The homotopy groups of the E(2)-local sphere at $p \geq 5$, revisited

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Chromatic Theory

$(\pi_*^s)_{(p)}$ = p-local stable htpy gps of spheres

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- Admits a filtration (chromatic filtration)
- k^{th} layer exhibits periodic behavior (v_k-periodicity)

$$|v_k| = 2(p^k - 1)$$

Chromatic Theory

$(\pi_*^s)_{(p)}$ = p-local stable htpy gps of spheres

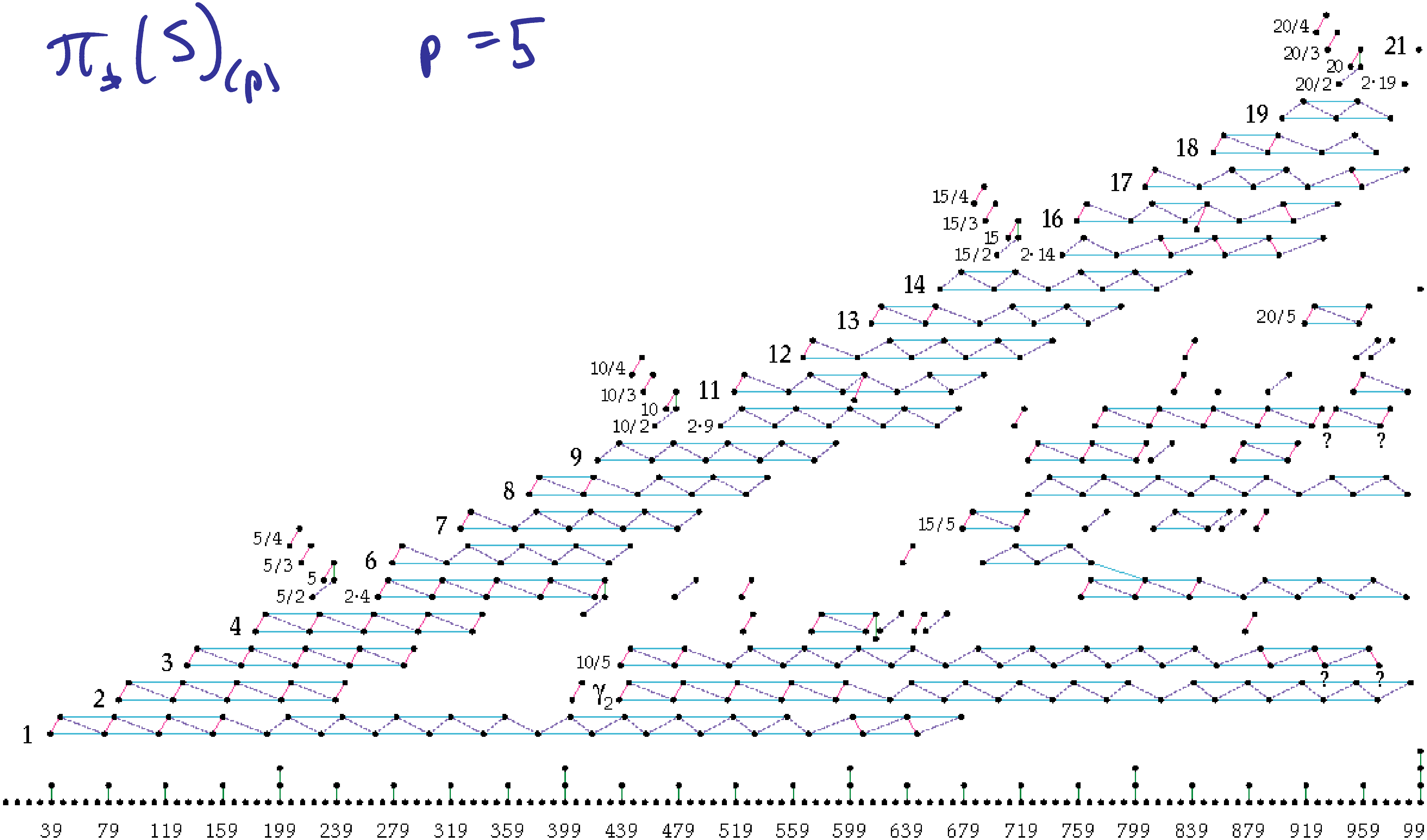
- Admits a filtration (chromatic filtration)

$$S_{(p)} \to \cdots \to S_{E(2)} \to S_{E(1)} \to S_{\mathbb{Q}}$$

- k^{th} layer exhibits periodic behavior (v_k-periodicity)

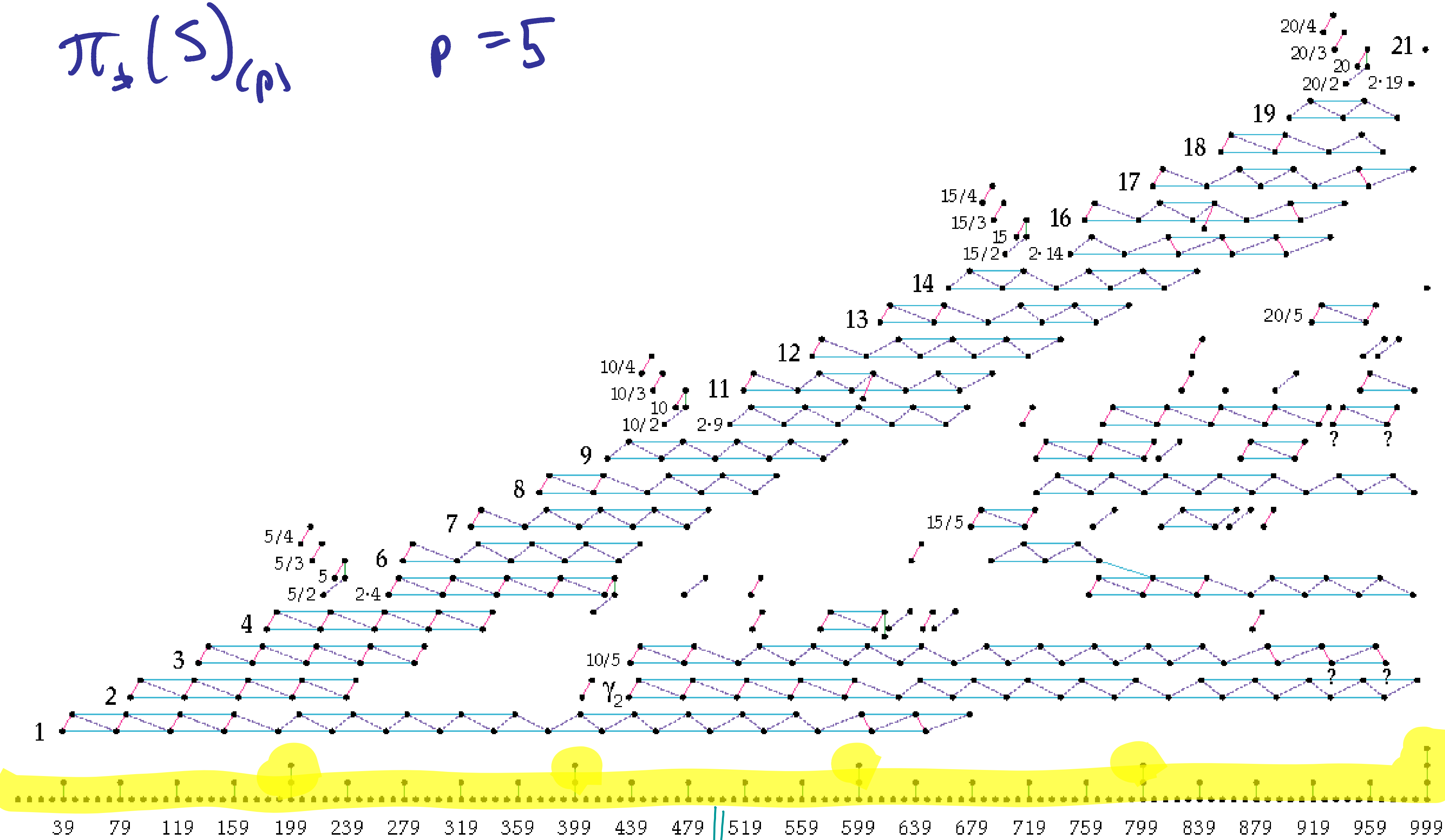
"k^{th} layer" $M_k S \longrightarrow S_{E(k)} \longrightarrow S_{E(k-1)}$

