Modular Forms in Topology

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Algebraic Topology:

Geometry

Algebra

Algebraic Topology:

Geometry

Manifolds,
topological spaces

Algebra

numbers,
groups

Algebraic Topology:

$$\text{Geometry} \xrightarrow{\text{INVARIANTS}} \text{Algebra}$$

Manifolds,
topological spaces

numbers,
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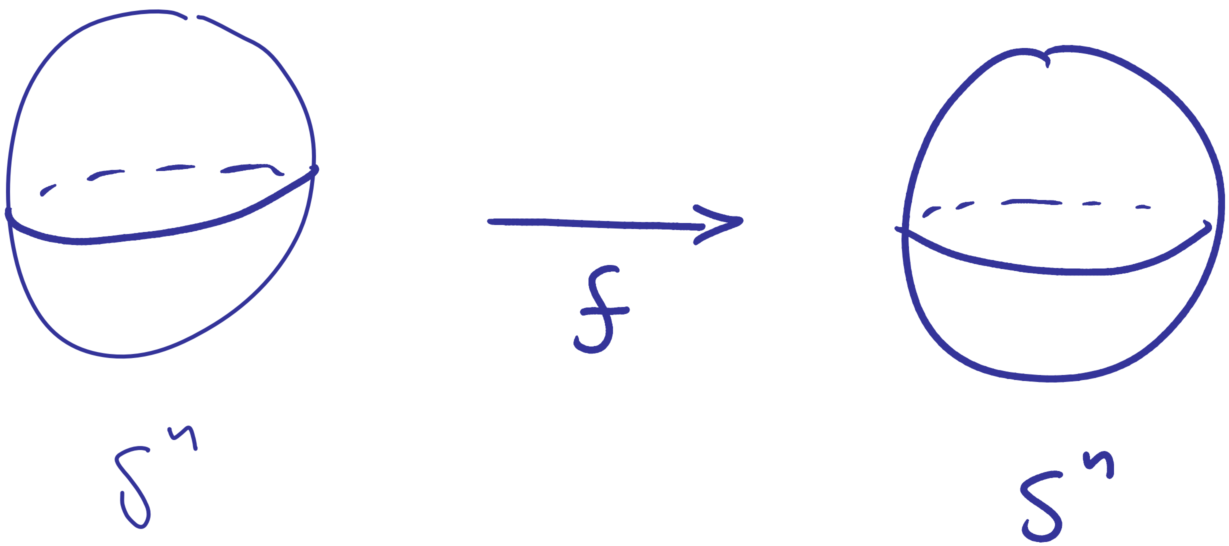
Algebraic Topology:

Geometry $\overset{\text{INVARIANTS}}{\rightsquigarrow}$ Algebra

Geometry	Algebra
Manifolds, topological spaces	numbers, groups
(continuous)	(discrete)

Homotopy $\longmapsto$ Equality

Example: degree of a mapping



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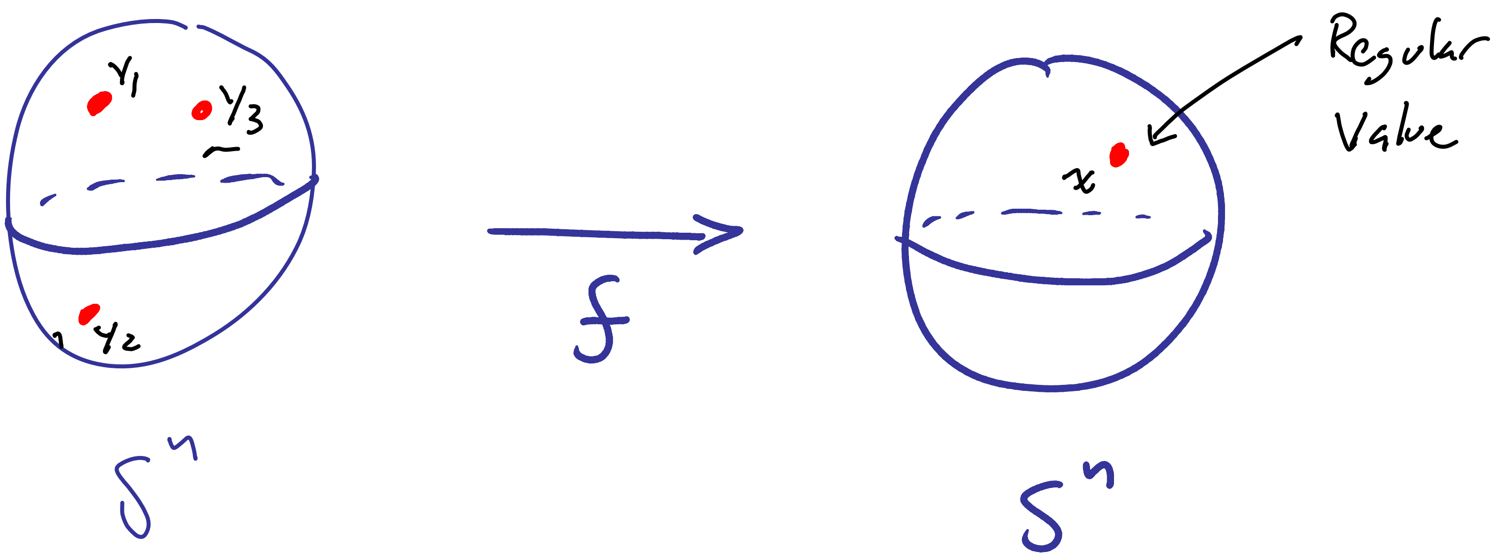
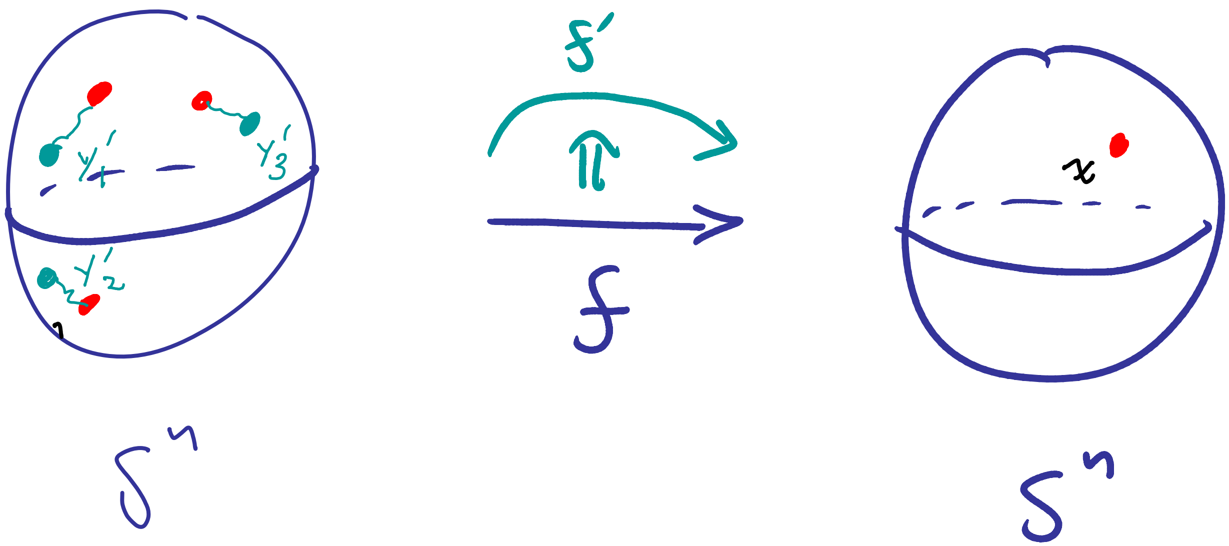


$$f^{-1}(x) = \{y_1, y_2, y_3\}$$

$$d(f) = \# |f^{-1}(x)|$$

Example: degree of a mapping

perturb $f \rightsquigarrow f'$ (small homotopy)



