} TMF$$



Exotic spheres from β-family

- $\beta_k = \beta_{k/1,1}$ exists for $p \geq 5$ and $k \geq 1$ [Smith-Toda]

  $\Theta_n \neq 0$ for $n \equiv -2(p-1) - 2 \; mod \; 2(p^2 - 1)$

- β_k exists for $p = 3$ and $k \equiv 0,1,2,3,5,6 \; (9)$ [B-Pemmaraju]

  [Shimomura]

  $\Theta_n \neq 0$ for $n \equiv -6, 10, 26, 42, 74, 90 \text{ mod } 144$

$$\Sigma^{144} M^0_{1,1} \xrightarrow[v_2^9]{} M^0_{1,1}$$ [uses TMF]

Exotic spheres from β-family

- $\beta_k = \beta_{k/1,1}$ exists for $p \geq 5$ and $k \geq 1$ [Smith-Toda]

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- β_k exists for $p = 3$ and $k \equiv 0,1,2,3,5,6 \; (9)$ [B-Pemmaraju]

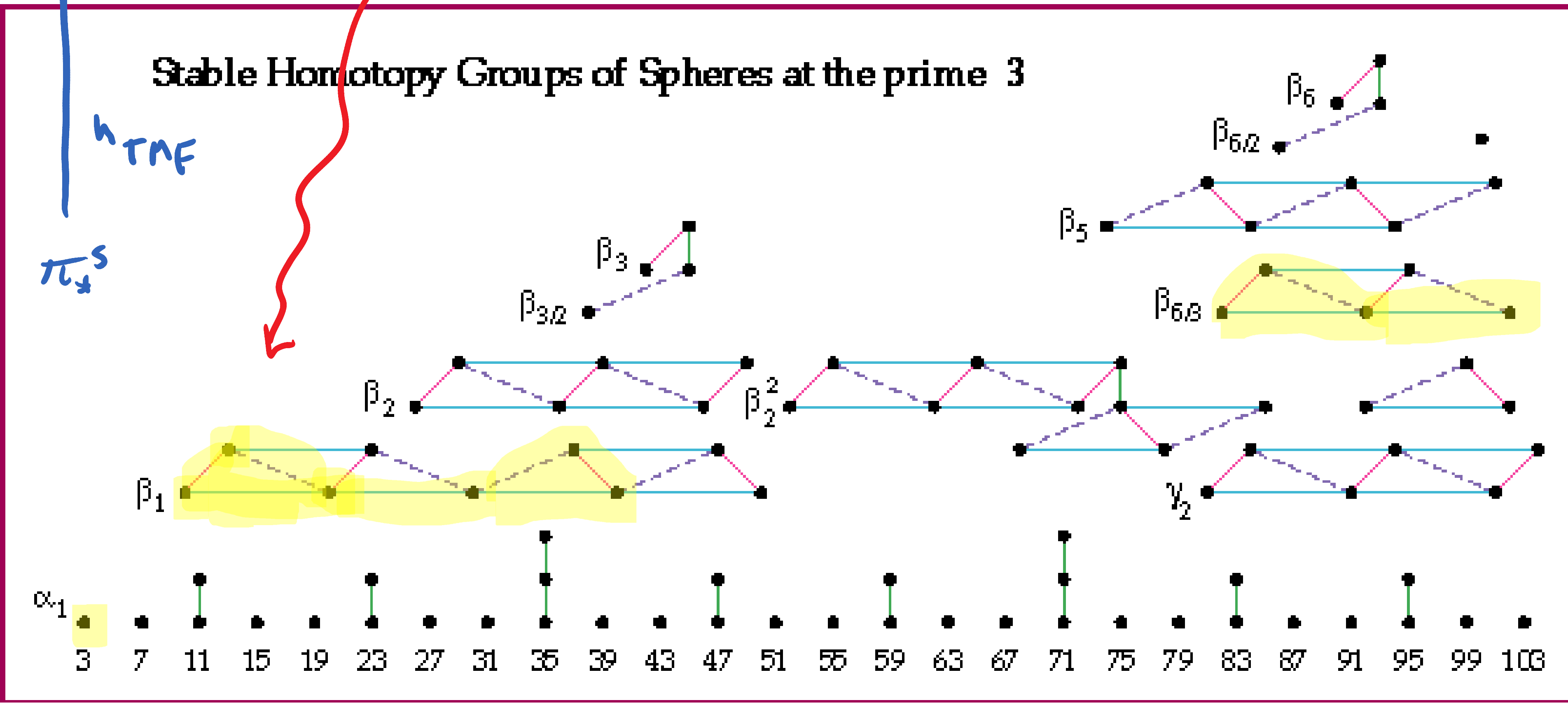
  [Shimomura]