$\pi_*(S)_{(p)}$ $\quad p=5$



picture: Hatcher
computation: Ravenel

$\pi_*(S)_{(p)}$ $p=5$

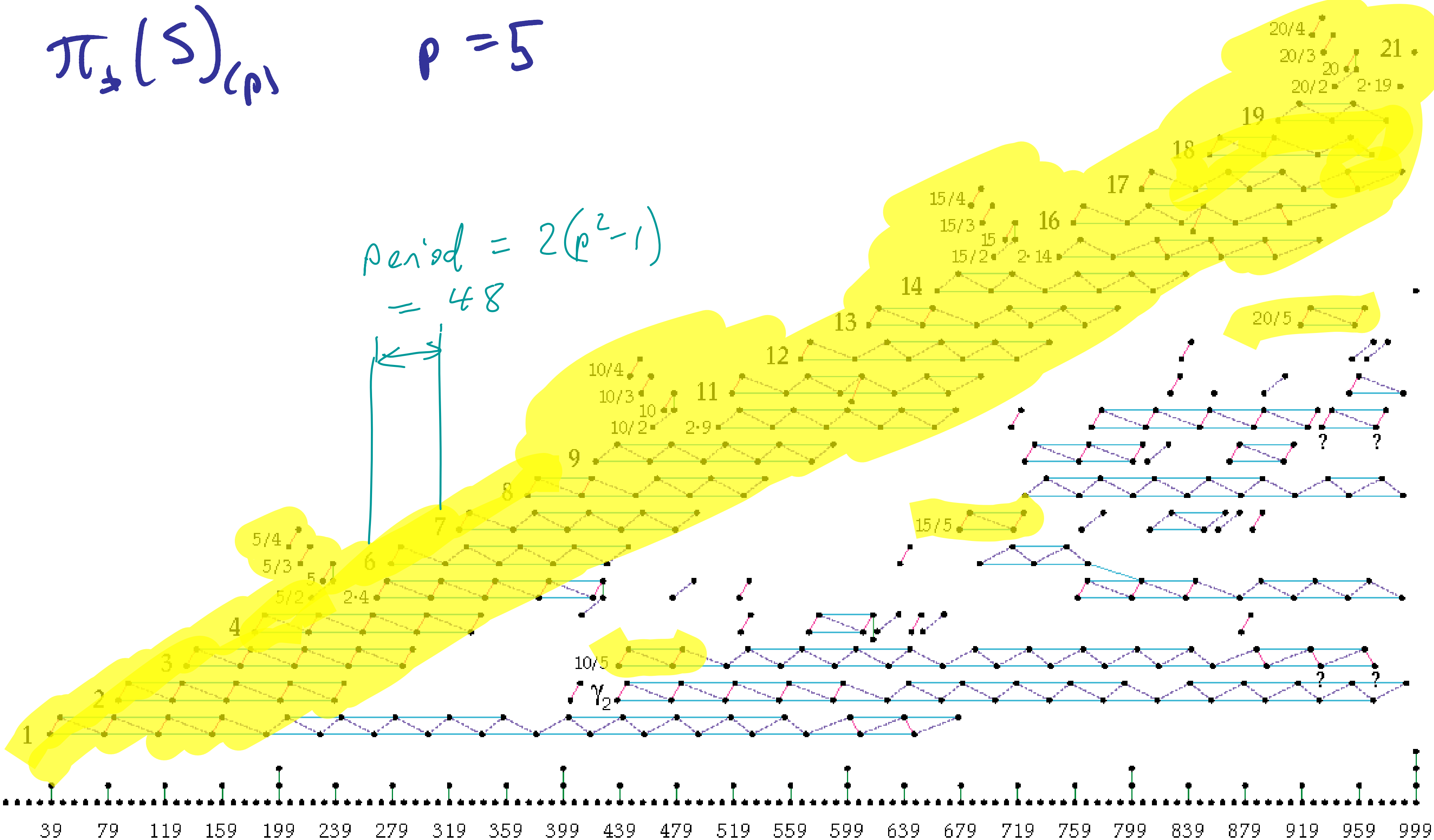


period $= 2(p-1) = 8$

picture: Hatcher
computation: Ravenel

v_1-periodic
$=$ ImJ

$\pi_*(S)_{(p)}$ $p=5$

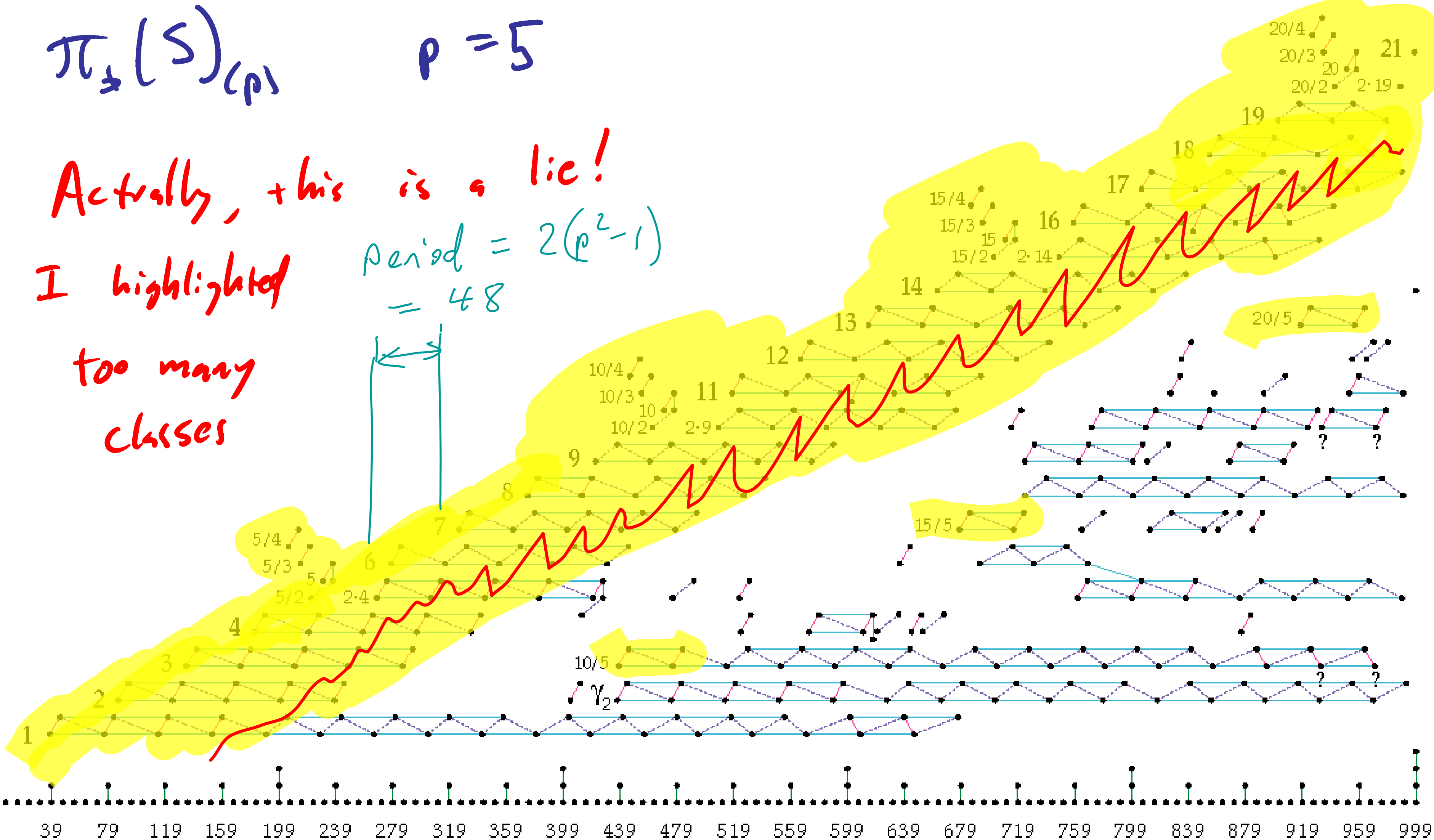


picture: Hatcher
computation: Ravenel

v_2-periodic

$\pi_*(S)_{(p)}$ $\quad p = 5$

Actually, this is a lie! I highlighted too many classes

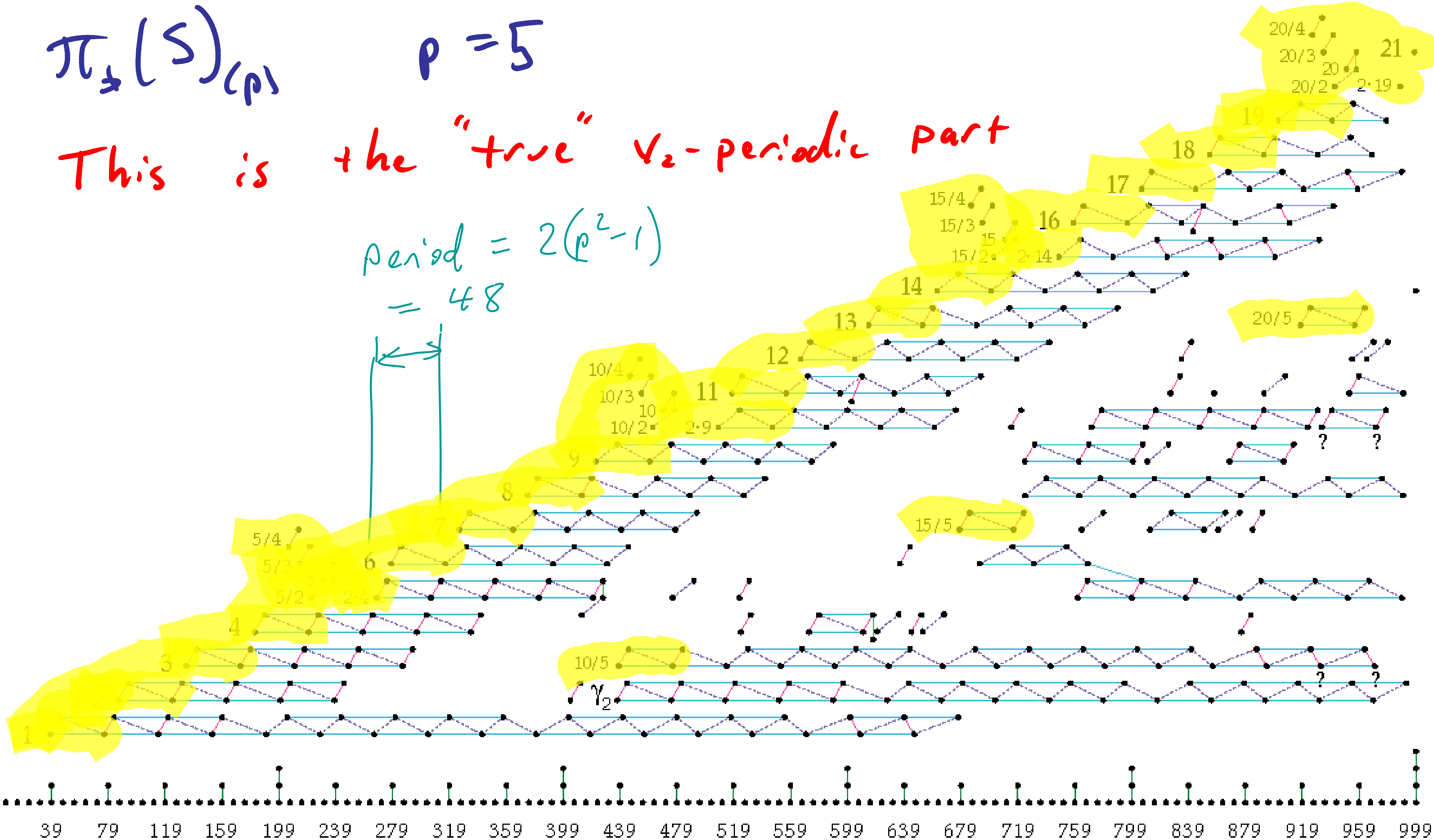


picture: Hatcher
computation: Ravenel

v_2-periodic

$\pi_*(S)_{(p)}$ $\quad p=5$

This is the "true" v_2-periodic part

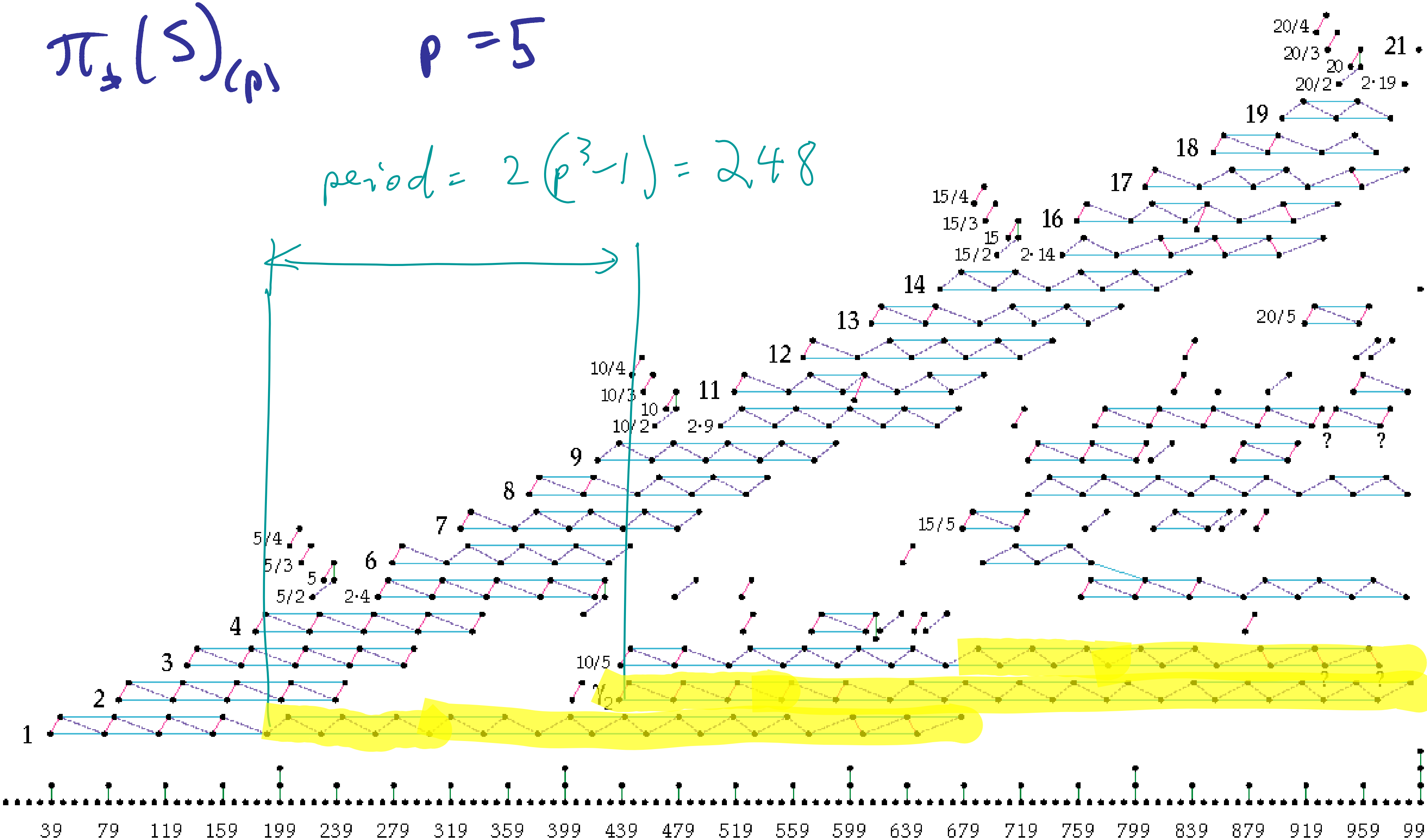


v_2-periodic

picture: Hatcher
computation: Ravenel

$\pi_*(S)_{(p)}$ $p=5$

period $= 2(p^3-1) = 248$

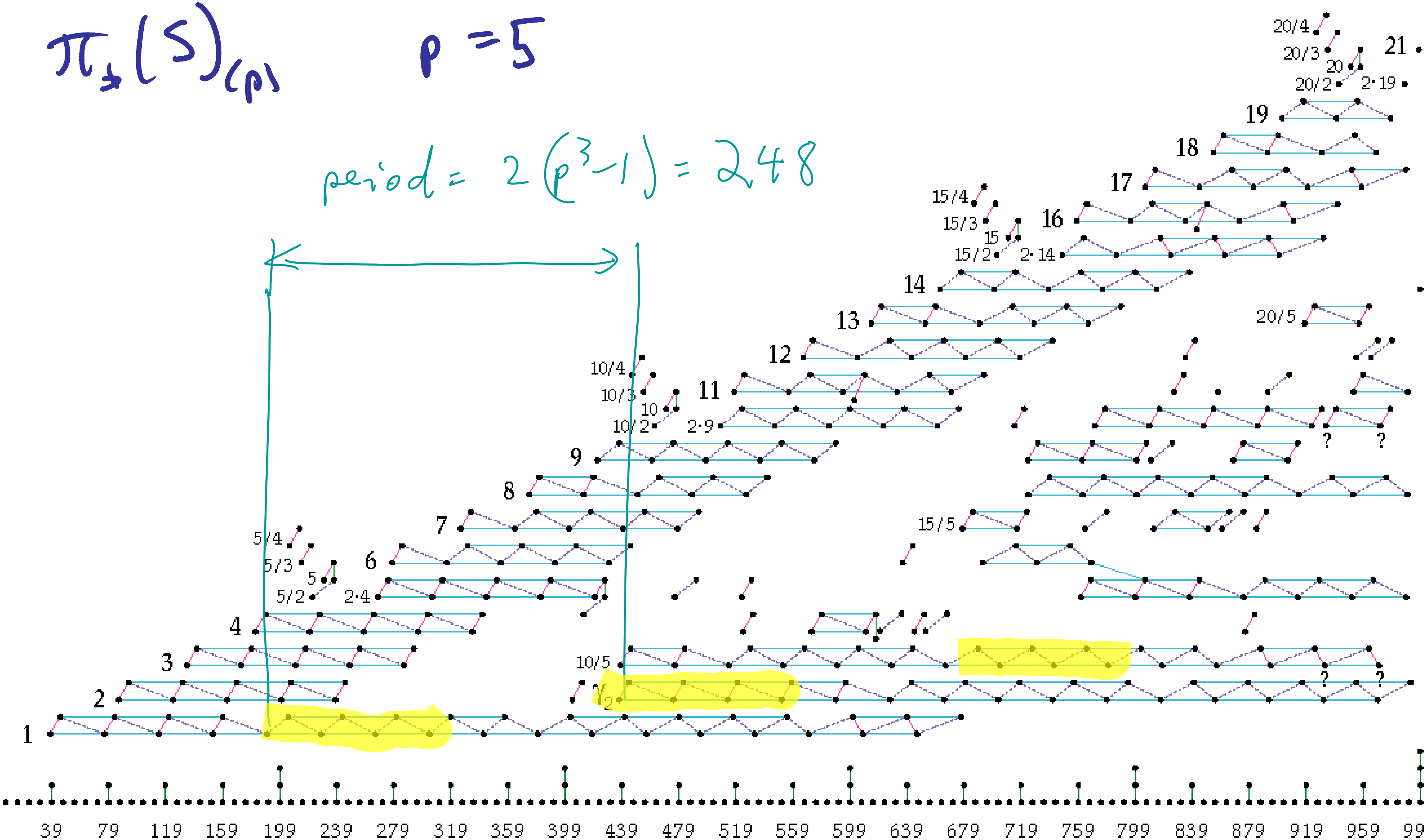


picture: Hatcher
computation: Ravenel

v_3-periodic

$\pi_*(S)_{(p)}$ $\quad p=5$

period $= 2(p^3-1) = 248$



picture: Hatcher
computation: Ravenel

v_3-periodic [Actually, probably just this!]