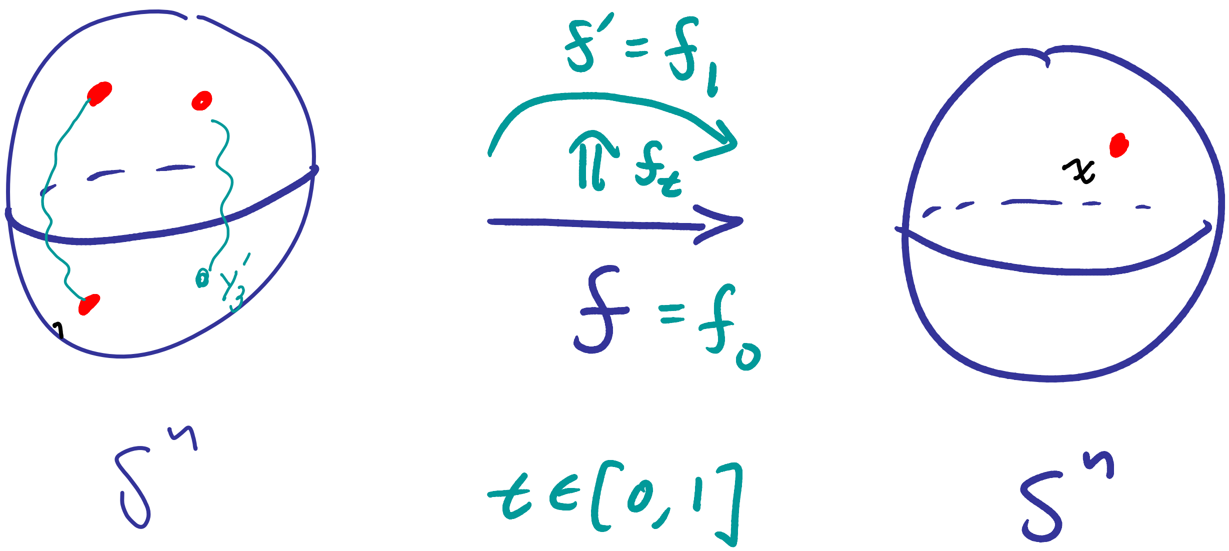
$$(f')^{-1}(x) = \{y_1', y_2', y_3'\}$$

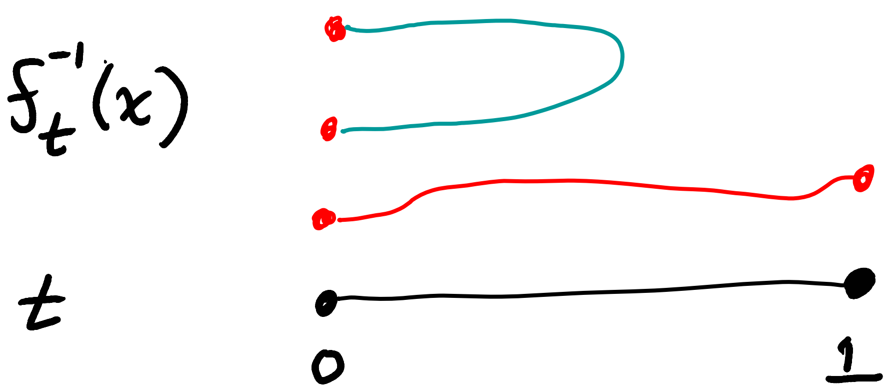
$$d(f') = d(f) = \# | f^{-1}(x) |$$

Example: degree of a mapping

perturb $f \rightsquigarrow f'$

BIG HOMOTOPY





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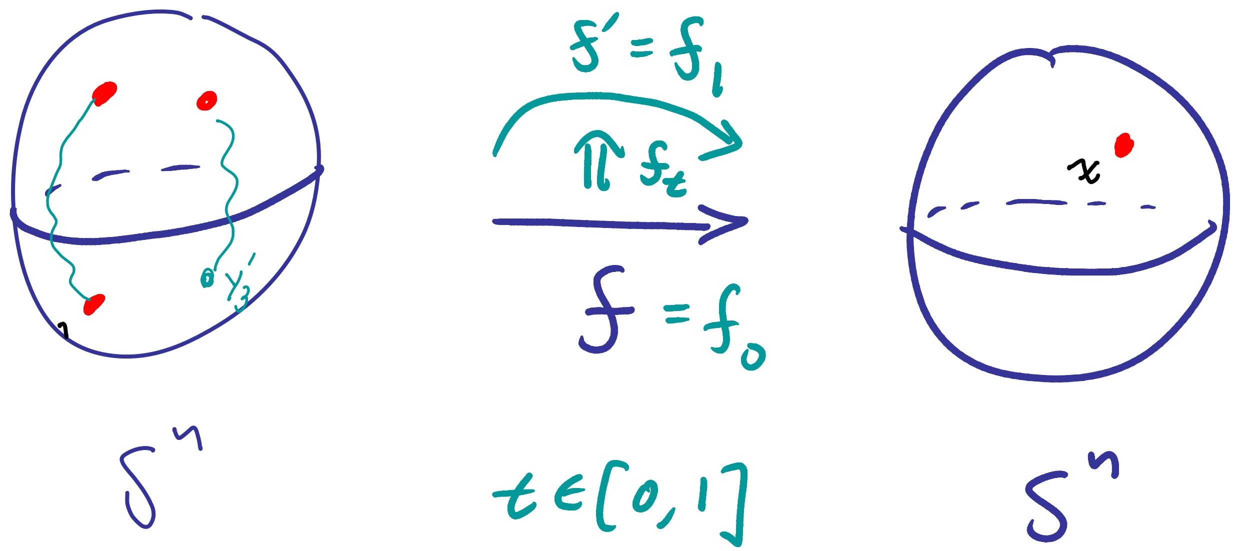
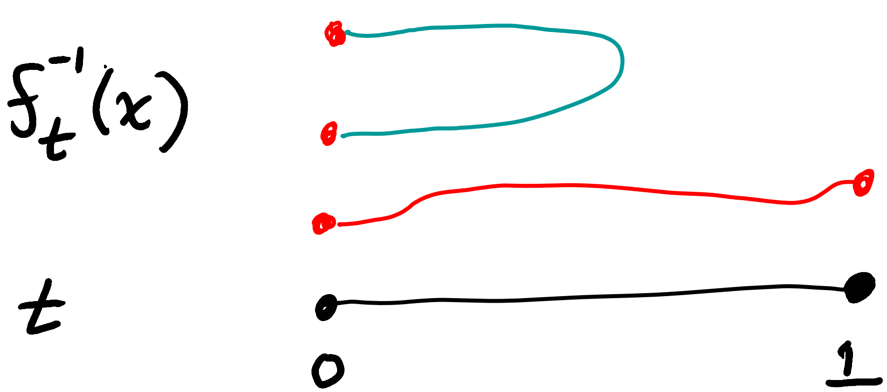




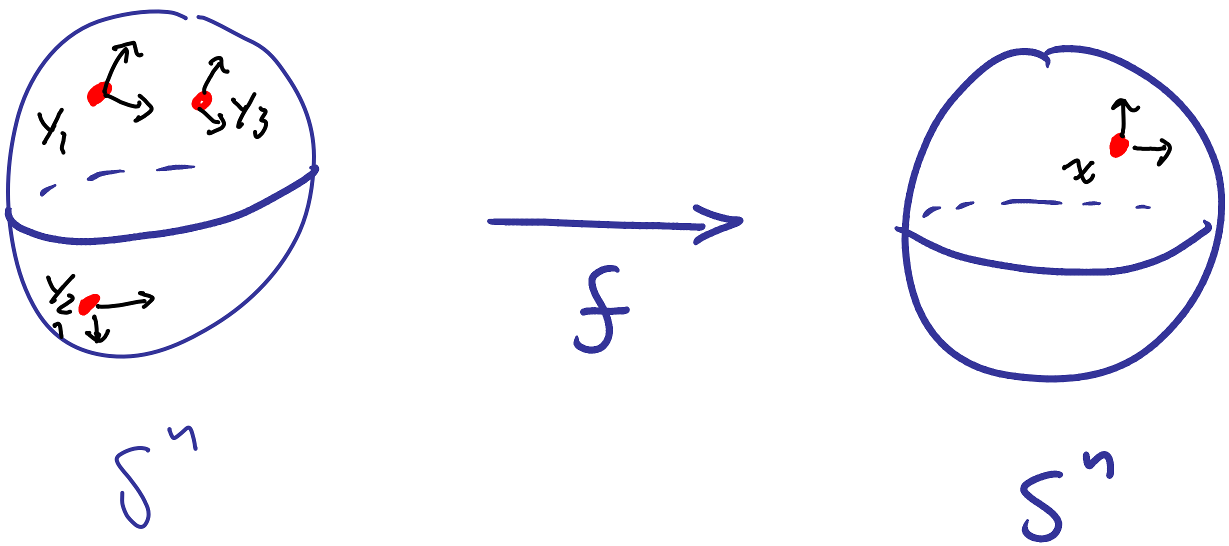
$$d(f) \equiv d(f') \mod 2$$

$$d \in \mathbb{Z}/2$$

Oriented degree:

x = 0-dim'l embedded submanifold

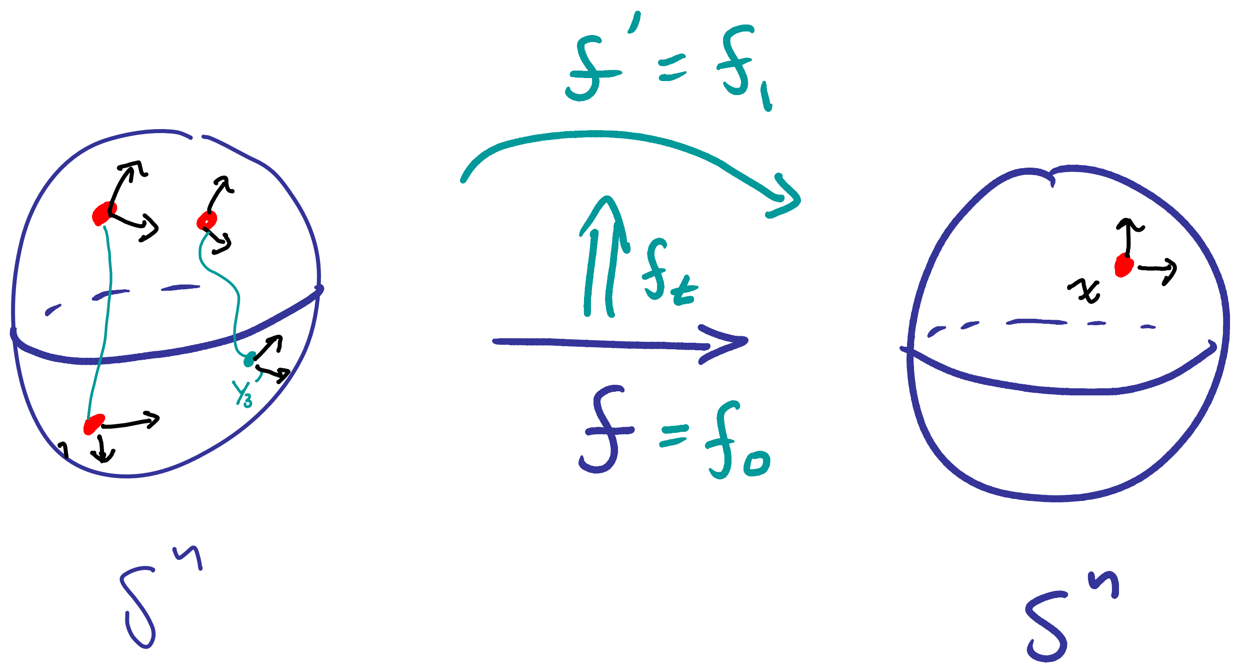
orientation on S^n ⇝ orientation on normal bundle of x



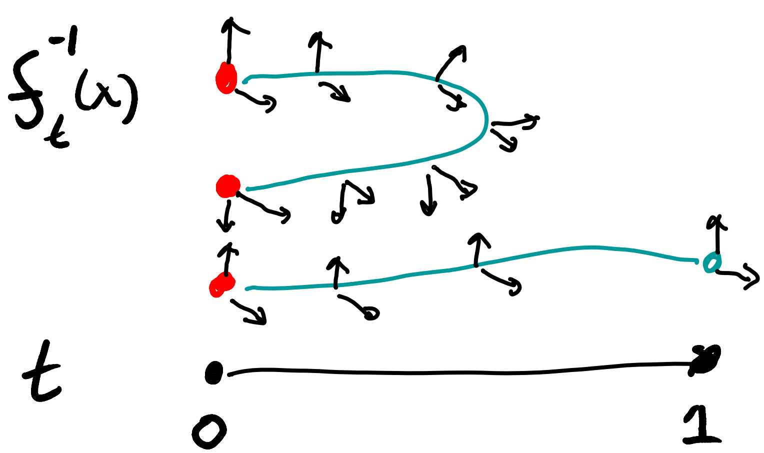
$$\tilde{d}(f) = 1 - 1 + 1$$

$$\tilde{d} \in \mathbb{Z}$$

Oriented degree:



$$\tilde{d}(f') = 1 = \tilde{d}(f) = 1 - 1 + 1$$



$$\tilde{d} \in \mathbb{Z}$$

What have we done?

Map	Manifold	Invariant
f $\rightsquigarrow$	$f^{-1}(x)$ $\xrightarrow[\text{cardinality}]{}$	$d(f) \in \mathbb{Z}/2$
f $\rightsquigarrow$	$f^{-1}(x)$ w/ oriented normal bundle $\xrightarrow[\text{cardinality}]{\text{signed}}$	$\tilde{d}(f) \in \mathbb{Z}$

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f $\rightsquigarrow$	$f^{-1}(x)$ w/ oriented normal bundle	$\xrightarrow[\text{cardinality}]{\text{signed}}$	$\tilde{d}(f) \in \mathbb{Z}$
Homotopy	Cobordism		Equality
$f \simeq f'$ $\rightsquigarrow$	$\partial W = f^{-1}(x) \amalg (f')^{-1}(x)$	$\rightsquigarrow$	$d(f) = d(f')$

We will generalize this picture

$$\begin{tikzcd}
\text{Homotopy groups of spheres} \arrow[r, squiggly, "\text{degree}"'] & \text{Cobordism of Manifolds} \arrow[r, squiggly, "\text{genus}"'] & \text{Cohomology theories} \arrow[r, squiggly, "\text{evaluate on point}"'] & \text{Rings} \\
\pi_k(S^n) & \Omega_k^G & E^*(-) & E^*(pt)
\end{tikzcd}$$

We will generalize this picture

Homotopy groups of spheres	$\rightsquigarrow$ degree	Cobordism of Manifolds	$\rightsquigarrow$ genus	Cohomology theories	$\rightsquigarrow$ evaluate on point	Rings
$\pi_k(S^n)$		Ω_k^G		$E^*(-)$		$E^*(pt)$

Hierarchy of structure

Periodicity			
v_0-periodicity	Ω_*^{SO}	H^*	$\mathbb{Z}$
v_1-periodicity	Ω_*^{Spin}	K^*	$\mathbb{Z}[\beta^{\pm 1}]$
v_2-periodicity	Ω_*^{String}	TMF^*	Modular forms

Homotopy groups

X = pointed space

S^n = n-sphere w/ fixed basepoint.