  $\Theta_n \neq 0$ for $n \equiv -\underbrace{6, 10, 26, 42, 74, 90}_{\text{all} \equiv 2 \; (8)} \bmod 144$

Hurewicz image of TMF (p = 3)

144 periodic

$$\pi_*^s \xrightarrow{h_{TMF}} \pi_* TMF_{(3)}$$



Coker J

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4	0	0	0
8	ε	0	0
12	0	0	0
16	η4	0	0
20	κbar	β1^2	0
24	h4 ε η	0	0
28	ε κbar	0	0
32	q	0	0
36	t	β2 β1	0
40	κbar^2	β1^4	0
44	g2	0	0
48	e0 r	0	0
52	κbar q	β2^2	0
56	κbar t	0	0
60	kbar^3	0	0
64		0	0
68		<α1,β3/2,β2>	0
72		β2^2 β1^2	0
76		0	β1^2
80		0	0
84		β5 β1	0
88		0	0
92		β6/3 β1	0
96		0	0
100		β2 β5	0
104			0
108			0
112			0
116			0
120			0
124			β2 β1
128			0
132			0
136			0
140			0
144			0
148			0
152			β1^4
156			0
160			0

n = -2 mod 8 (including Kervaire Inv 1)

Stem	p = 2	p = 3	p = 5
6	v^2	0	0
14	k	0	0
22	ε k	0	0
30	θ4	β1^3	0
38	y	β3/2	β1
46	w η	β2 β1^2	0
54	v2^8 v^2	0	0
62	h5 n	β2^2 β1	0
70		0	0
78		β2^3	0
86		β6/2	β2
94		β5	0
102		β6/3 β1^2	0
110			0
118			0
126			0
134			β3
142			0
150			0
158			0
166			0
174		β1^3	0
182		β3/2	β4
190		β2 β1^2	β1^5
198			0
206		β2^2 β1	β5/4
214			β5/3
222		β2^3	β5/2
230		β6/2	β5
238		β5	β2 β1^4
246		β6/3 β1^2	0
254			0
262			0
270			0
278			β1
286			β3 β1^4
294			0
302			0
310			0
318		β1^3	0

n = 2^k - 3 (where Θ_n^bp = 0 because of Kervaire class)

Stem	p = 2	p = 3	p = 5
1	0	0	0
5	0	0	0
13	0	β1 α1	0
29	0	β2 α1	0
61	0	β4 α1	0
125?			0

v_2-periodicity at the prime 2

Thm [B-Hill-Hopkins-Mahowald]

$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{1,4} \longrightarrow M^0_{1,4}$$

Uses TMF

v_2-periodicity at the prime 2

Thm [B-Hill-Hopkins-Mahowald]

$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{1,4} \longrightarrow M^0_{1,4}$$

Uses TMF

Thm [B-Mahowald]