Greek letter elements

The most fundamental v_n-periodic elts are the GREEK LETTER ELTS

Greek letter elements

The most fundamental v_n-periodic elts are the GREEK LETTER ELTS

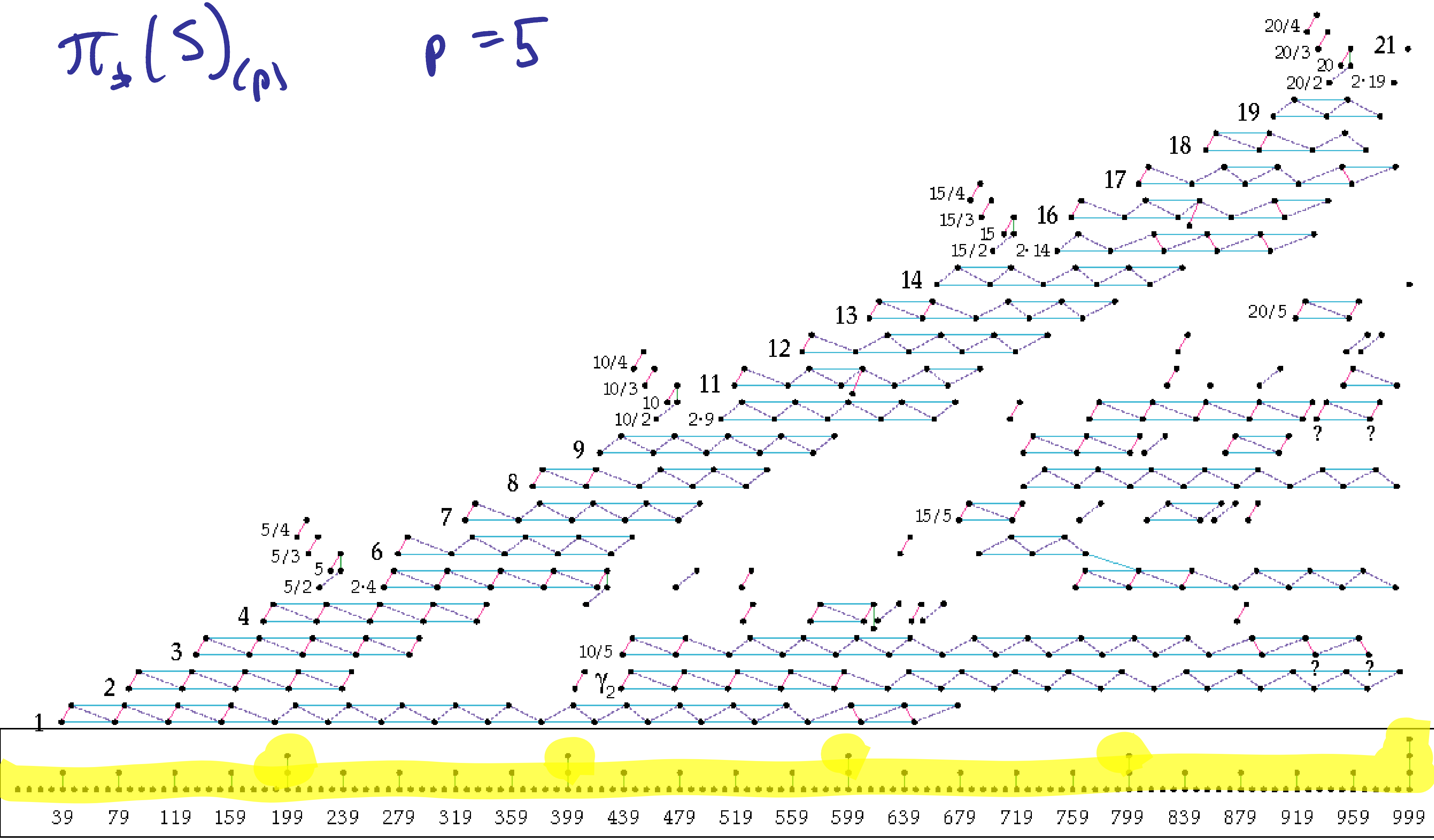
Notation

v_1-periodic : $\alpha_{i/j}$

v_2-periodic : $\beta_{i/j,k}$

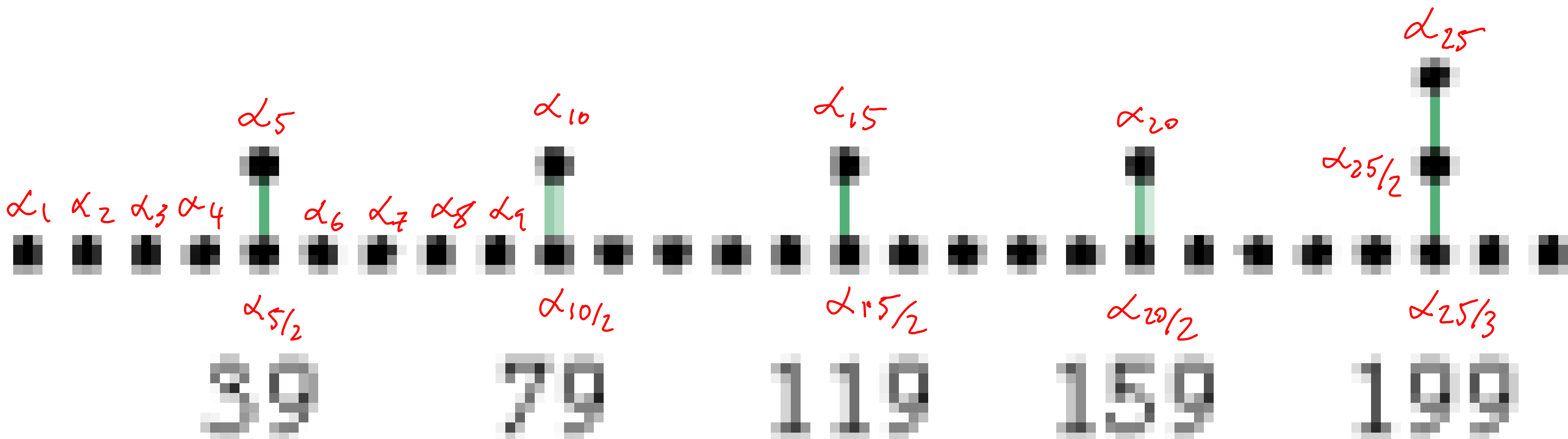
v_3-periodic : $\gamma_{i/j,k,l}$

$\pi_*(S)_{(p)} \qquad p=5$



v_1-periodic: α-family

Greek letter notation: $\alpha_{i/j} \in (\pi^S_{2(p-1)i-1})$

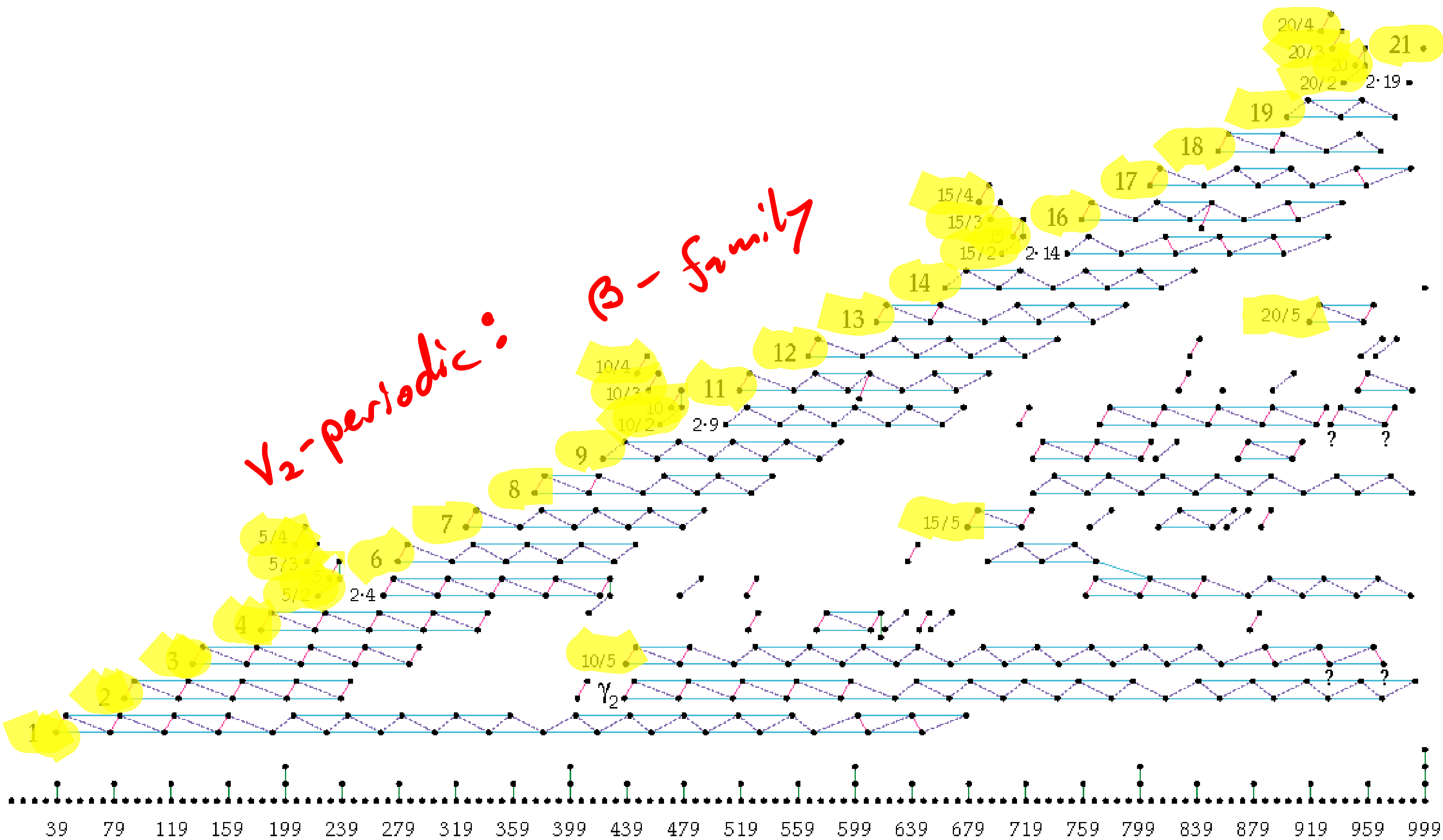


$\alpha_{i/j}$ is p^j – torsion

"ImJ pattern"