Homotopy groups

X = pointed space

S^n = n-sphere w/ fixed basepoint.

$$\pi_n(X) := \mathrm{Map}_*(S^n, X) \Big/ \text{pointed homotopy}$$

basepoint preserving maps

Most fundamental computation:

$$\pi_n(S^k)$$

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$$\pi_n(S^k)$$

- $\pi_n(S^n) = \mathbb{Z}$ (using $\check{d} : \pi_n(S^n) \xrightarrow[\cong]{} \mathbb{Z}$)
- $\pi_{<n}(S^n) = 0$
- $\pi_{>n}(S^n) = ???$

$\pi_i(S^n)$

$n \downarrow$ / $i \longrightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

$\pi_i(S^n)$

$n \downarrow$ / $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
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5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

$\pi_{<n}(S^n)$ $\qquad$ $\pi_n(S^n)$ $\qquad$ $\pi_{>n}(S^n)$

$\pi_i(S^n)$

	$i \longrightarrow$ 1	2	3	4	5	6	7	8	9	10	11	12
$n \downarrow$ 1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
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7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

- finitely generated abelian gps
- mostly torsion

$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
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5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

- finitely generated abelian gps
- mostly torsion
- Stabilizes along diagonals
- only stable non-torsion comes from $\pi_n(S^n)$

$\pi_i(S^n)$

$n \downarrow$ / $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
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5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
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8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

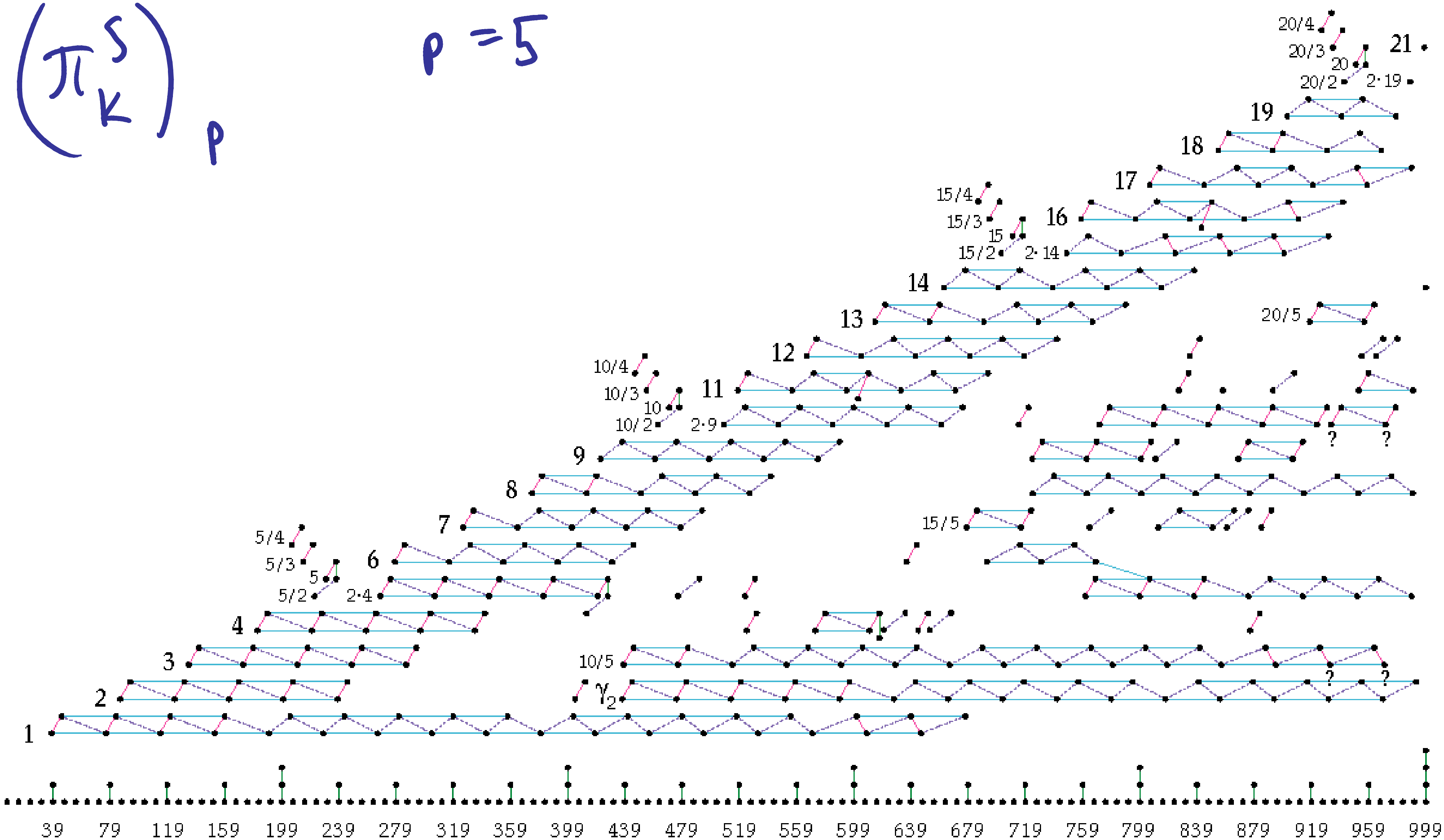
π^s_0 π^s_1 π^s_2 π^s_3 π^s_4

Stable homotopy groups:

$$\pi^s_k := \lim_{n\to\infty} \pi_{n+k}(S^n)$$

$k>0$ $\quad \pi^s_k \cong \bigoplus_{p \text{ prime}} (\pi^s_k)_p \leftarrow$ finite gp p-torsion