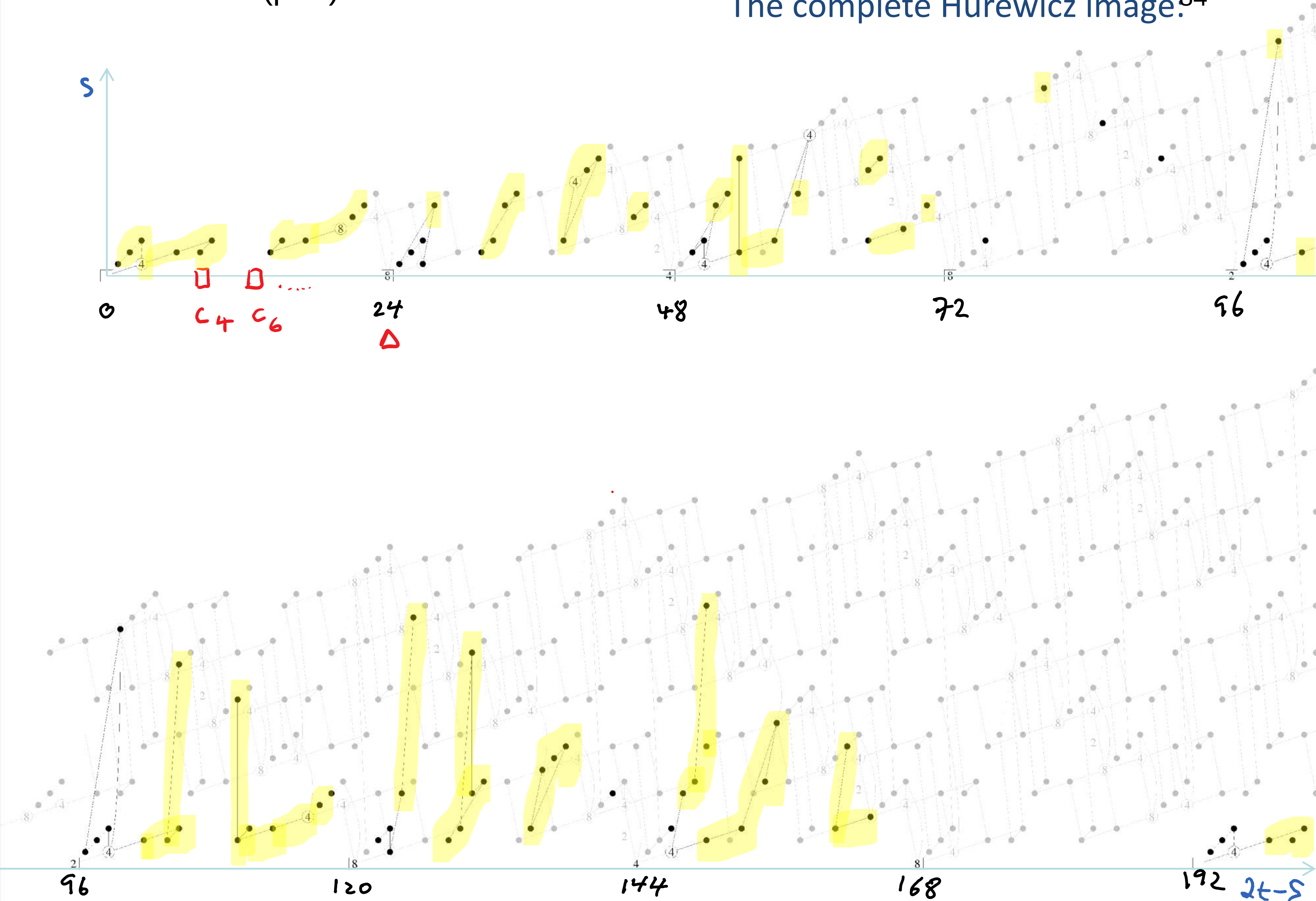
$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{3,8} \longrightarrow M^0_{3,8}$$

Allows for complete determination of Hurewicz image $p=2$

The decent spectral sequence for TMF
(p=2)

Thm: (B-Mahowald)

The complete Hurewicz image:

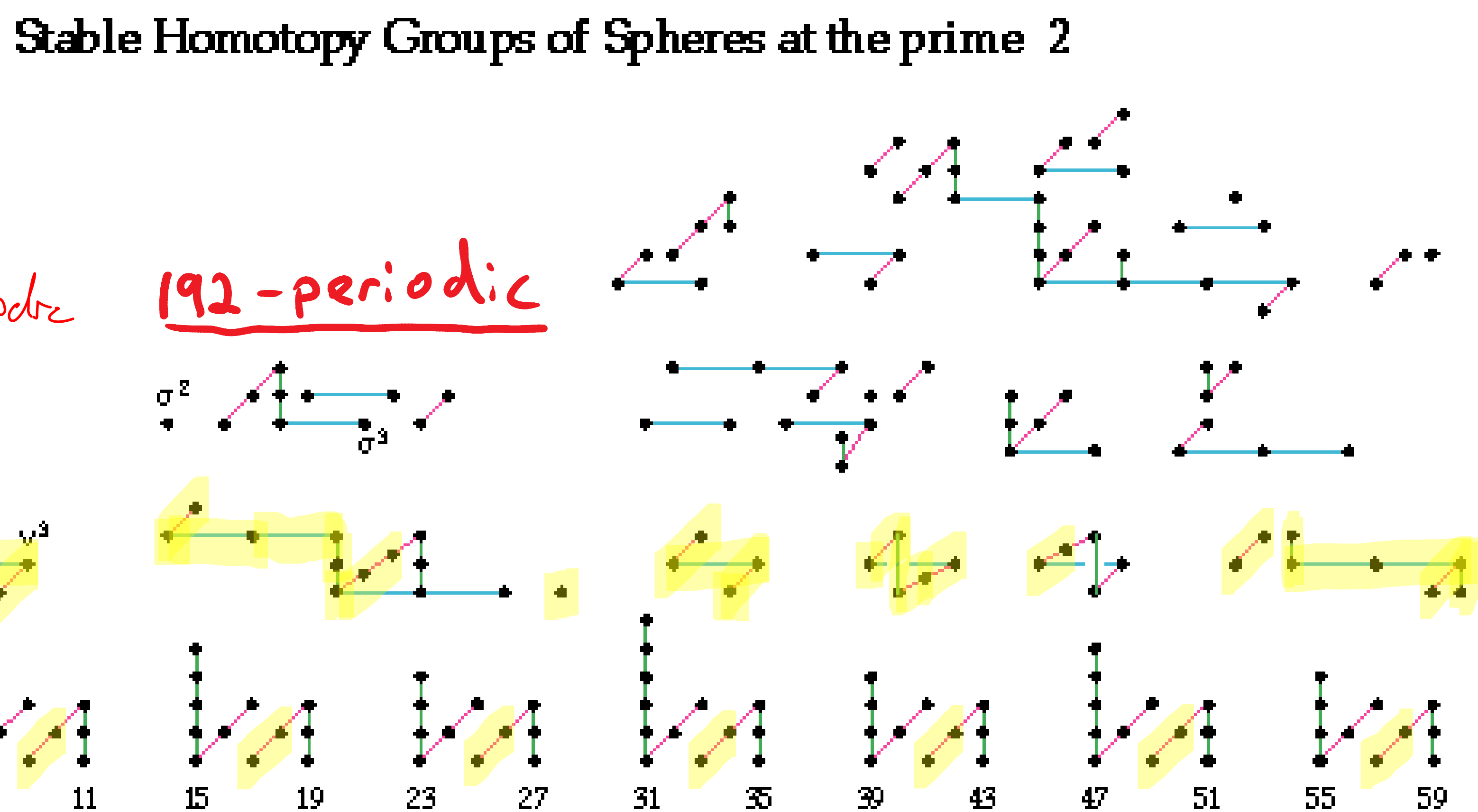


Hurewicz image of TMF (p = 2)

$\pi_* TMF_{(2)}$

$\uparrow h_{TMF}$

π_*^S



192-periodic

v_2-periodic

v_1-periodic

Coker J

n = 0 mod 4

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8	ε	0	0
12	0	0	0
16	η4	0	0
20	κbar	β1^2	0
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28	ε κbar	0	0
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40	κbar^2	β1^4	0
44	g2	0	0
48	e0 r	0	0
52	κbar q	β2^2	0
56	κbar t	0	0
60	kbar^3	0	0
64		0	0
68	v2^8 k v^2	<α1,β3/2,β2>	0
72		β2^2 β1^2	0
76		0	β1^2
80	kbar^4	0	0
84		β5 β1	0
88		0	0
92		β6/3 β1	0
96		0	0
100	kbar^5	β2 β5	0
104	v2^16 ε		0
108			0
112			0
116	2v2^16 kbar		0
120			0
124	v2^16 k^2		β2 β1
128	v2^16 q		0
132			0
136	<v2^16 k kbar,2,v^2>		0
140			0
144		v2^9	0
148	v2^16 ε kbar		0
152			β1^4
156	<Δ^6 v^2,2v,η^2>		0
160			0

n = -2 mod 8 (including Kervaire Inv 1)

Stem	p = 2	p = 3	p = 5
6	v^2	0	0
14	k	0	0
22	ε k	0	0
30	θ4	β1^3	0
38	y	β3/2	β1
46	w η	β2 β1^2	0
54	v2^8 v^2	0	0
62	h5 n	β2^2 β1	0
70	<kbar w,v,η>	0	0
78		β2^3	0
86		β6/2	β2
94		β5	0
102	v2^16 v^2	β6/3 β1^2	0
110	v2^16 k		0
118	v2^16 η^2 kbar		0
126			0
134			β3
142	v2^16 η w		0
150	(v2^16 ε kbar)η^2	v2^9	0
158			0
166			0
174	beta32/8	β1^3	0
182	beta32/4	β3/2	β4
190		β2 β1^2	β1^5
198	v2^32 v^2		0
206	k	β2^2 β1	β5/4
214	ε k		β5/3
222		β2^3	β5/2
230		β6/2	β5
238	w η	β5	β2 β1^4
246	v2^8 v^2	β6/3 β1^2	0
254			0
262	<kbar w,v,η>		0
270			0
278			β1
286			β3 β1^4
294	v2^16 v^2	v2^18	0
302	v2^16 k		0
310	v2^16 η^2 kbar		0
318		β1^3	0

n = 2^k - 3 (where Θ_n^bp = 0 because of Kervaire class)

Stem	p = 2	p = 3	p = 5
1	0	0	0
5	0	0	0
13	0	β1 α1	0
29	0	β2 α1	0
61	0	β4 α1	0
125?	w kbar^4		0

= in tmf

= not in tmf, not known to be v2-periodic

= not in tmf, but v2-periodic

= Kervaire

= trivial