$(\alpha_i := \alpha_{i/1})$

$\pi_*(S)_{(p)}$ $p=5$



v_2-periodic: β-family
1
2
3
4
5/2
5/3
5/4
5
2·4
6
7
8
9
10/2
10/3
10/4
10
2·9
10/5
γ_2
11
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15/2
15/3
15/4
15
2·14
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17
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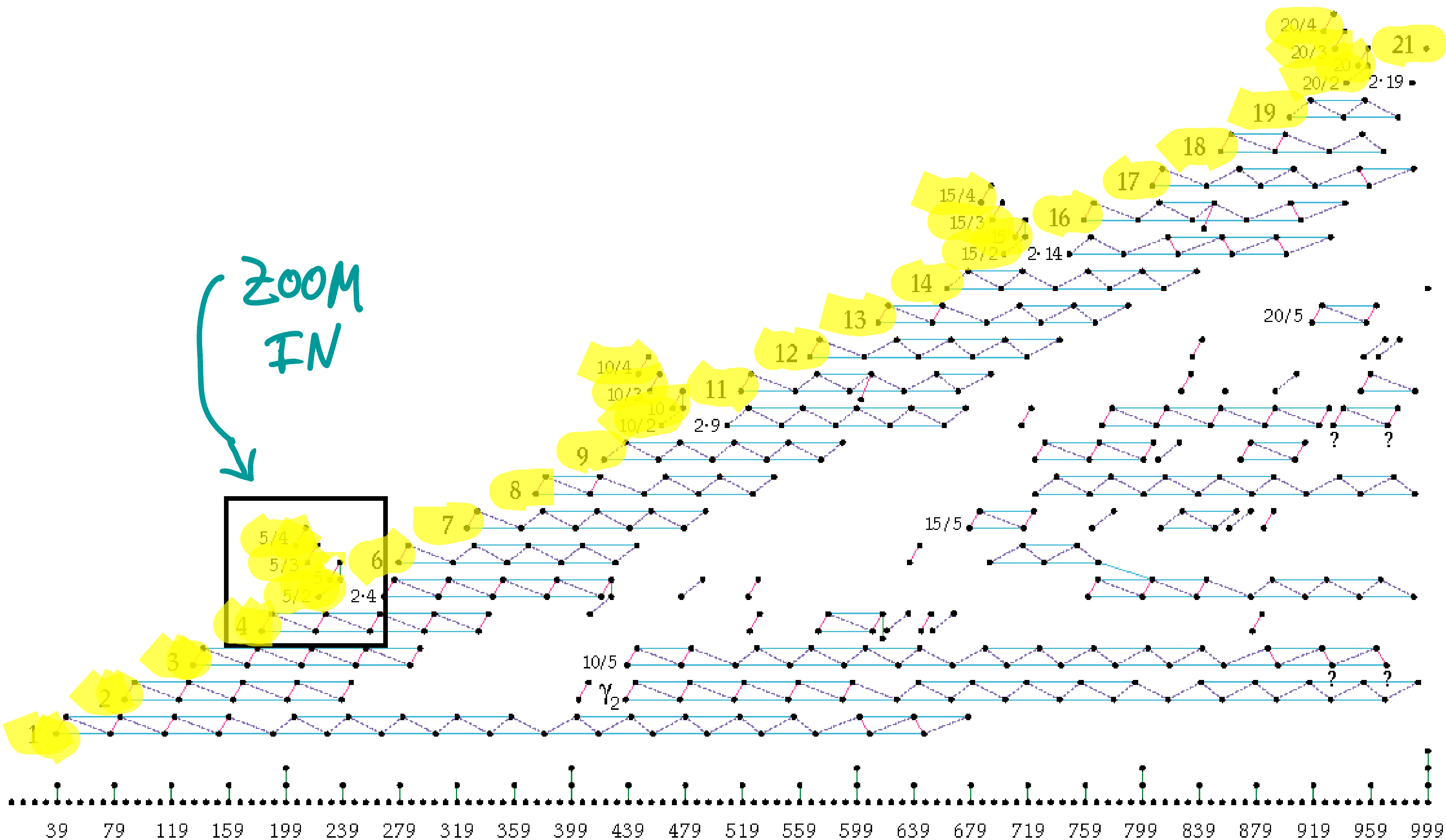
19
20/2
20/3
20/4
20
2·19
20/5
21
?
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

$\pi_*(S)_{(p)}$ $\quad p=5$


ZOOM IN
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5/2
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5/4
2·4
6
7
8
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10/2
10/3
10/4
2·9
10/5
γ_2
11
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14
15/2
15/3
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2·14
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18
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2·19
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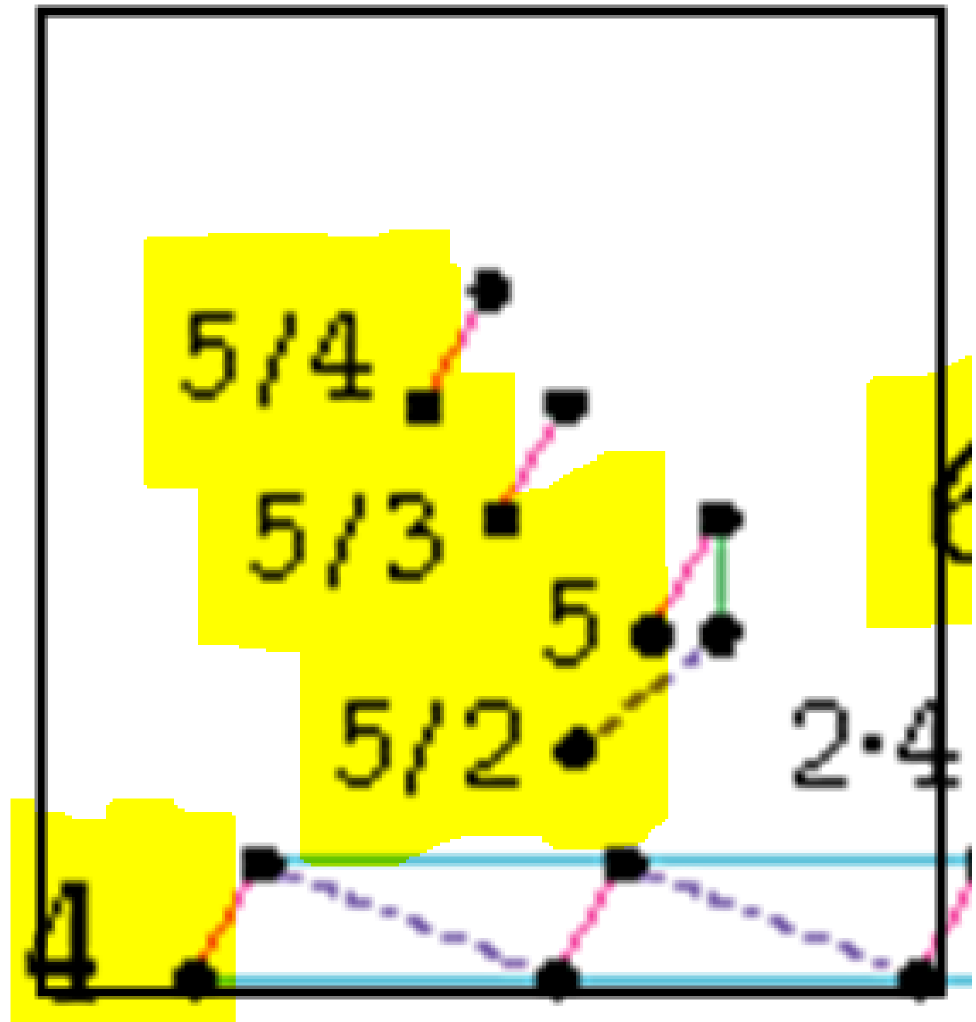
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

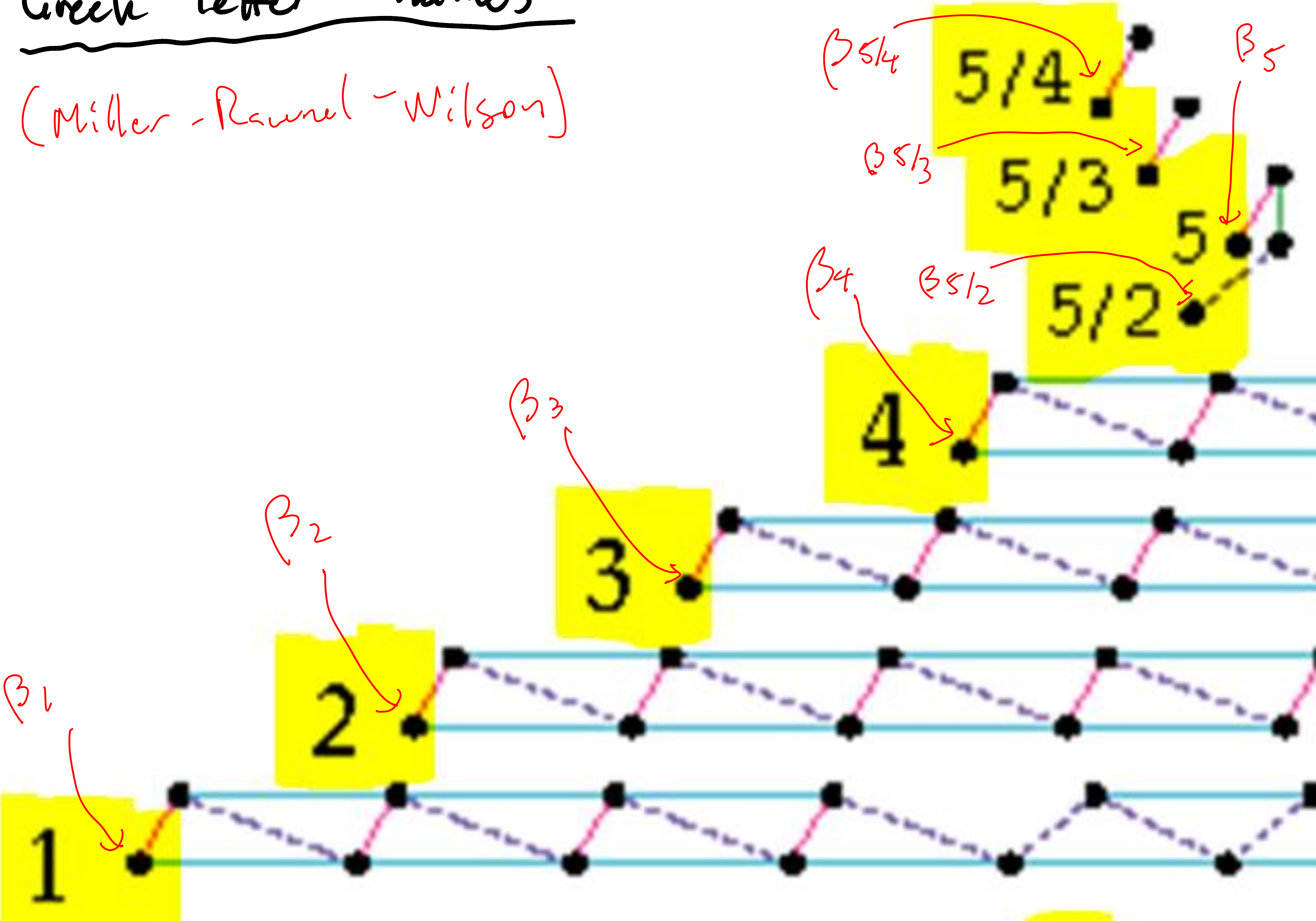
v_1-torsion in y_2-family



$$\text{“}5/4\text{”} \xrightarrow[v_1]{} \text{“}5/3\text{”} \xrightarrow[v_1]{} \text{“}5/2\text{”} \xrightarrow[v_1]{} \text{“}5\text{”} \xrightarrow[v_1]{} 0$$

"Greek letter names"

(Miller-Rawnsley-Wilson)