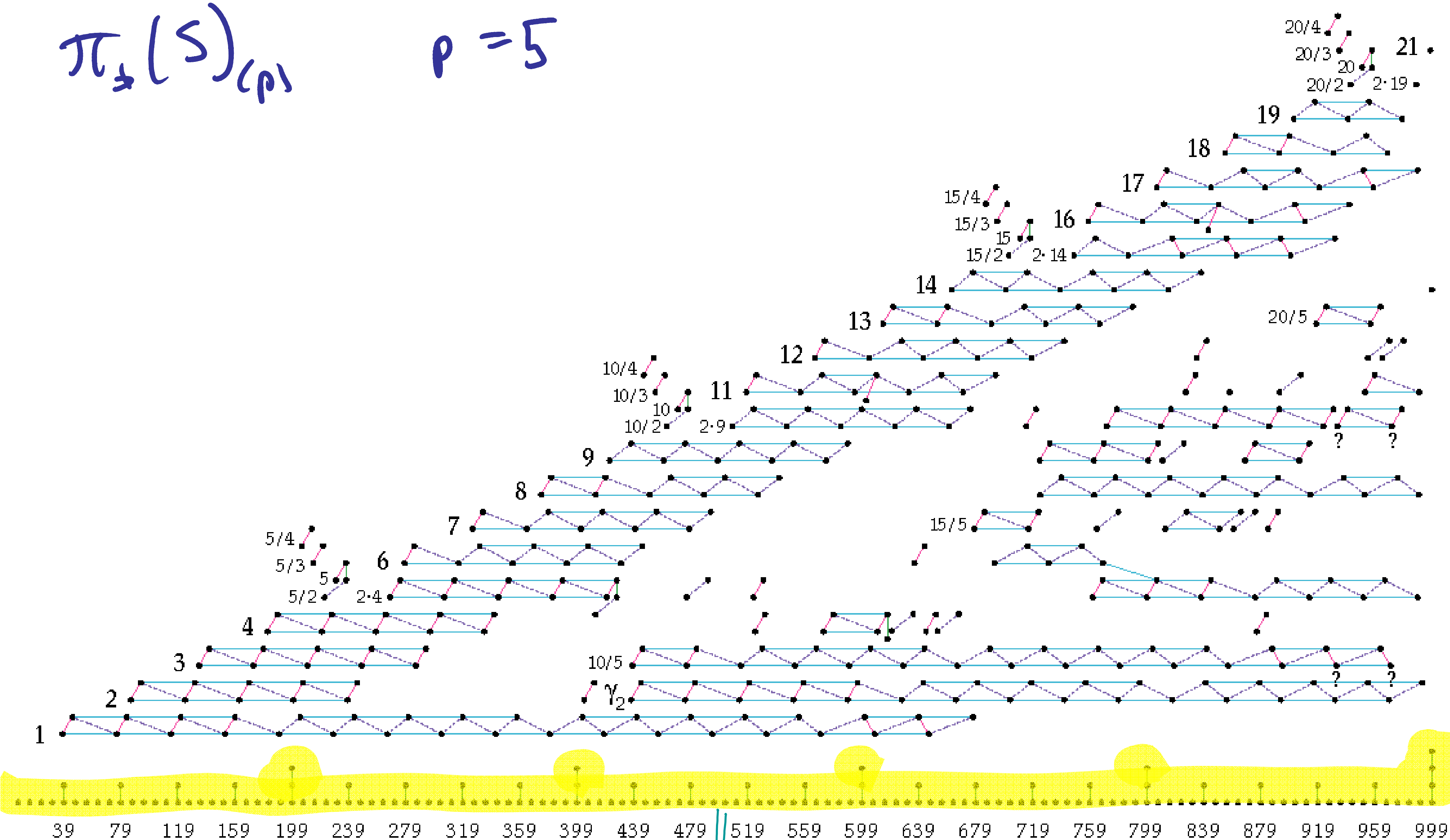
$\left(\pi_k^S\right)_p$ $p=5$



picture: Hatcher
computation: Ravenel

k

$\pi_*(S)_{(p)}$ $\quad p = 5$



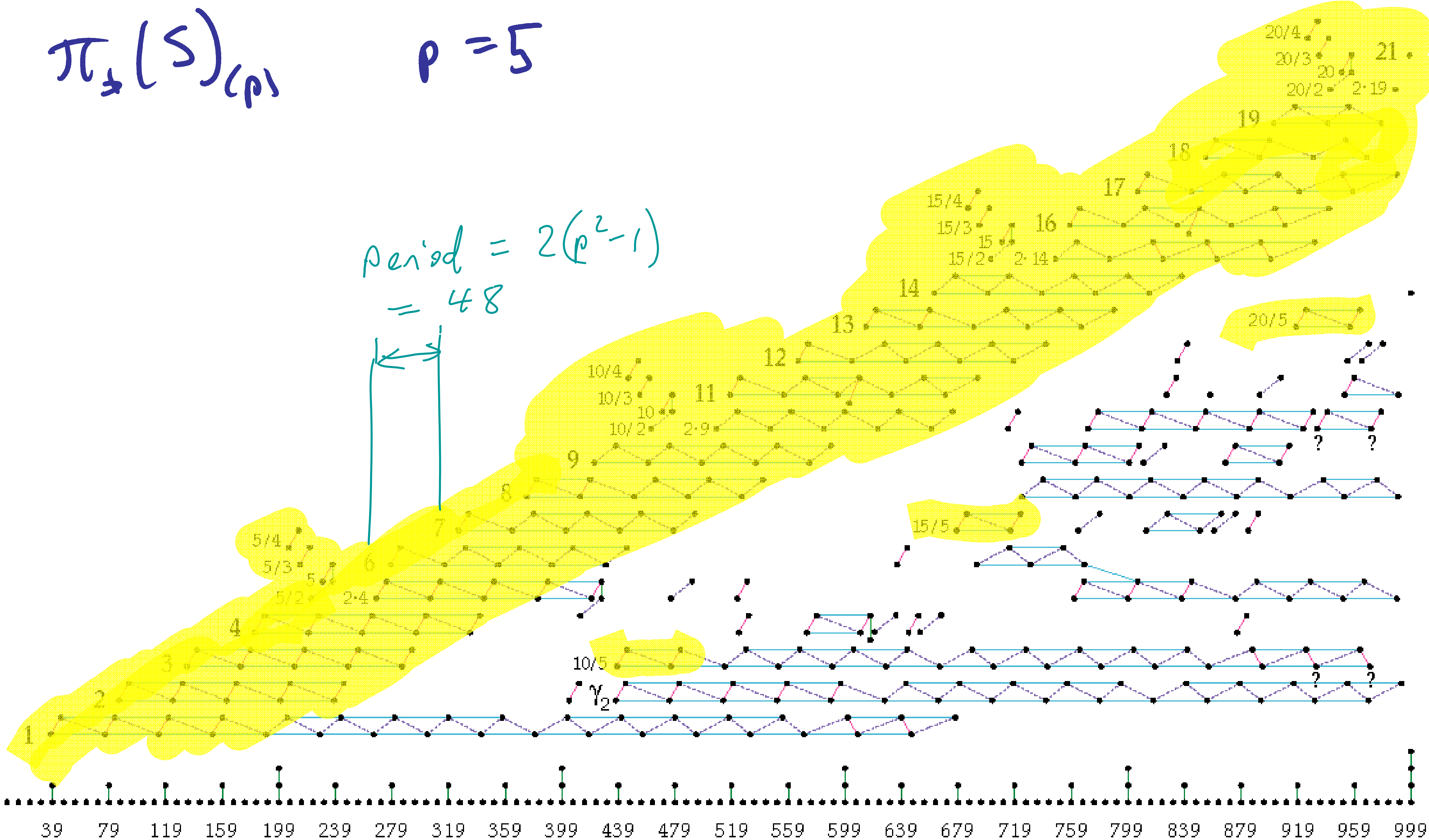
period $= 2(p-1) = 8$

v_1-periodic

picture: Hatcher
computation: Ravenel

$\pi_*(S)_{(p)}$ $\quad p = 5$

period $= 2(p^2-1)$
$= 48$

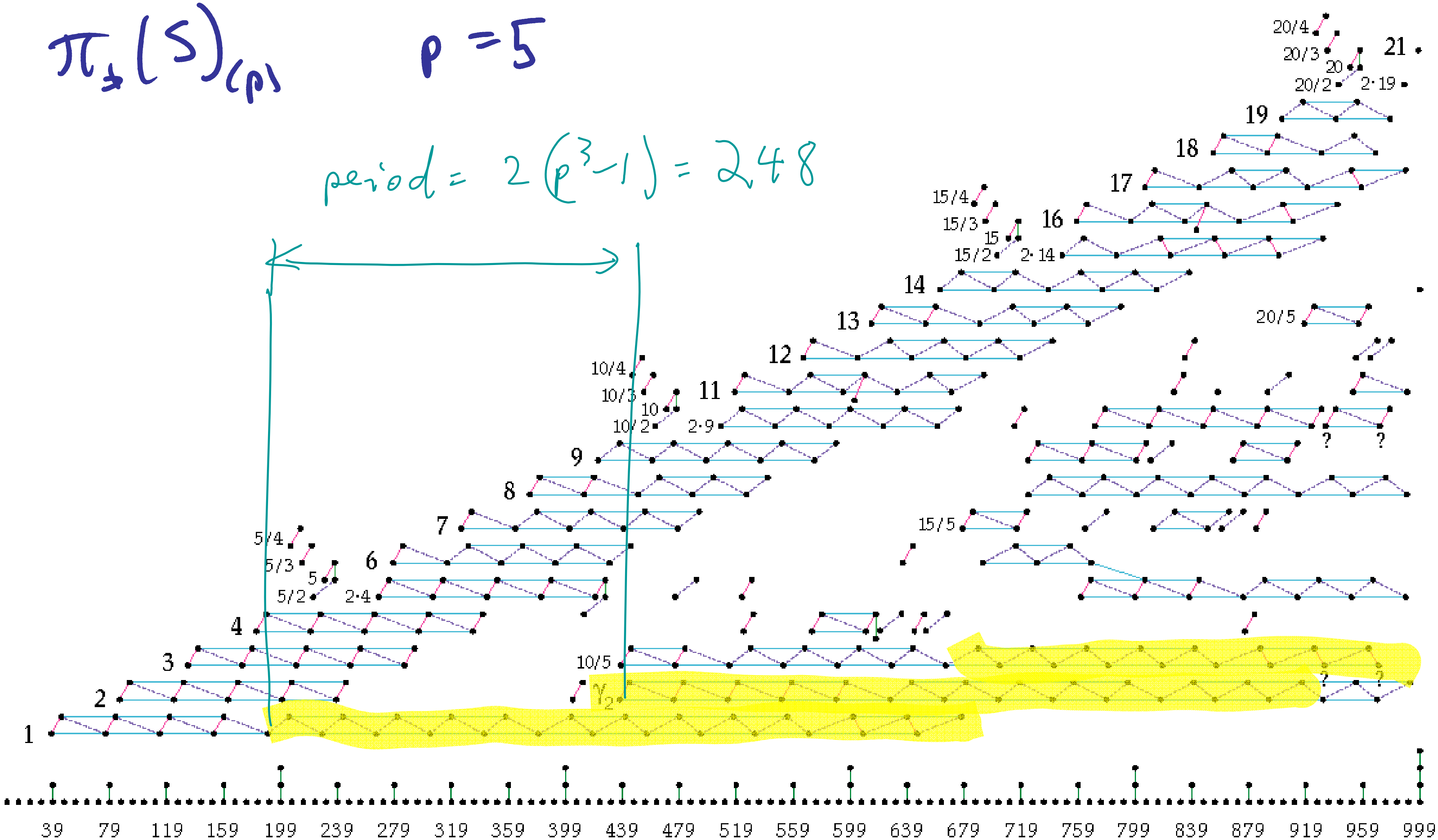


picture: Hatcher
computation: Ravenel

v_2-periodic

$\pi_*(S)_{(p)}$ $\quad p = 5$

period $= 2(p^3-1) = 248$



picture: Hatcher
computation: Ravenel

v_3-periodic

Cobordism

$\{G_n\}$ = sequence of topological gps

$\shortparallel$

G

$G_n \longrightarrow O_n(\mathbb{R})$

Cobordism

$\{G_n\}$ = sequence of topological gps

$\shortparallel$

G

$G_n \longrightarrow O_n(\mathbb{R})$

V rank k vector bundle

$$\begin{array}{ccc} V & & \\ \downarrow & & \\ X & \xleftarrow{\text{cover}} & \{U_k\} \end{array}$$

Transition functions

$g_{ij} : U_i \cap U_j \longrightarrow O_n(\mathbb{R})$

Def: A <u>G-structure</u> on V

is a lift

$$\begin{array}{ccc} & & G_n \\ & \overset{\tilde{g}_{ij}}{\nearrow} & \downarrow \\ U_i \cap U_j & \xrightarrow[g_{ij}]{} & O_n(\mathbb{R}) \end{array}$$

Cobordism

$\{G_n\}$ = sequence of topological gps

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$G_n \longrightarrow O_n(\mathbb{R})$

Transition functions

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V rank k vector bundle

$\downarrow$

$X \xleftarrow[\text{cover}]{} \{U_k\}$

Def: A G-structure on V is a lift

$$\begin{tikzcd} & G_n \arrow[d] \\ U_i \cap U_j \arrow[ur, dashed, "\tilde{g}_{ij}"] \arrow[r, "g_{ij}"'] & O_n(\mathbb{R}) \end{tikzcd}$$

"Stable" G-structure = G-structure on $V \oplus \mathbb{R}^N$, $N \gg 0$

Cobordism

M = manifold

$M \hookrightarrow \mathbb{R}^N$ embedding

ν_M = normal bundle of embedding

Def: A <u>G-structure</u> on M is a stable G-structure on ν_M

Examples

$SO = \{SO_n\}$ orientation on M $\rightsquigarrow$ SO-structure

$U = \{U_n\}$ complex structure on M $\rightsquigarrow$ U-structure

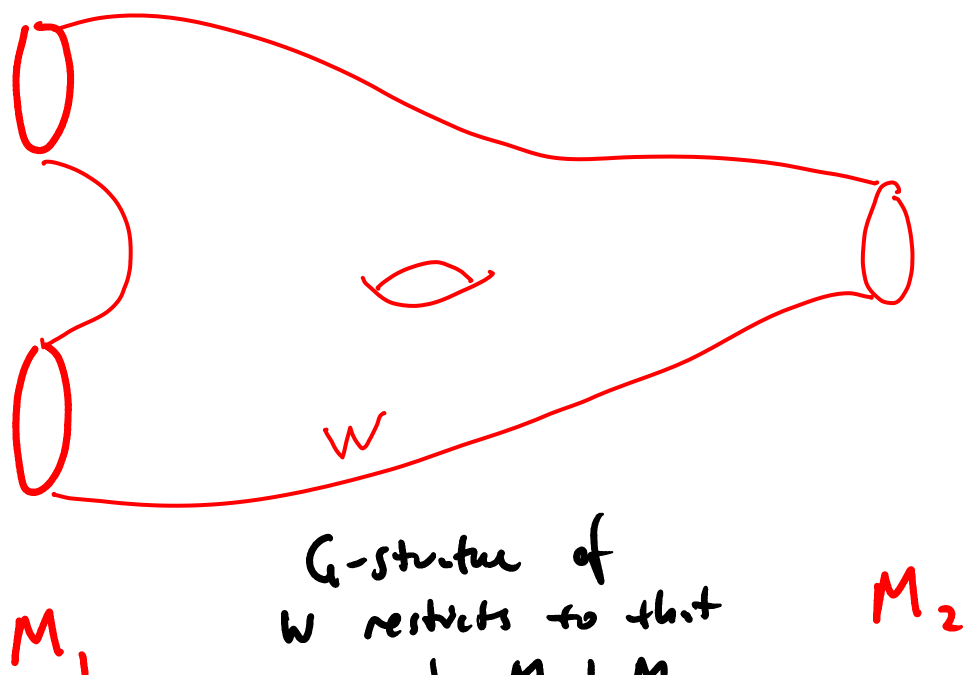
$Spin = \{Spin_n\}$ "Spin manifolds"

$e = \{e\}$ "framed manifolds" (trivialization of ν_M)

Cobordism

$$\Omega_k^G := \frac{\{\text{smooth cpt } k\text{-manifolds w/ } G\text{-structure}\}}{\text{cobordism}}$$

A cobordism between M_1 and M_2



G-structure of W restricts to that of M_1 + M_2

Cobordism

$$\Omega_k^G := \frac{\{\text{smooth cpt } k\text{-manifolds w/ } G\text{-structure}\}}{\text{cobordism}}$$

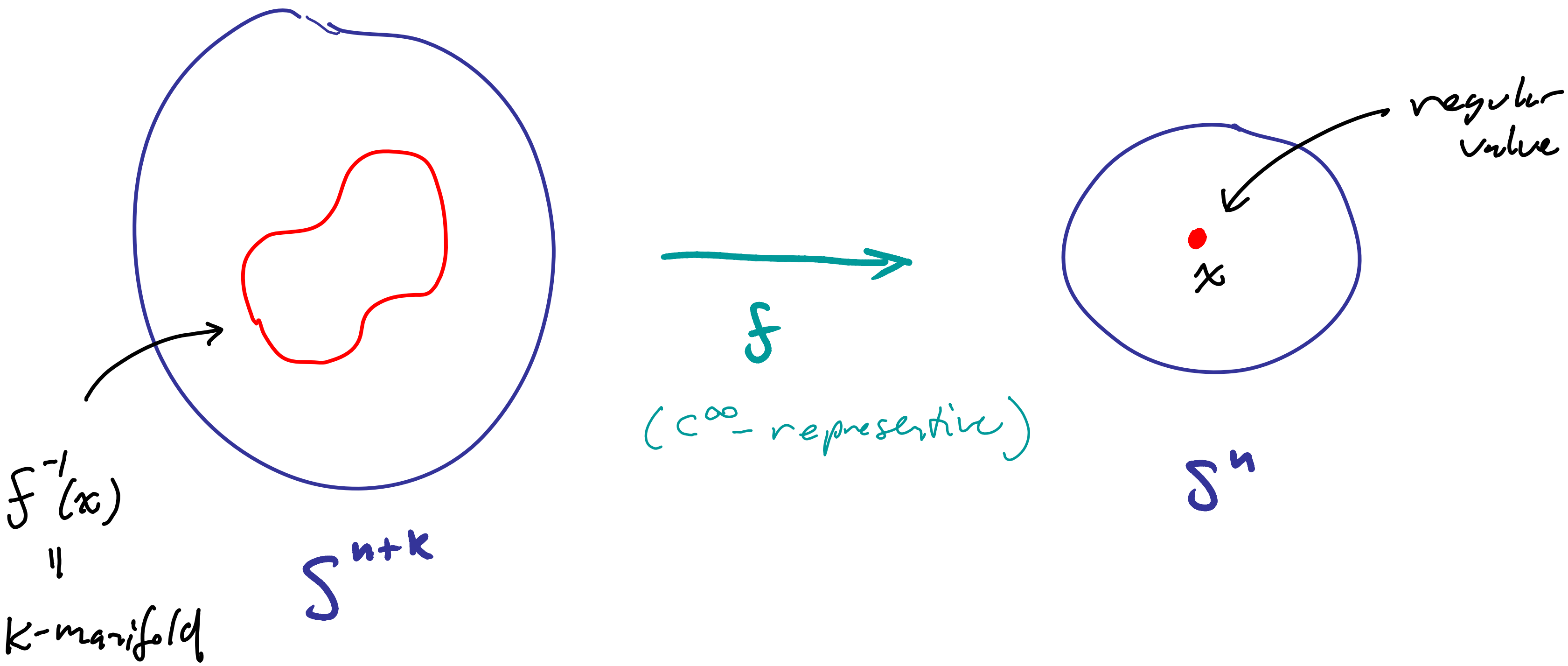
Ω_*^G is a graded ring:

$$[M_1] + [M_2] := [M_1 \amalg M_2]$$

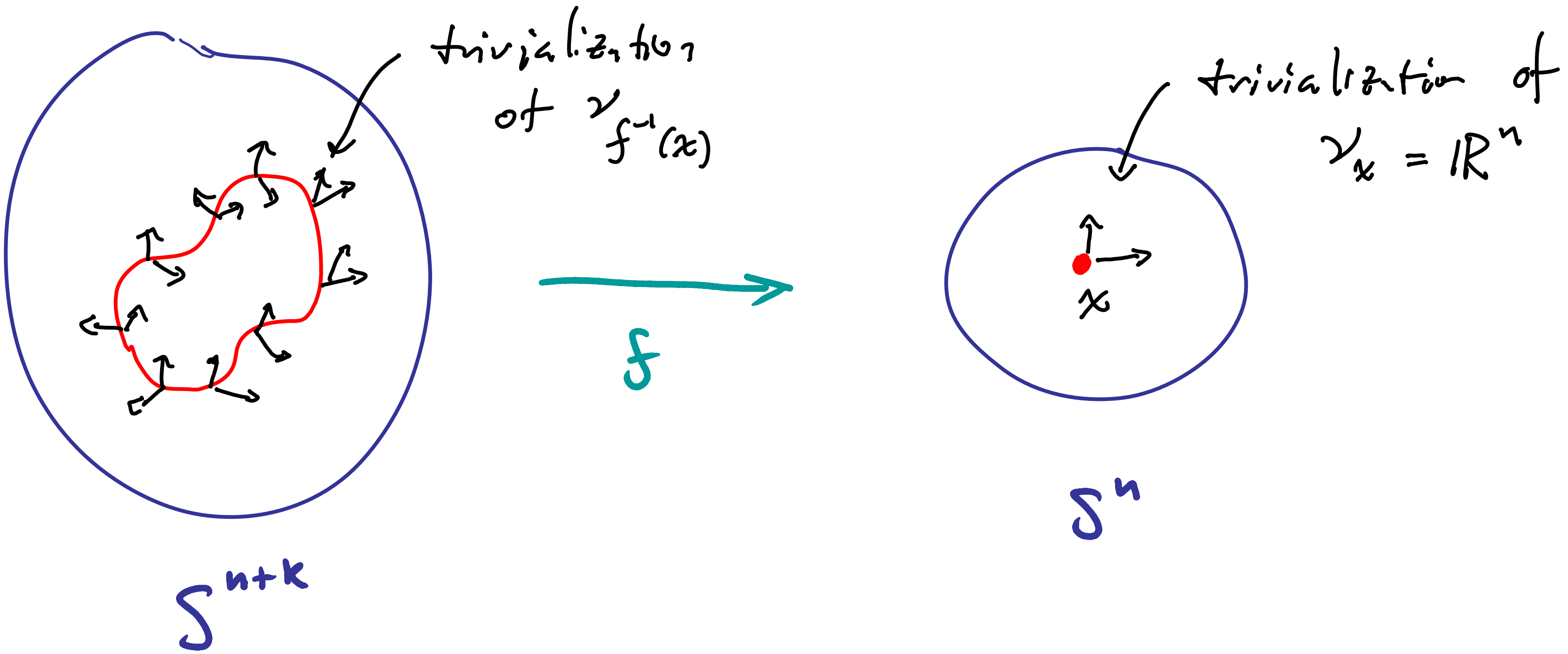
$$[M_1][M_2] := [M_1 \times M_2]$$

Homotopy ⟶ Cobordism

$[f] \in \pi_k^s$

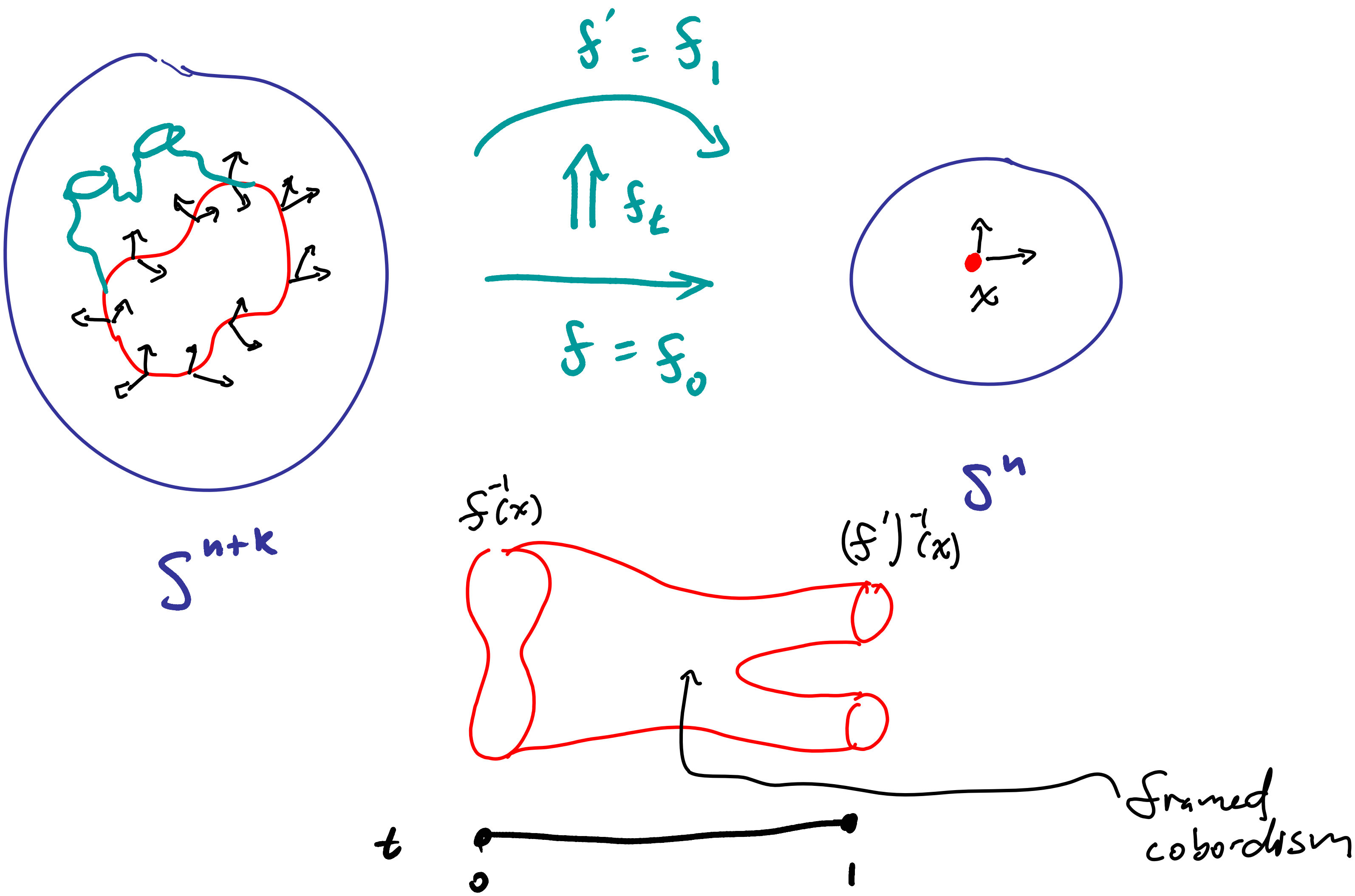


Homotopy ⟶ Cobordism



$\Rightarrow f^{-1}(x)$ is a framed k-manifold

Homotopy ⟶ Cobordism



We get:

$$\begin{array}{ccc} \text{stable htpy} & & \text{framed cobordism} \\ \pi_k^s & \longrightarrow & \Omega_k^e \\ [f] & \longmapsto & [f^{-1}(x)] \end{array}$$

We get:

stable htpy $\qquad\qquad$ framed cobordism

$$\pi_k^s \longrightarrow \Omega_k^e$$

$$[f] \longmapsto [f^{-1}(x)]$$

Thm (Pontryagin)

This gives an isomorphism

$$\pi_k^s \xrightarrow[\cong]{} \Omega_k^e$$

Bad News

Ω_*^e is really complicated!

Good News

Given any G, get $e \to G$

$\Longrightarrow$ there is a homomorphism

$$\pi_*^s \xrightarrow[\cong]{} \Omega_*^e \to \underbrace{\Omega_*^G}_{\text{simpler}}$$

$R = $ ring

Genera

An **R-valued genus** is a ring homomorphism

$$\Phi : \Omega_*^G \longrightarrow R$$

"Study Ω_*^G through its genera"

Complex Genera

Thm (Milnor)

$$\Omega^U_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \mathbb{C}P^3, \dots\dots]$$

polynomial ring

Complex Genera

Thm (Milnor)

$$\Omega_*^U \otimes \mathbb{Q} \;\hat{=}\; \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \mathbb{C}P^3, \ldots\ldots]$$

polynomial ring

Consequence: If $R \subset R \otimes \mathbb{Q}$

a genus $\Phi : \Omega_*^U \longrightarrow R$

is determined by $\{\Phi(\mathbb{C}P^n)\}_n$

Φ is determined by:

$$\log_{\Phi}(x) = \sum_{i} \frac{\Phi(cp^{n-1})}{n} x^n \qquad \text{(logarithm)}$$

– or equivalently by –

$$F_{\Phi}(x,y) = \log_{\Phi}^{-1}\left(\log_{\Phi}(x) + \log_{\Phi}(y)\right) \qquad \text{(Formal group)}$$

Φ is determined by:

$$\log_\Phi(x) = \sum \frac{\Phi(p^{n-1})}{n} x^n \in R\otimes\mathbb{Q}[[x]] \quad (\text{logarithm})$$

– or equivalently by –

$$F_\Phi(x,y) = \log_\Phi^{-1}\left(\log_\Phi(x) + \log_\Phi(y)\right) \quad (\text{Formal group})$$

$$F_\Phi(x,y) \in R[[x,y]] \longleftarrow (\text{integral})$$

- $F_\Phi(0,x) = F_\Phi(x,0) = x$ — unital
- $F_\Phi(x,y) = F_\Phi(y,x)$ — commutative
- $F_\Phi(x, F_\Phi(y,z)) = F_\Phi(F_\Phi(x,y), z)$ — associative

Cohomology theories

Def: a cohomology theory is a contravariant functor:

$$E^* : \text{Spaces} \longrightarrow \mathbb{Z}\text{-graded Rings}$$
$$X \longmapsto E^*(X)$$

S.t.