β-family notation

$$\beta_{i/j,k} \in \left(\pi^S_{2(p^2-1)i-2(p-1)j-2}\right)_{(p)}$$

p^k-torsion

Conversion

$\beta_{i/j,1} =: \beta_{i/j}$

$\beta_{i/1} =: \beta_i$

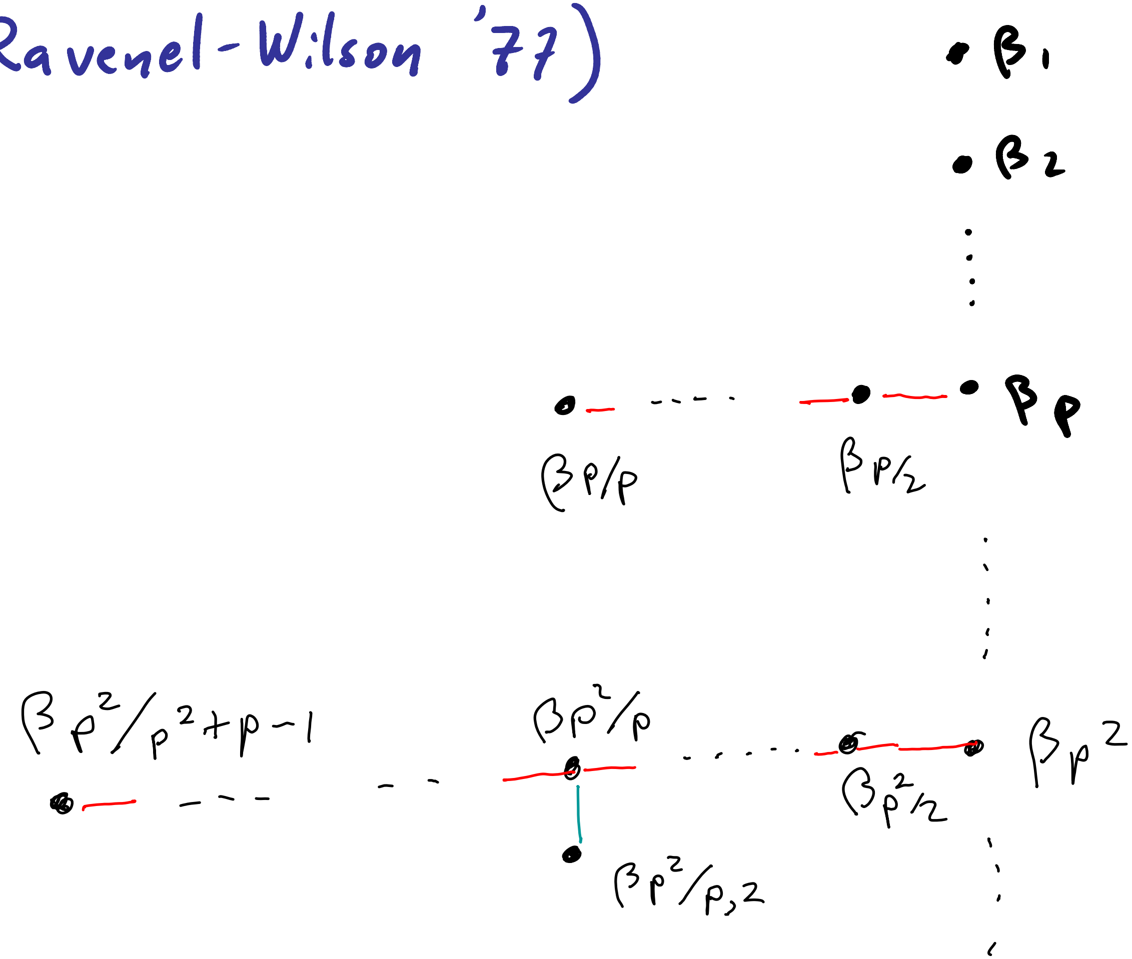
$$v_2 \beta_{i/j,k} = \beta_{i+1/j,k}$$

$$v_1 \beta_{i/j,k} = \beta_{i/j-1,k}$$

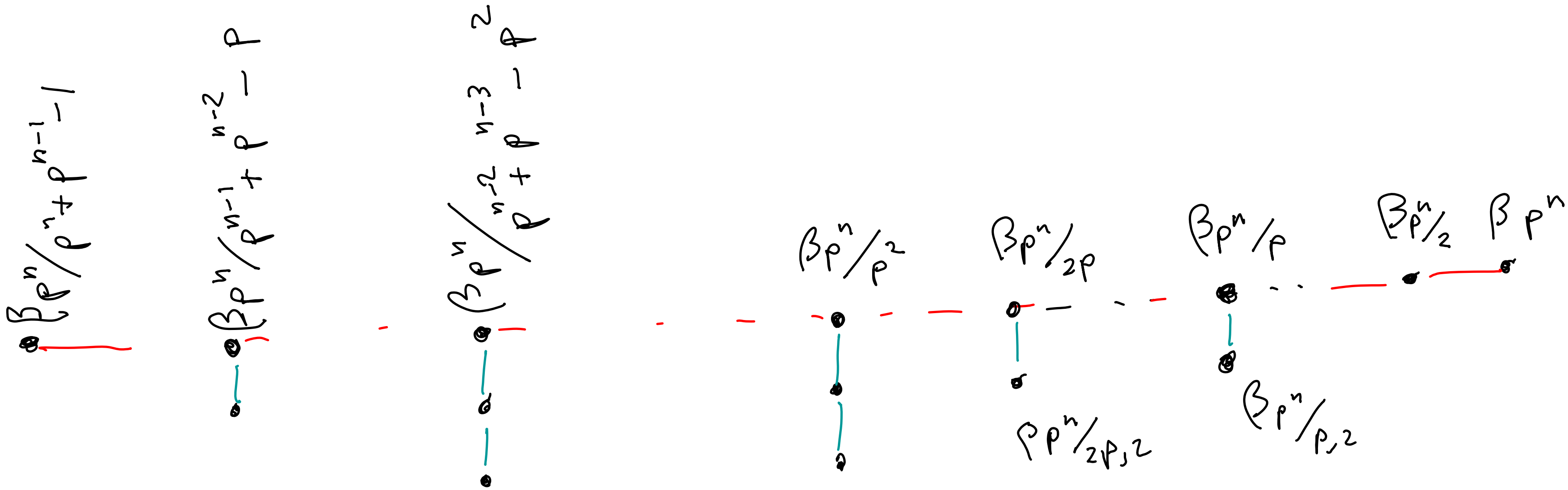
$$p\, \beta_{i/j,k} = \beta_{i/j,k-1}$$

Description of β family

(Miller-Ravenel-Wilson '77)



Description of β family



ANSS:

$$\mathrm{Ext}^{s,t}_{BP_*BP}\Big(BP_*,\ \overbrace{BP_*M_2(s)}^{\frac{BP_*}{(p^\infty,\, v_1^\infty)}[v_2^{-1}]}\Big) \Rightarrow \pi_{t-s}M_2(s)$$

$$\|$$

$$H^{s,t}(M_0^2)$$

ANSS:

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*M_2(S)) \Rightarrow \pi_{t-s} M_2(S)$$

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*M_2(S)) \cong H^{s,t}(M_0^2)$$

- $(p \geq 5)$ $H^{s,t}(M_0^2) = 0$ for $s > 4$
- $H^{s,t}(M_0^2) = 0$ for $t \not\equiv 0 \mod \overset{2(p-1)}{q}$

ANSS:

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*M_2(s)) \Rightarrow \pi_{t-s} M_2(s)$$

$$\begin{array}{c} \| \\ H^{s,t}(M_0^2) \end{array}$$

$(p \geq 5)$

- $H^{s,t}(M_0^2) = 0$ for $s > 4$

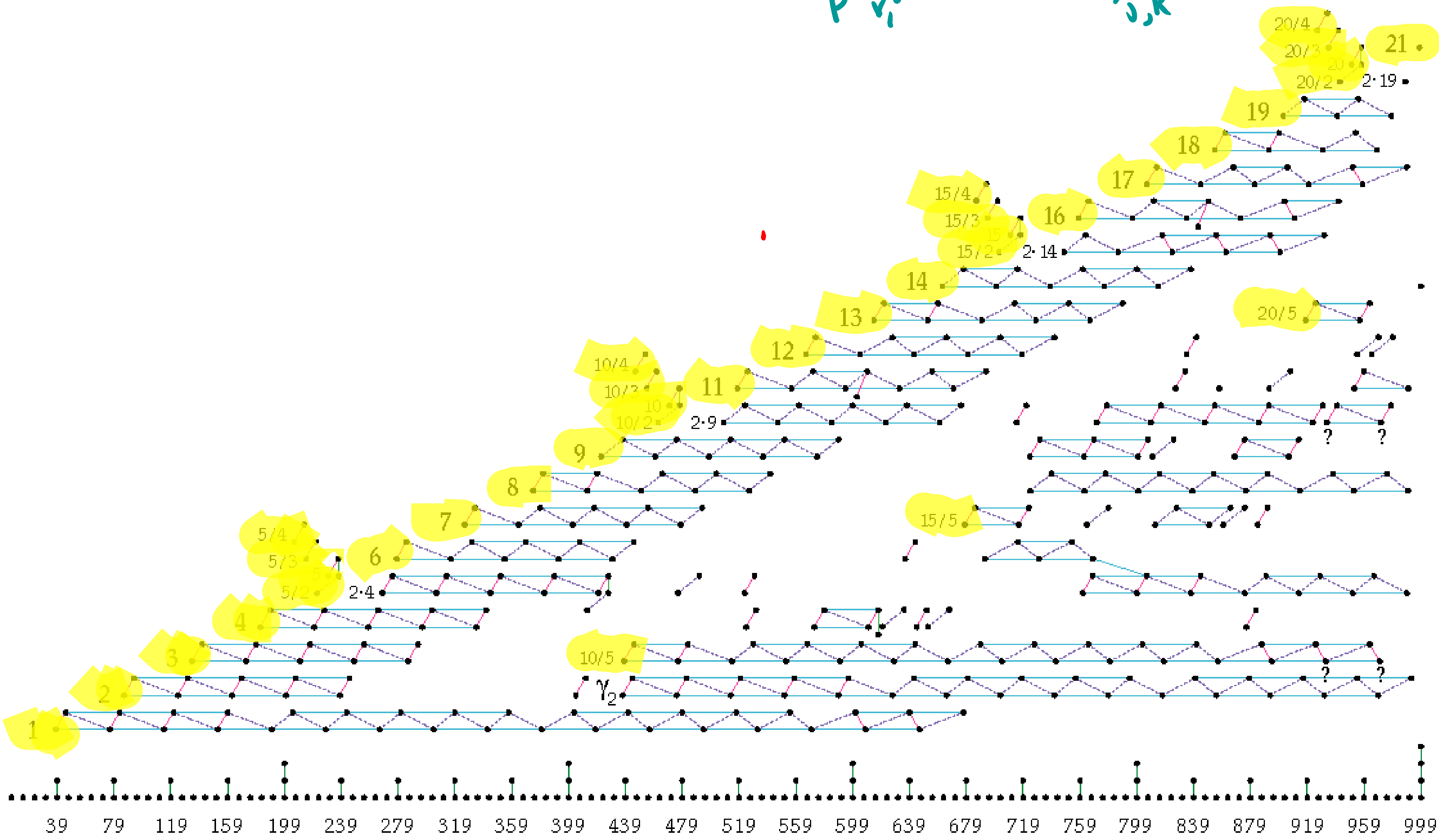
$2(p-1)$

- $H^{s,t}(M_0^2) = 0$ for $t \not\equiv 0 \mod q$

$\Longrightarrow$ ANSS collapses

$\pi_*(S)_{(p)}$ $p = 5$

= β-family = $H^{0,*}(M_0^2) \ni \dfrac{v_2^{c'}}{p^k v_1^j} \longleftrightarrow \beta_{i/j,k}$

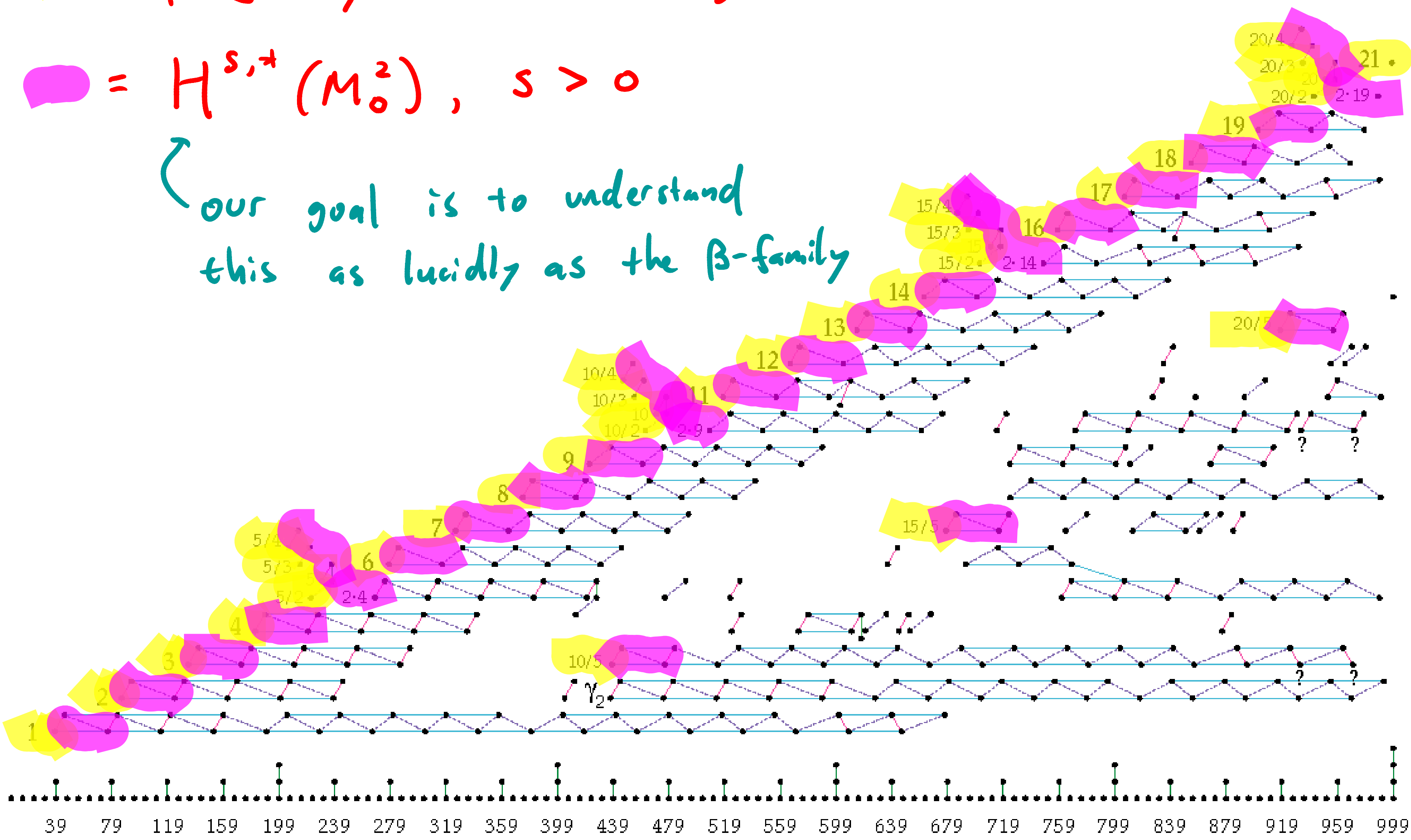


$\pi_*(S)_{(p)} \qquad p=5$

= β-family = $H^{0,*}(M_0^2)$

= $H^{s,*}(M_0^2)$, $s>0$

(our goal is to understand this as lucidly as the β-family



Strategy for Computation of $H^* M_0^2$

[Miller - Ravenel - Wilson]

$$H^*(M_0^2) \cong \mathrm{Ext}\left(\frac{BP_*}{(p^\infty, v_1^\infty)}[v_2^{-1}]\right)$$

Strategy for Computation of $H^* M_0^2$

[Miller-Ravenel-Wilson]

$$\begin{array}{ccc} H^*(M_1^1) & & H^*(M_0^2) \\ \| & & \| \\ \mathrm{Ext}\left(\frac{BP_*}{(p, v_1^\infty)}[v_2^{-1}]\right) & \xRightarrow[v_0\text{-BSS}]{} & \mathrm{Ext}\left(\frac{BP_*}{(p^\infty, v_1^\infty)}[v_2^{-1}]\right) \end{array}$$

Strategy for Computation of $H^*M_0^2$

[Miller-Ravenel-Wilson]

$$
\begin{array}{ccccc}
H^*(M_2^0) & & H^*(M_1^1) & & H^*(M_0^2) \\
\Vert & & \Vert & & \Vert \\
\mathrm{Ext}\left(\dfrac{BP_*}{(p,v_1)}[v_2^{-1}]\right) & \overset{}{\underset{v_1\text{-BSS}}{\Longrightarrow}} & \mathrm{Ext}\left(\dfrac{BP_*}{(p,v_1^\infty)}[v_2^{-1}]\right) & \overset{}{\underset{v_0\text{-BSS}}{\Longrightarrow}} & \mathrm{Ext}\left(\dfrac{BP_*}{(p^\infty,v_1^\infty)}[v_2^{-1}]\right)
\end{array}
$$

Strategy for Computation of $H^*M^2_0$

[Miller-Ravenel-Wilson]

$$\begin{array}{ccccc}
H^*(M_2^0) & & H^*(M_1^1) & & H^*(M_0^2) \\
\| & & \| & & \| \\
\mathrm{Ext}\left(\dfrac{BP_*}{(p,v_1)}[v_2^{-1}]\right) & \underset{v_1\text{-BSS}}{\Longrightarrow} & \mathrm{Ext}\left(\dfrac{BP_*}{(p,v_1^\infty)}[v_2^{-1}]\right) & \underset{v_0\text{-BSS}}{\Longrightarrow} & \mathrm{Ext}\left(\dfrac{BP_*}{(p^\infty,v_1^\infty)}[v_2^{-1}]\right)
\end{array}$$

$\cong$ Morava change of rings

$$H^*(\mathbb{S}_2; \mathbb{F}_p[v_2^{\pm 1}]) \Big\} \text{ "Computable"}$$

Morava Stabilizer gp

Strategy for Computation of $H^*M_0^2$

[Miller-Ravenel-Wilson]

$$\begin{tikzcd}
H^*(M_2^0) \arrow[d, equal] & H^*(M_1^1) \arrow[d, equal] & H^*(M_0^2) \arrow[d, equal] \\
\mathrm{Ext}\left(\frac{BP_*}{(p,v_1)}[v_2^{-1}]\right) \arrow[r, Rightarrow, "v_1\text{-BSS}"'] \arrow[d, "\cong"] & \mathrm{Ext}\left(\frac{BP_*}{(p,v_1^\infty)}[v_2^{-1}]\right) \arrow[r, Rightarrow, "v_0\text{-BSS}"'] & \mathrm{Ext}\left(\frac{BP_*}{(p^\infty,v_1^\infty)}[v_2^{-1}]\right) \\
\underbrace{H^*(\mathbb{S}_2; \mathbb{F}_p[v_2^{\pm 1}])}_{\text{Ravenel '77}} & \text{Shimomura '86} & \text{Shimomura-Yabe '95}
\end{tikzcd}$$

Strategy for Computation of $H^*M_0^2$

[Miller-Ravenel-Wilson]

$$
\begin{array}{ccccc}
H^*(M_2^0) & & H^*(M_1^1) & & H^*(M_0^2) \\
\parallel & & \parallel & & \parallel \\
\mathrm{Ext}\left(\frac{BP_*}{(p,v_1)}[v_2^{-1}]\right) & \xRightarrow{v_1\text{-BSS}} & \mathrm{Ext}\left(\frac{BP_*}{(p,v_1^\infty)}[v_2^{-1}]\right) & \xRightarrow{v_0\text{-BSS}} & \mathrm{Ext}\left(\frac{BP_*}{(p^\infty,v_1^\infty)}[v_2^{-1}]\right) \\
\cong & & & & \\
\underbrace{H^*(\mathbb{S}_2;\, \mathbb{F}_p[v_2^{\pm 1}])}_{\text{Ravenel '77}} & & & &
\end{array}
$$

v_1-BSS: Shimomura '86 "Understandable"

v_0-BSS: Shimomura-Yabe '95 "Incomprehensible"

Ravenel '77 "Easy"

From [SY95]

THEOREM 2.3. *The module $H^*M_0^2$ is isomorphic to*

$$(X_\infty^\infty \oplus Y_{\infty,C}^\infty \oplus G_0^\infty) \otimes E(\zeta) \oplus X^\infty \oplus X\zeta_C^\infty \oplus Y_{0,C}^\infty \oplus Y_{1,C}^\infty \oplus Y_C^\infty \oplus G^\infty.$$

Here the modules are defined by

$$X^\infty = \mathbf{Z}_{(p)}\{v_2^{sp^n}/p^{i+1}v_1^j : n \geq 0,\ s \in \mathbf{Z} - p\mathbf{Z},\ i \geq 0,$$

$$j \geq 1 \text{ with } p^i | j \leq a_{n-i} \text{ and either } p^{i+1} \not| j \text{ or } a_{n-i-1} < j\}$$

$$X_\infty^\infty = \mathbf{Z}_{(p)}\{1/p^{i+1}v_1^j : i = \nu_p(j) \geq 0\}$$

for dimension 0,

$$X\zeta_C^\infty = \mathbf{Z}_{(p)}\{v_2^{sp^n}\zeta/p^{i+1}v_1^j : s \in \mathbf{Z} - p\mathbf{Z},\ j > 0,\ p^i | j \leq a_{n-i}$$

$$\text{either } p^{i+1} \not| j \text{ or } j > a_{n-i-1}, \text{ and } p^{i+1} | j \text{ if } p^{k+1} | j \text{ for } s = tp^{k+1} - 1 \text{ with } k \geq 0\}$$

$$Y_{0,C}^\infty = \mathbf{Z}_{(p)}\{v_2^{sp^n}h_0/p^{i+1}v_1^{kp^i+1} : p \not| s(s+1), \text{ for } k = 0,\ i = n, \text{ and for } k > 0,$$

$$kp^i + 1 \leq A_{n-i} + 2,\ kp^i + 1 > a_{n-i} \text{ if } p \not| k, \text{ and } > A_{n-i-1} + 2 \text{ otherwise}\}$$

$$Y_{1,C}^\infty = \mathbf{Z}_{(p)}\{v_2^{(tp^2-1)p^n}h_0/p^l v_1^{kp^i+1} : l = n+1 \text{ if } k = 0; \text{ for } k > 0 \text{ with } kp^i > a_{n-i},$$

$$l = i > 0 \text{ for } p^{n+2} - p^n < kp^i < p^{n+2} - p^n + A_{n-i+1} + 2 \text{ and}$$

$$p^{n+2} - p^n + A_{n-i} + 2 \leq kp^i \text{ if } p | k$$

$$l = i + 1 \text{ for } i = 0 \text{ and } p \not| (k + p^{n-i}), \text{ for } kp^i = (p^2 - 1)p^n \text{ or}$$

$$\text{for } kp^i < p^{n+2} - p^n,\ p \not| (k + p^{n-i}) \text{ and } 0 < i \leq n$$

$$l = n + 2 \text{ for } i = n,\ k \leq p^2 - 1,\ p | (k+1) \text{ and } k \neq p^2 - p - 1; \text{ and}$$

$$l = n + 3 \text{ if } i = n \text{ and } k = p^2 - p - 1\}$$

$$Y_C^\infty = \mathbf{Z}_{(p)}\{v_2^{tp-1}h_1/p^l v_1^j : l = 1 \text{ if } j < p - 1, \text{ and } l = 2 \text{ if } p | t \text{ and } j = p - 1\}$$

$$Y_{\infty,C}^\infty = \mathbf{Q}/\mathbf{Z}_{(p)} \text{ generated by the set } \{h_0/p^j v_1 : j > 0\}$$

ctd ...

for dimension 1, and

$$G^{\infty} = G^{\infty}_{C} \oplus Y\zeta^{\infty}_{C}$$

$$Y\zeta^{\infty}_{C} = (Y^{\infty,G}_{0,C} \oplus Y^{\infty,G}_{1,C}) \otimes \mathbf{Z}_{(p)}\{\zeta\}$$

for

$$Y^{\infty,G}_{0,C} = \mathbf{Z}_{(p)}\{v_2^{sp^n} h_0/p^{i+1} v_1^{kp^{i+1}+1} : p \nmid s(s+1),$$
$$k \neq 0,\ A_{n-i-1} + 1 < kp^{i+1} \le A_{n-i} + 1 \text{ for } i \ge 0\}$$

$$Y^{\infty,G}_{1,C} = \mathbf{Z}_{(p)}\{v_2^{(tp^2-1)p^n} h_0/p^{i+1} v_1^{kp^{i+1}+1} : k \neq 0,$$
$$p^{n+2} - p^n + A_{n-i-1} + 1 < kp^{i+1} \le p^{n+2} - p^n + A_{n-i} + 1 \text{ for } i \ge 0\}$$

$$G^{\infty}_{C} = \mathbf{Z}_{(p)}\{v_2^{sp^n} g_0/p^{n+1} v_1,\ v_2^{sp^n - (p^{n-1}-1)/(p-1)} g_1/p^l v_1^j :$$
$$p \nmid (s+1),\ 0 < j \le a_n,\ p^{i+1} \nmid (j + A_{n-i-1} + 1) \text{ if } s = up^i \in \mathbf{Z}(0),$$
$$p^i \nmid (j + A_{n-i} + 1) \text{ if } s = up^i \in \mathbf{Z}(2), \text{ and } l = i + 1 \text{ if } n = 0 \text{ and } v_p(s) = i;$$
$$l = i + 1 \text{ if } n \ge 1 \text{ and } v_p(j + A_{n-1} + 1) = i\}$$

$G^{\infty}_{0} = \mathbf{Q}/\mathbf{Z}_{(p)}$ generated by the set $\{g_0/p^j v_1 : j > 0\}$.

$$a_n = p^n + p^{n-1} - 1$$

$$A_n = (p^{n-1} + p^{n-2} + \cdots + 1)(p+1)$$

ctd ....

for dimension 1, and

$$G^\infty = G_C^\infty \oplus Y\zeta_C^\infty$$
$$Y\zeta_C^\infty = (Y_{0,C}^{\infty,G} \oplus Y_{1,C}^{\infty,G}) \otimes \mathbf{Z}_{(p)}\{\zeta\}$$

for

$$Y_{0,C}^{\infty,G} = \mathbf{Z}_{(p)}\{v_2^{sp^n} h_0/p^{i+1} v_1^{kp^{i+1}+1} \colon p \nmid s(s+1),$$
$$k \neq 0,\ A_{n-i-1} + 1 < kp^{i+1} \leq A_{n-i} + 1 \text{ for } i \geq 0\}$$
$$Y_{1,C}^{\infty,G} = \mathbf{Z}_{(p)}\{v_2^{(tp^2-1)p^n} h_0/p^{i+1} v_1^{kp^{i+1}+1} \colon k \neq 0,$$
$$p^{n+2} - p^n + A_{n-i-1} + 1 < kp^{i+1} \leq p^{n+2} - p^n + A_{n-i} + 1 \text{ for } i \geq 0\}$$
$$G_C^\infty = \mathbf{Z}_{(p)}\{v_2^{sp^n} g_0/p^{n+1} v_1,\ v_2^{sp^n - (p^{n-1}-1)/(p-1)} g_1/p^l v_1^j \colon$$
$$p \nmid (s+1),\ 0 < j \leq a_n,\ p^{i+1} \nmid (j + A_{n-i-1} + 1) \text{ if } s = up^i \in \mathbf{Z}(0),$$
$$p^i \nmid (j + A_{n-i} + 1) \text{ if } s = up^i \in \mathbf{Z}(2), \text{ and } l = i+1 \text{ if } n = 0 \text{ and } v_p(s) = i;$$
$$l = i + 1 \text{ if } n \geq 1 \text{ and } v_p(j + A_{n-1} + 1) = i\}$$
$$G_0^\infty = \mathbf{Q}/\mathbf{Z}_{(p)} \text{ generated by the set } \{g_0/p^j v_1 \colon j > 0\}.$$