(1) $f \simeq g \implies E^*(f) = E^*(g)$

(2) Meyer Vietoris $X = U \cup V$, $A = U \cap V$

$$\cdots \to E^n(X) \longrightarrow E^n(U) \oplus E^n(V) \longrightarrow E^n(A) \longrightarrow E^{n+1}(X) \to \cdots$$

Examples

1) Singular Cohomology $E = HR$

R = Ring

$$HR^n(X) = H^n(X;R)$$

2) K-theory $E = K$

$$K^0(X) = \{\text{stable complex vector bundles on } X\}$$

(Can also define $K_{\mathbb{R}} \leftrightarrow$ stable real vector bundles)

$$\begin{array}{ccc} \text{Cohomology theories} & \longrightarrow & \text{Rings} \\ E^* & \longmapsto & E^*(\text{point}) \end{array}$$

Cohomology theories $\longrightarrow$ Rings

$$E^* \longmapsto E^{-*}(\text{point})$$

$$HR^{-*}(\text{point}) = \cdots \underset{-2}{0} \quad \underset{-1}{0} \quad \underset{0}{R} \quad \underset{1}{0} \quad \underset{2}{0} \cdots$$

$$K^{-*}(\text{point}) = \cdots \underset{-2}{\mathbb{Z}} \quad \underset{-1}{0} \quad \underset{0}{\mathbb{Z}} \quad \underset{1}{0} \quad \underset{2}{\mathbb{Z}} \quad \underset{3}{0} \cdots$$

2-period

8-periodic

$$K_{\mathbb{R}}^{-*}(pt) = \cdots \underset{0}{\mathbb{Z}} \quad \underset{1}{\mathbb{Z}/2} \quad \underset{2}{\mathbb{Z}/2} \quad \underset{3}{0} \quad \underset{4}{\mathbb{Z}} \quad \underset{5}{0} \quad \underset{6}{0} \quad \underset{7}{0} \quad \underset{8}{\mathbb{Z}} \cdots$$

Genus of families

$E^* =$ cohomology theory

$$\begin{array}{l} M \\ \downarrow \\ S \end{array}$$

family of n-dim'l G-manifolds (parametrized by S)

$$\Phi(M/S) \in E^{-n}(S)$$

Note: Get a genus when $S =$ point

$$\Phi : \Omega^G_* \longrightarrow E^{-*}(\text{point})$$

$$\Phi(M) = \Phi(M/\text{point}) \in E^{-n}(\text{point})$$

Examples

1) $\mathrm{Card}: \Omega^{0}_{*} \longrightarrow H\mathbb{Z}/2^{-*}$

cardinality mod 2
of 0-manifolds

2) $\widetilde{\mathrm{Card}}: \Omega^{SO}_{*} \longrightarrow H\mathbb{Z}^{-*}$

signed cardinality
of <u>oriented</u> 0-mflds

3) $\mathrm{Todd}: \Omega^{u}_{*} \longrightarrow K^{-*}$

4) $\hat{A}: \Omega^{Spin}_{*} \longrightarrow K_{\mathbb{R}}^{-*}$

Examples

1) $\mathrm{Card} : \Omega^{0}_{*} \longrightarrow H\mathbb{Z}/2^{-*}$

cardinality mod 2
of 0-manifolds

$$\left.\begin{array}{l} F_{card}(x,y) \\ F_{\widetilde{card}}(x,y) \end{array}\right\} = x+y$$

additive

2) $\widetilde{\mathrm{Card}} : \Omega^{SO}_{*} \longrightarrow H\mathbb{Z}^{-*}$

signed cardinality
of oriented 0-mflds

3) $\mathrm{Todd} : \Omega^{U}_{*} \longrightarrow K^{-*}$

$$F_{Todd}(x,y) = x+y-xy$$

multiplicative

4) $\hat{A} : \Omega^{Spin}_{*} \longrightarrow K_{\mathbb{R}}^{-*}$

Elliptic Curves

- An elliptic curve C is a genus 1 alg curve over a ring R
  (with basepoint)

- Elliptic curves are <u>abelian</u> <u>groups</u>.
  basepoint = identity

- $C \rightsquigarrow F_C(x,y) \in R[[x,y]]$ (formal gp)
  power series expansion of $+: C \times C \to C$ at basepoint

Elliptic cohomology

A cohomology theory E^* is <u>elliptic</u> if

- C/R elliptic curve
- $E^{-*}(\text{point}) = \cdots \underset{-2}{R} \quad \underset{-1}{0} \quad \underset{0}{R} \quad \underset{1}{0} \quad \underset{2}{R} \quad \underset{3}{0} \cdots$ (2-periodic)
- $\Phi : \Omega^u_* \longrightarrow E^{-*}$ genus of families

$$F_\Phi = F_C$$

Modular forms

A modular form f of weight k is an assignment

$$C/_{\mathbb{R}} \longmapsto f(C) \in \mathbb{R}$$

elliptic curve

such that

$$\alpha : C \xrightarrow[\cong]{} C'$$

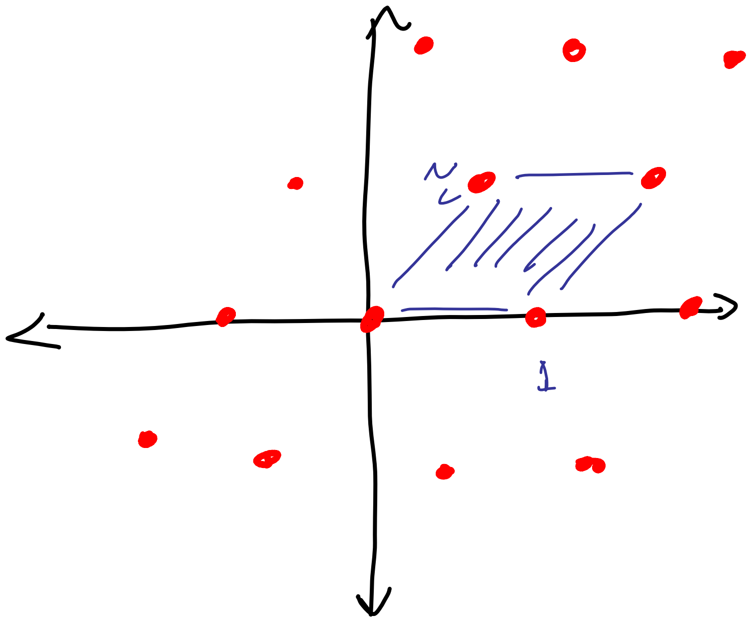
$$\alpha'(0)^k f(C') = f(C)$$

Analytic theory

$\tau \in \mathcal{H}$ (upper half plane) $\rightsquigarrow$ $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ lattice

$\cap$

$\mathbb{C}$



$\rightsquigarrow$ $\mathbb{C}/\Lambda_\tau$

elliptic curve$/\mathbb{C}$

$$f(\tau) = f(\mathbb{C}/\Lambda_\tau)$$

$$\frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

τ-expansion

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in SL_2(\mathbb{Z}) \implies f(\tau) = f(\tau+1)$$

$\implies$ Fourier expansion:

$$f(\tau) = \sum_n a_n q^n \qquad q = e^{2\pi i \tau}$$

τ-expansion

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in SL_2(\mathbb{Z}) \implies f(\tau) = f(\tau+1)$$

$\implies$ Fourier expansion:

$$f(\tau) = \sum a_n q^n \qquad q = e^{2\pi i \tau}$$

Example: Eisenstein series

B_{2k} = Bernoulli number

$$E_{2k}(\tau) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(c\tau+d)^{2k}}$$

weight 2k

$$= -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \Big(\sum_{d|n} d^{2k-1} \Big) q^n$$

Topological Modular Forms

A <u>topological modular form</u> f of weight k

is an assignment

$$(E^*, C, \Phi) \longmapsto f(E^*, C, \Phi) \in E^{-2k}(\text{point})$$

Elliptic cohomology theory

such that

$$\alpha : C \xrightarrow[\cong]{} C'$$

$$\begin{tikzcd}
 & E^{-*} \\
\Omega^U_* \arrow[ur, "\Phi"] \arrow[dr, "\Phi'"'] & \\
 & E^{-*} \arrow[uu, "\alpha^*"']
\end{tikzcd}$$

$$\alpha^* f(E^*, C, \Phi) = f(E^*, C', \Phi')$$

TMF $\quad \underbrace{\text{Topological Modular Forms}}$

There is a cohomology theory TMF^*

Hopkins
Miller
Goerss
Lurie
Mahowald

$$f \in TMF^*(S)$$

$$\rightsquigarrow \qquad f(E^*, C, \Phi) \in E^*(S)$$

$\forall$ elliptic (E, C, Φ)

$$C \underset{\alpha}{\xrightarrow{\cong}} C' \quad \rightsquigarrow \quad \alpha^* f(E^*, C', \Phi') = f(E^*, C, \Phi)$$

Witten Genus

"Birth of TMF"

$$\begin{tikzcd}
\text{String} \arrow[d] & \pi_0 = \pi_1 = \pi_3 = 0 \\
\text{Spin} \arrow[d] & \pi_0 = \pi_1 = 0 \\
SO \arrow[d] & \pi_0 = 0 \\
O &
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega^{String} \arrow[r, "W"] \arrow[d] & TMF^* \\
\Omega^{Spin} \arrow[r, "\hat{A}"'] \arrow[d] & K\mathbb{R}^* \\
\Omega^{SO} \arrow[r, "\mathrm{C\tilde{a}rd}"'] \arrow[d] & H\mathbb{Z}^* \\
\Omega^{O}_* \arrow[r, "\mathrm{Card}"'] & H\mathbb{Z}/2^*
\end{tikzcd}$$