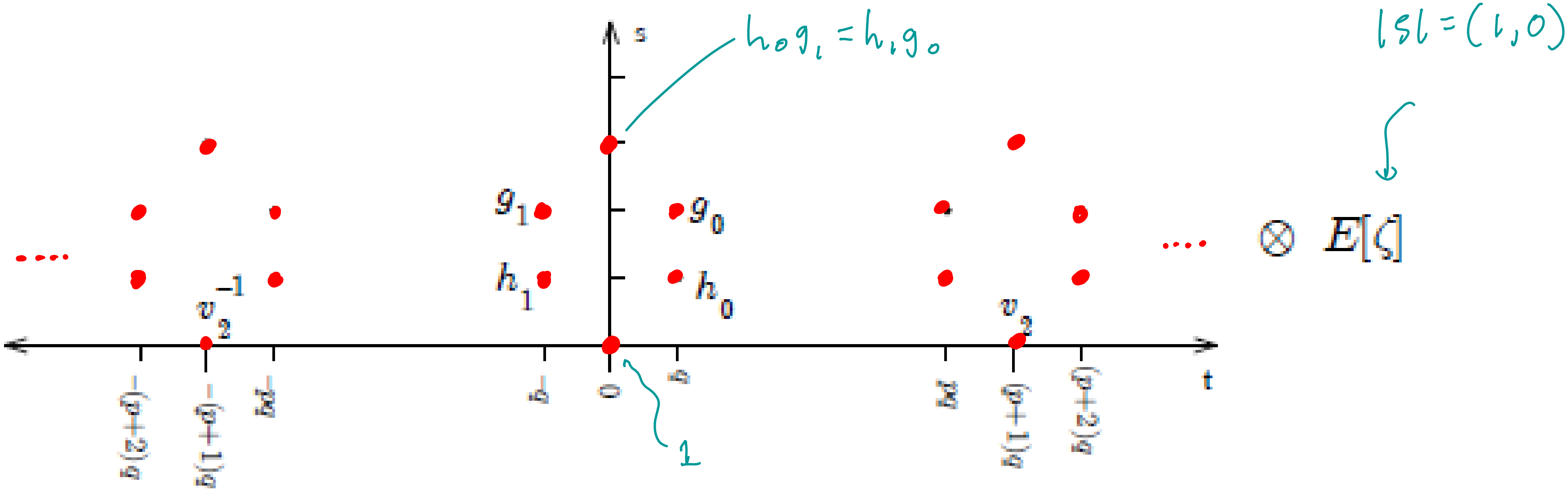
$$a_n = p^n + p^{n-1} - 1$$

$$A_n = (p^{n-1} + p^{n-2} + \cdots + 1)(p+1)$$

We shall give a simpler presentation of this answer

[correct some errors]

$H^{s,t}(M_2^0)$ [Ravenel]



v_1 - BSS

$$H^{\bullet\bullet}(M_2^0) \otimes \frac{\mathbb{F}_p[v_1]}{v_1^\infty} \Rightarrow H^{\bullet\bullet}(M_1^1)$$

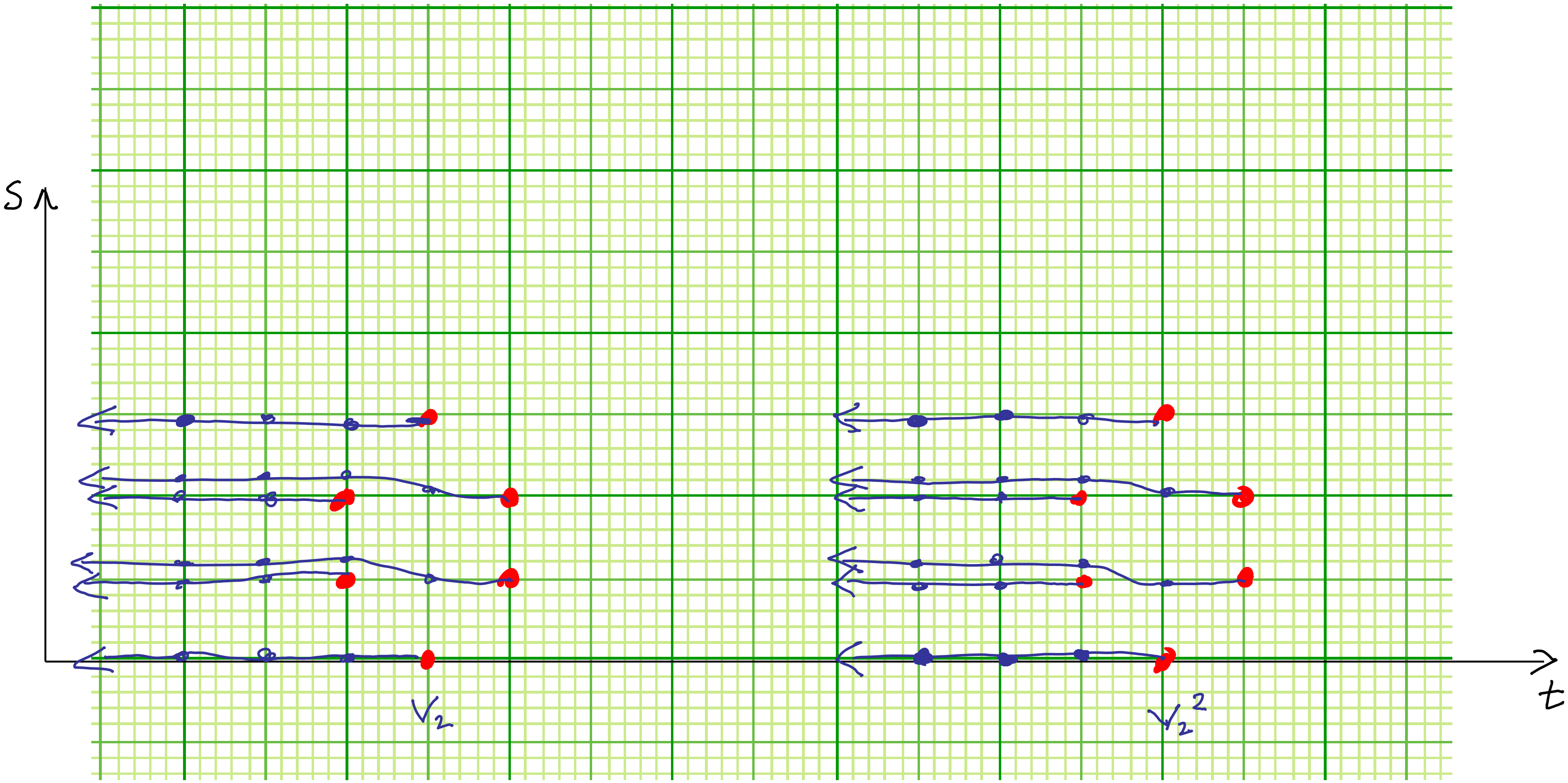
v_1-BSS

$$H^{**}(M_2^0) \otimes \frac{\mathbb{F}_p[v_1]}{v_1^\infty} \Longrightarrow H^{**}(M_1^1)$$

$$\Downarrow$$

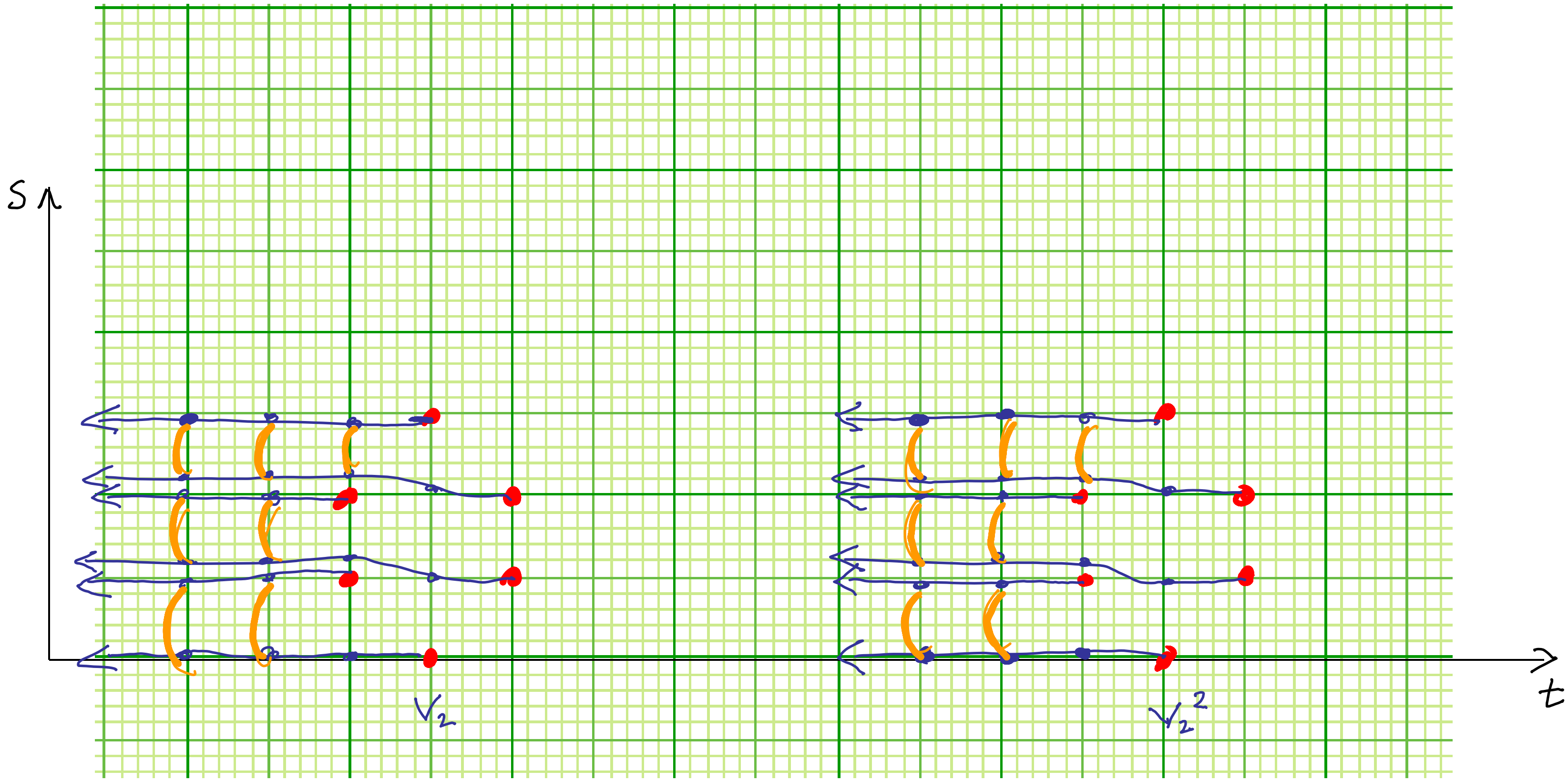
$$\frac{x}{v_1^j}, \quad j > 0, \quad x \in H^{**}(M_2^0)$$

$\underline{v_1\text{-BSS}}$: $H^{**}(M_2^0) \otimes \frac{\mathbb{F}_p[v_1]}{v_1^\infty} \Rightarrow H^{**}(M_1^1)$



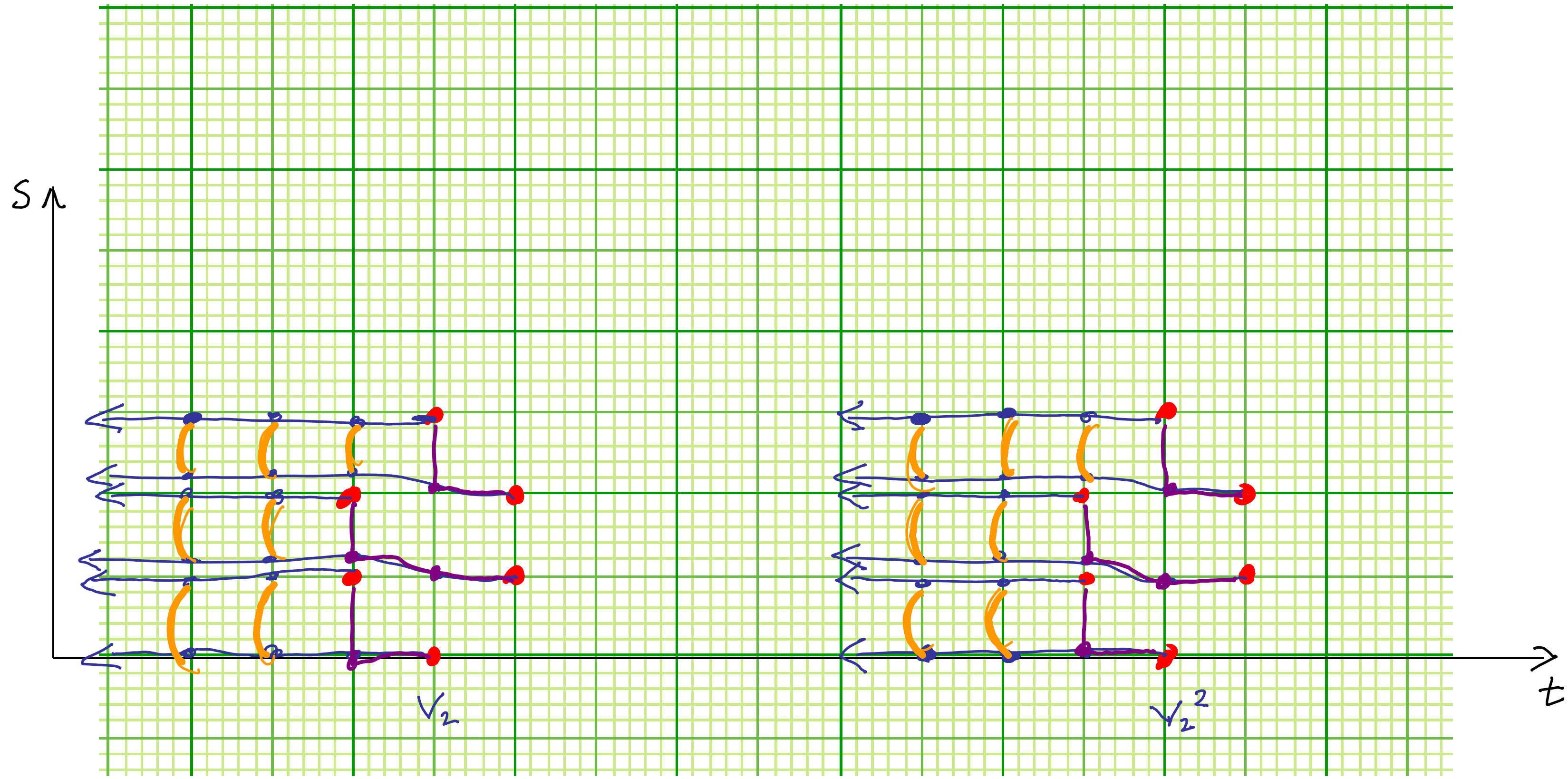
Note: We have omitted ζ factor

$\underline{v_1\text{-BSS}}$: $H^{**}(M_2^0) \otimes \dfrac{\mathbb{F}_p[v_1]}{v_1^\infty} \Rightarrow H^{**}(M_1^1)$