Witten

Landweber

Ravenel

Stong

Ando

Hopkins

Strickland

Rezk

Generalized degree

Start with: E^* = cohomology theory

$\Phi : \Omega^G_* \longrightarrow E^{-*}$ genus

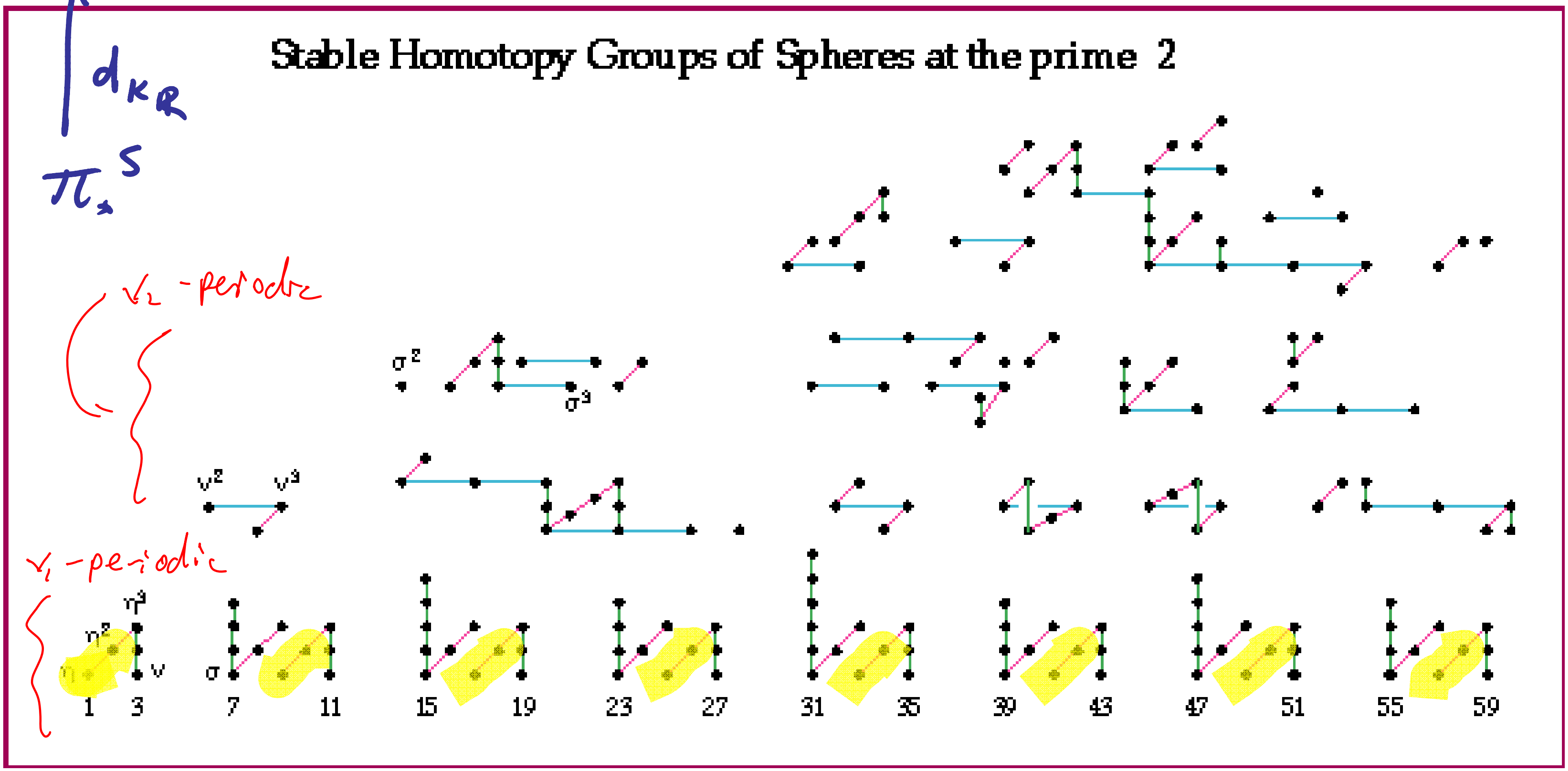
$$\pi_n^S \xrightarrow{d_E} E^{-n}(\text{point})$$

$f : S^{n+k} \longrightarrow S^n \quad \longmapsto \quad M = f^{-1}(x) \quad \longmapsto \quad \Phi(M)$

$$(E, \Phi) = (K_{\mathbb{R}}, \hat{A})$$

$$K_{\mathbb{R}}^{-*}(pt) = \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \dots$$

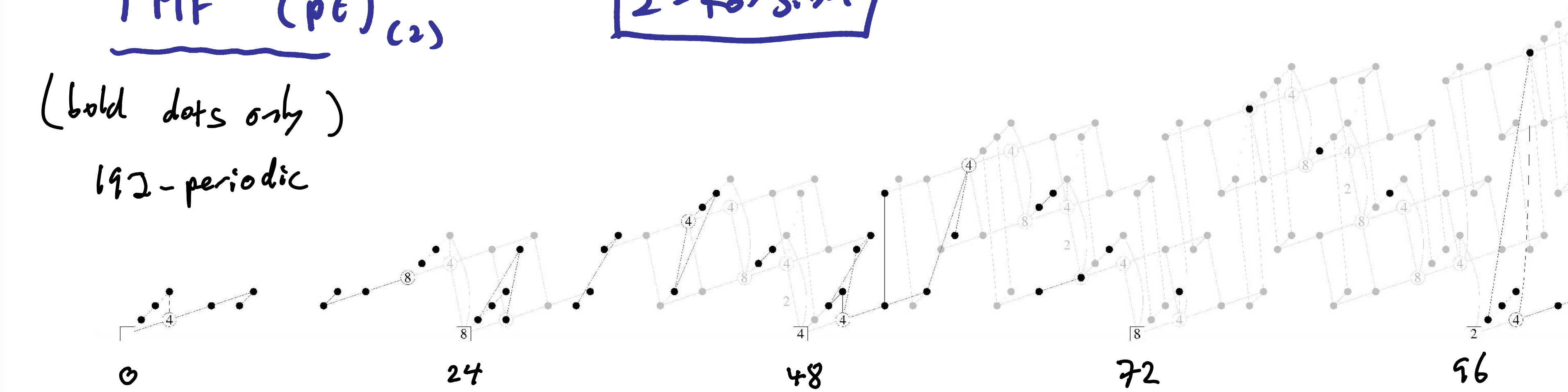
$$\pi_*^S \xrightarrow{d_{K_{\mathbb{R}}}} K_{\mathbb{R}}^{-*}(pt)$$

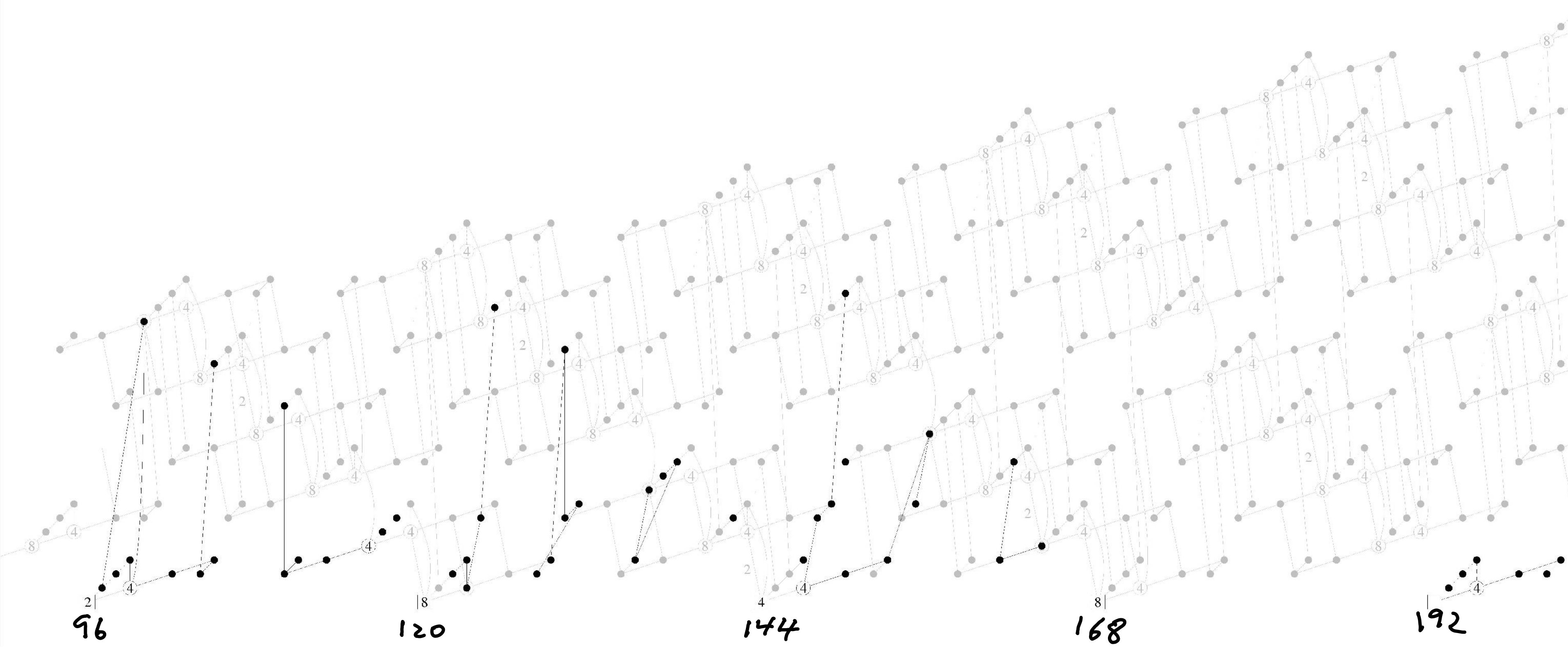


$\underline{TMF^{-*}(pt)}_{(2)}$ $\boxed{2\text{-torsion}}$

(bold dots only)

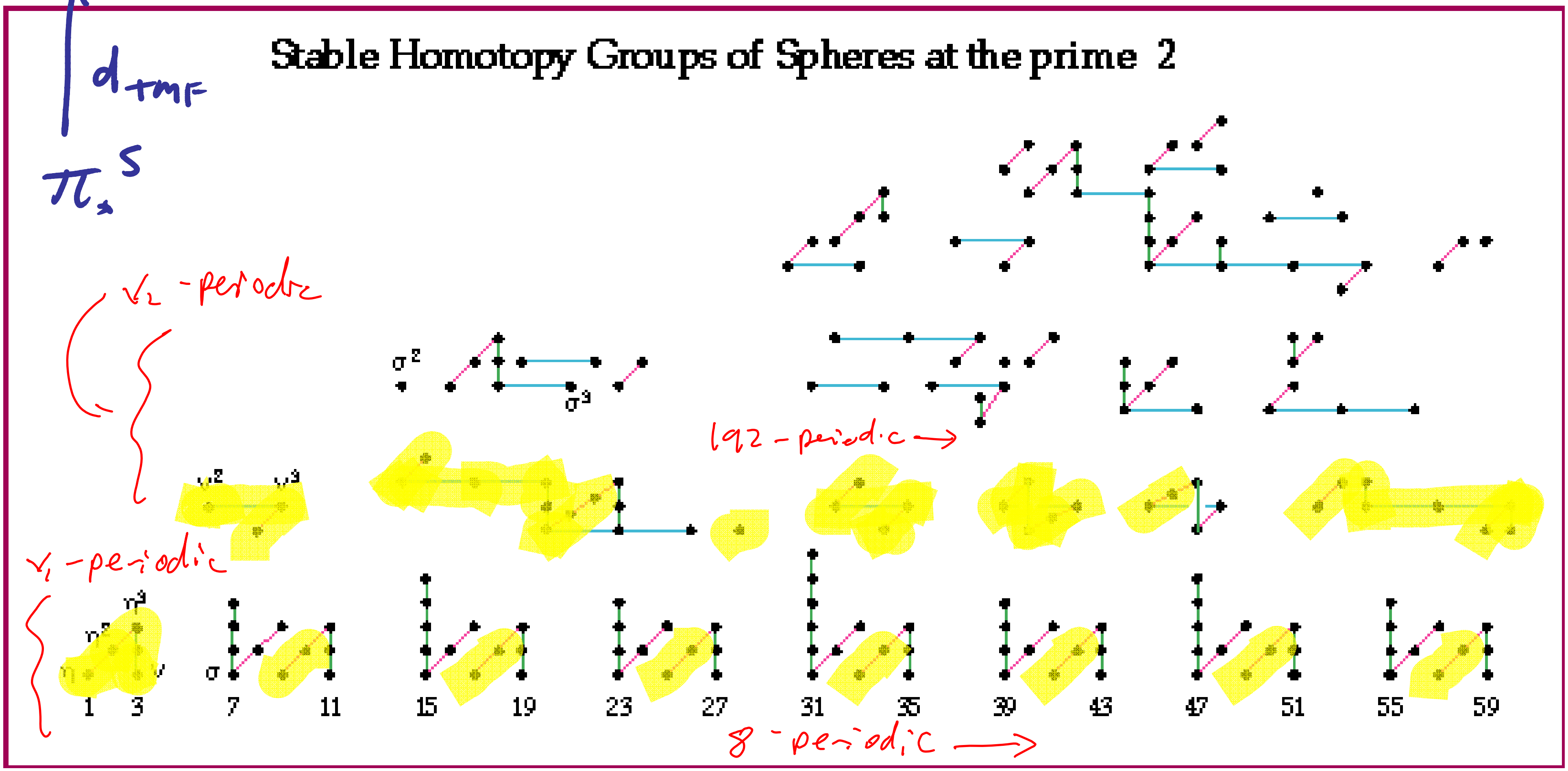
192-periodic





$(E, \Phi) = (TMF, \text{Witten})$

$$TMF^{-*}(pt) \xleftarrow{d_{tmf}} \pi_*^S$$



Torsion invariants

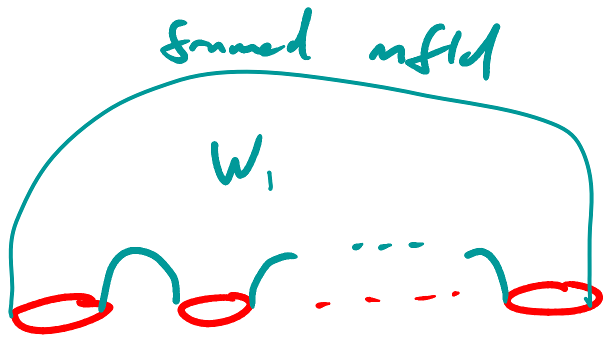
①

$$[f] \in \pi_n^s \, , \qquad N[f] = 0 \qquad N \in \mathbb{Z}$$

Suppose $M = f^{-1}(x)$ $\qquad [M] \in \Omega_n^G$

$$\overset{\shortparallel}{0} \quad ②$$

①



$\coprod_N M$

M

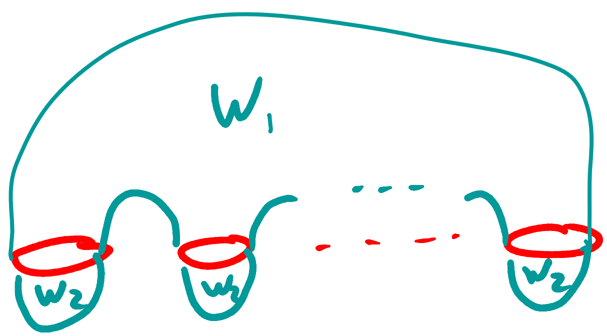
G-mfld

Torsion invariants

$[f] \in \pi_n^s$, $N[f] = 0$ $N \in \mathbb{Z}$

Suppose $M = f^{-1}(x)$ $[M] \in \Omega_n^G$

$= 0$



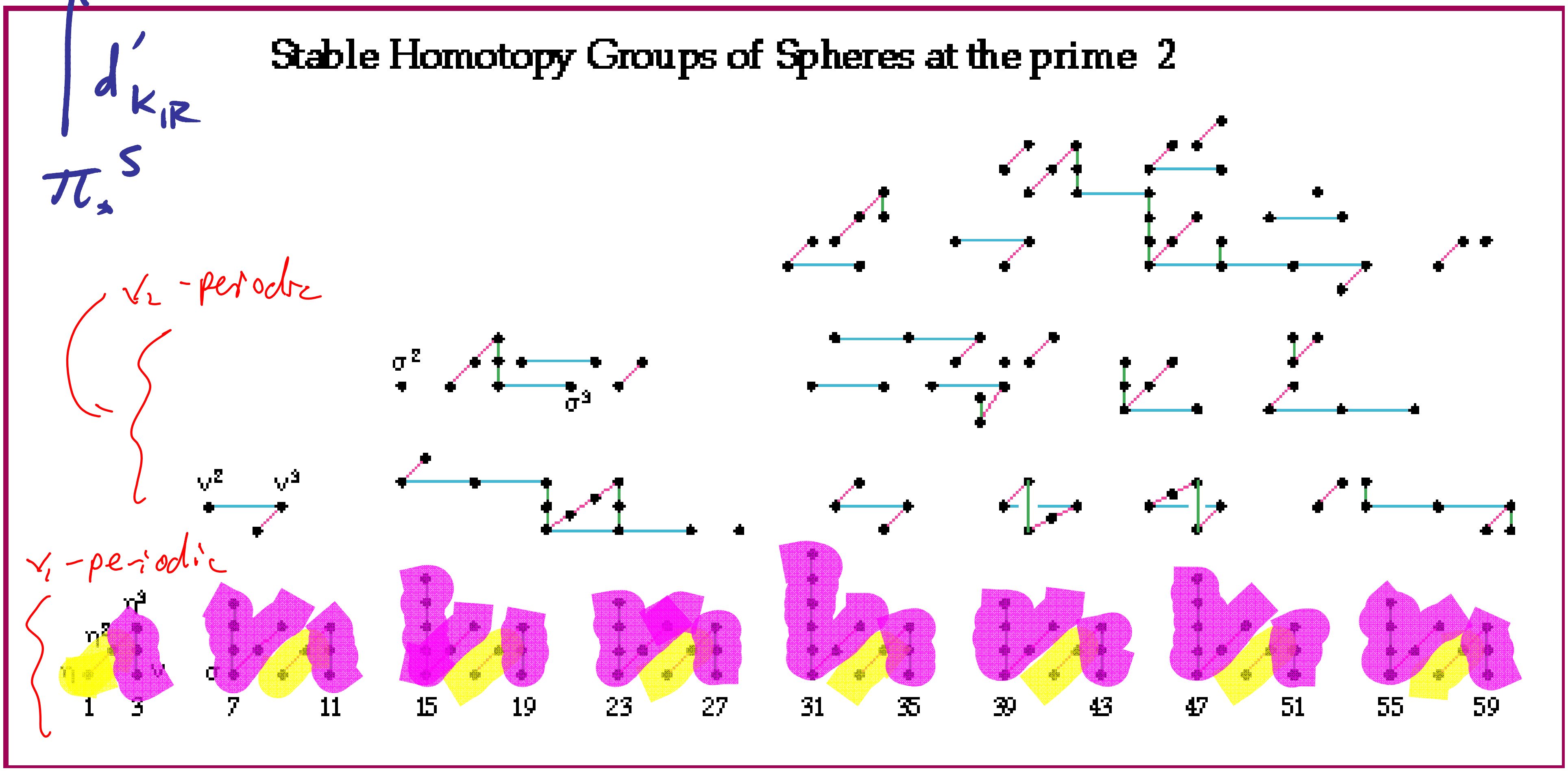
$\in \Omega_{n+1}^G$

$$d_E' : (\pi_n^s)_{torsion} \longrightarrow \Omega_{n+1}^G \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow E^{-(n+1)}(pt) \otimes \mathbb{Q}/\mathbb{Z}$$

$$(E, \Phi) = (K_{\mathbb{R}}, \hat{A})$$

$$K_{\mathbb{R}}^{-(*+1)}(pt)\otimes \mathbb{Q}/\mathbb{Z} = \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Q}/\mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Q}/\mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Q}/\mathbb{Z} \cdots$$

$$\pi_*^S \xrightarrow{d'_{K_{\mathbb{R}}}} K_{\mathbb{R}}^{-(*+1)}(pt)\otimes \mathbb{Q}/\mathbb{Z}$$

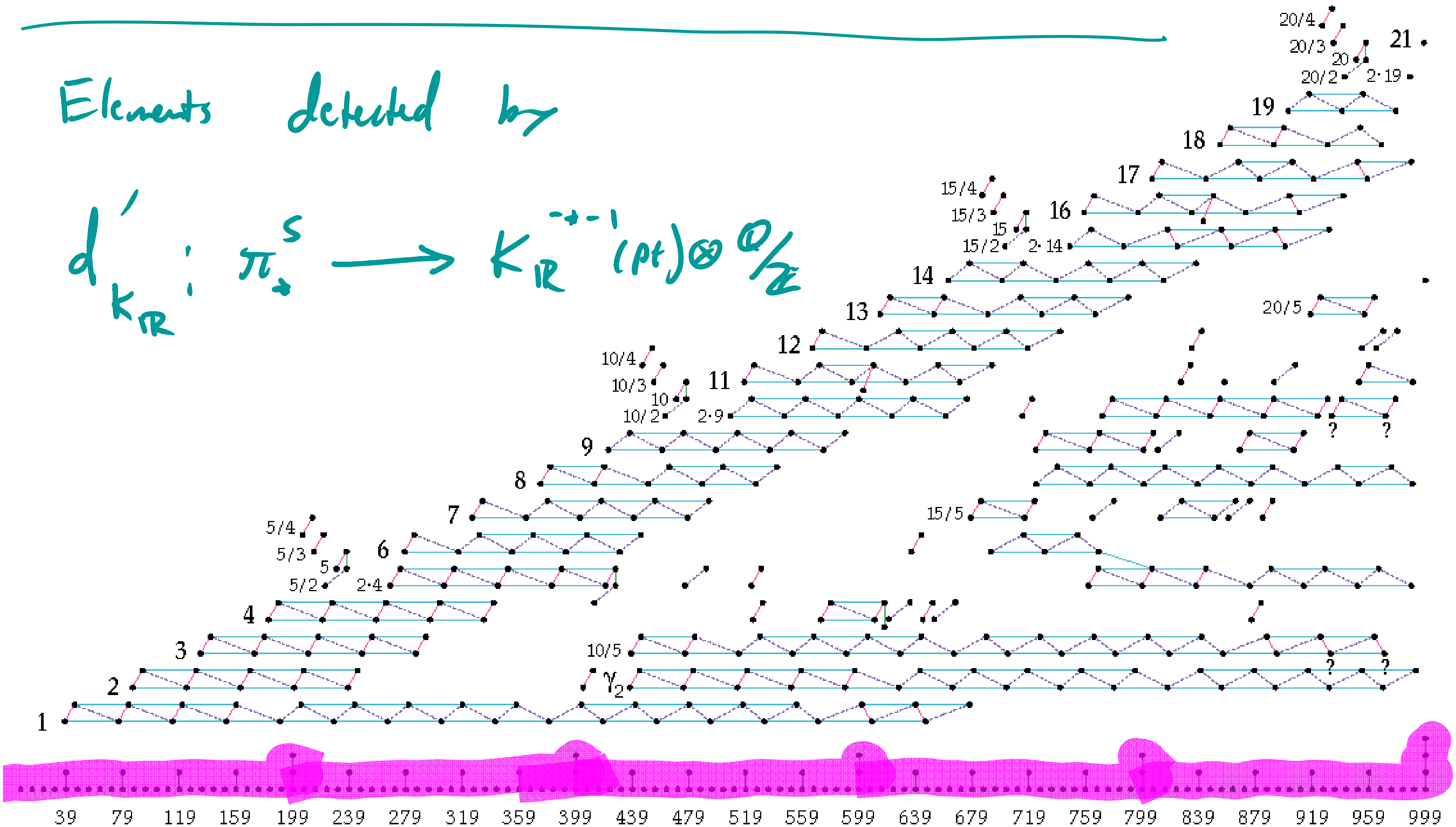


5-torsion

$$(\pi_*^s)_{(5)}$$

Elements detected by

$$d'_{K_{\mathbb{R}}} : \pi_*^s \longrightarrow K_{\mathbb{R}}^{-*-1}(pt) \otimes \mathbb{Q}/\mathbb{Z}$$



Thm (Laures, Baker)

Image $d'_{TMF} : (\pi^s_{2k-1})_{torsion} \longrightarrow TMF^{-2k}(pt) \otimes \mathbb{Q}/\mathbb{Z}$

is generated by

$$G_k(\tau) = \frac{B_{2k}}{4k} E_{2k}(\tau) = \frac{B_{2k}}{4k} - \sum_{n=1}^{\infty} \Big(\sum_{d|n} d^{2k-1} \Big) q^n$$

Consequence (originally due to Adams)

v_1-periodic $(\pi^s_*) \Longleftrightarrow$ denominators of Bernoulli numbers

(odd primary)

Similar techniques (Lawres, B)

$$d''_{TMF} : v_2\text{-periodic}\,(\pi_*^S)_{(p)} \longrightarrow \frac{\text{Modular forms}}{\text{congruence of } q\text{-expansion}}$$

Thm (B)

$f \in$ image d''_{TMF}

$\langle \ell \rangle = \mathbb{Z}_p^\times$

$\Updownarrow$

f Not congruent mod p^j to form of lower weight | $g(\tau) = f(\tau) - f(\ell\tau)$ Is congruent mod p^j to a form of lower weight

Elements of $(\pi_*^S)_{(5)}$ detected by d''_{TMF}