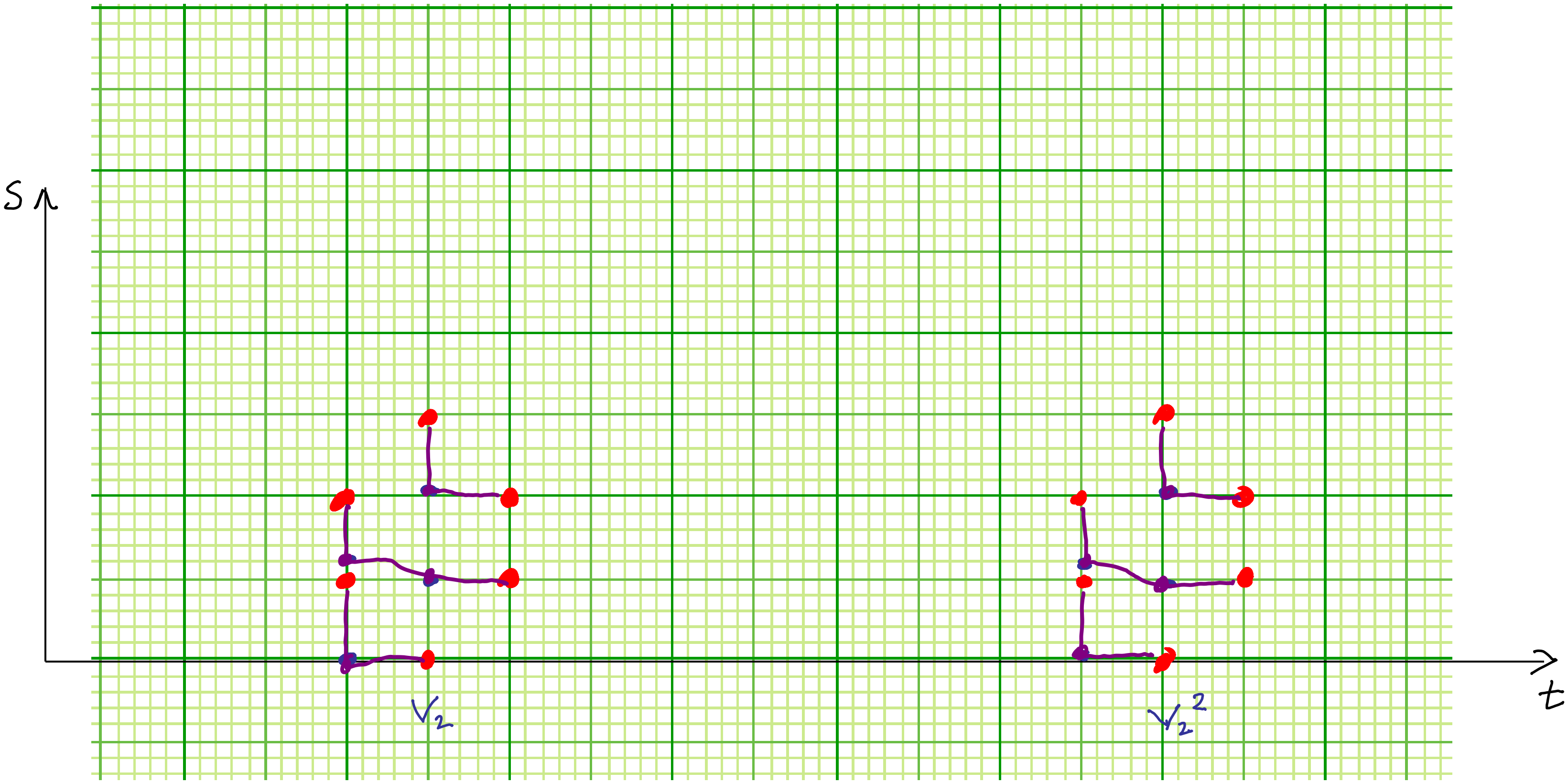
Differentials

$\underline{v_1\text{-BSS}}$: $H^{**}(M_2^0) \otimes \dfrac{\mathbb{F}_p[v_1]}{v_1^\infty} \Rightarrow H^{**}(M_1^1)$



Graphical Shorthand

$\underline{v_1\text{-BSS}}$: $H^{**}(M_2^0) \otimes \dfrac{\mathbb{F}_p[v_1]}{v_1^\infty} \Rightarrow H^{**}(M_1^1)$



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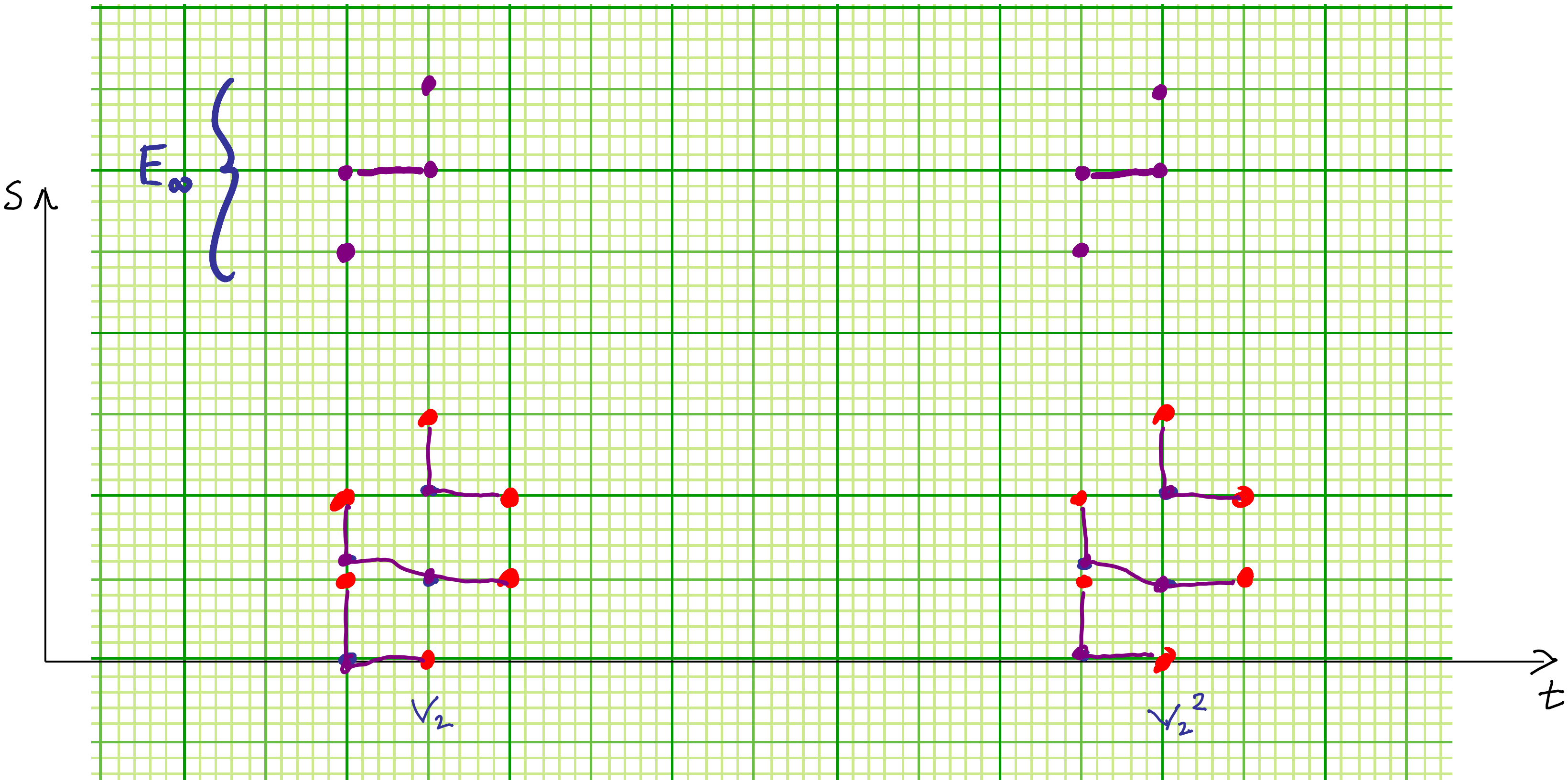
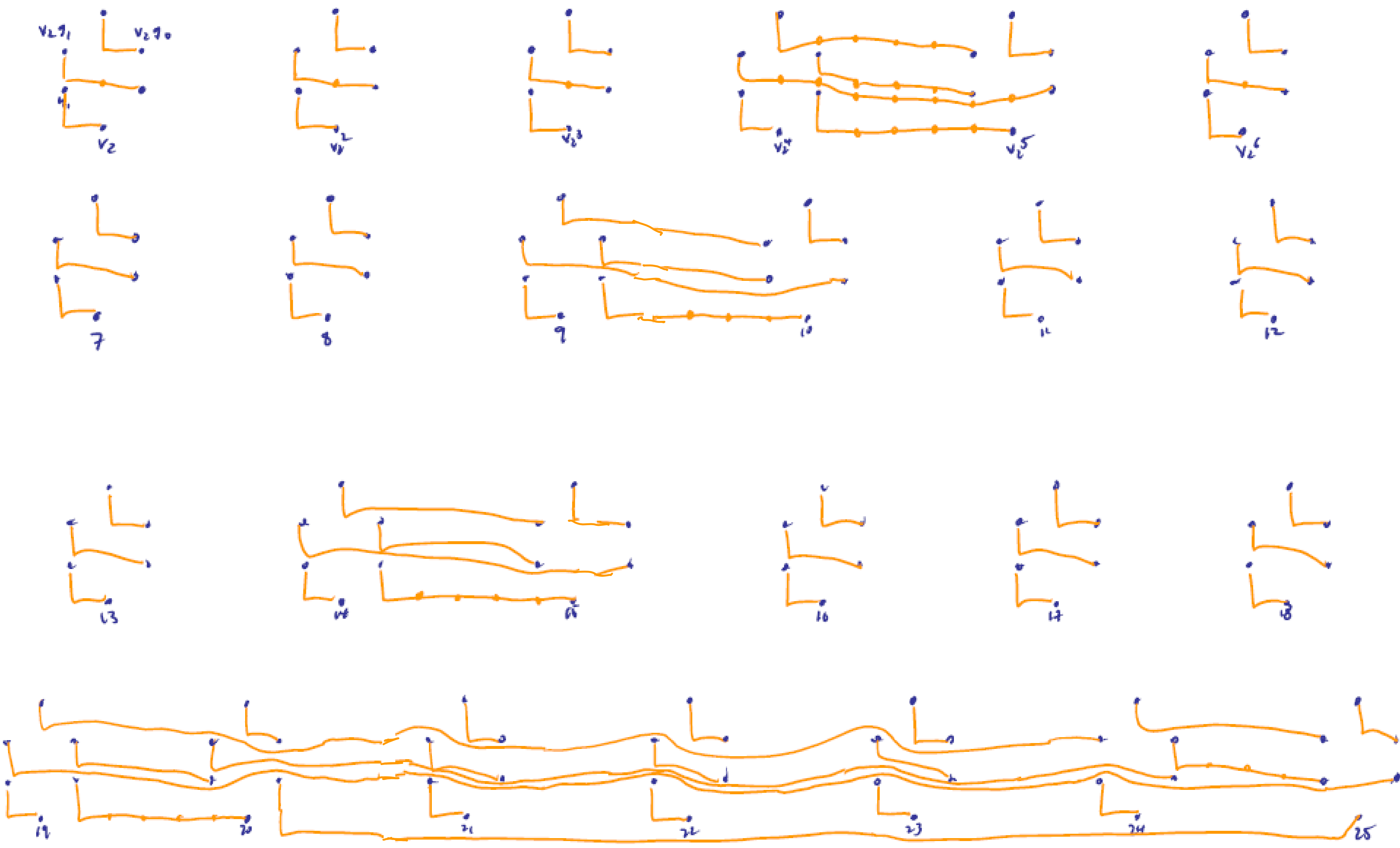


$(p=5)$



v2 q1
v2 q0
q1
v2
v2^2
v2^3
v2^4
v2^5
v2^6
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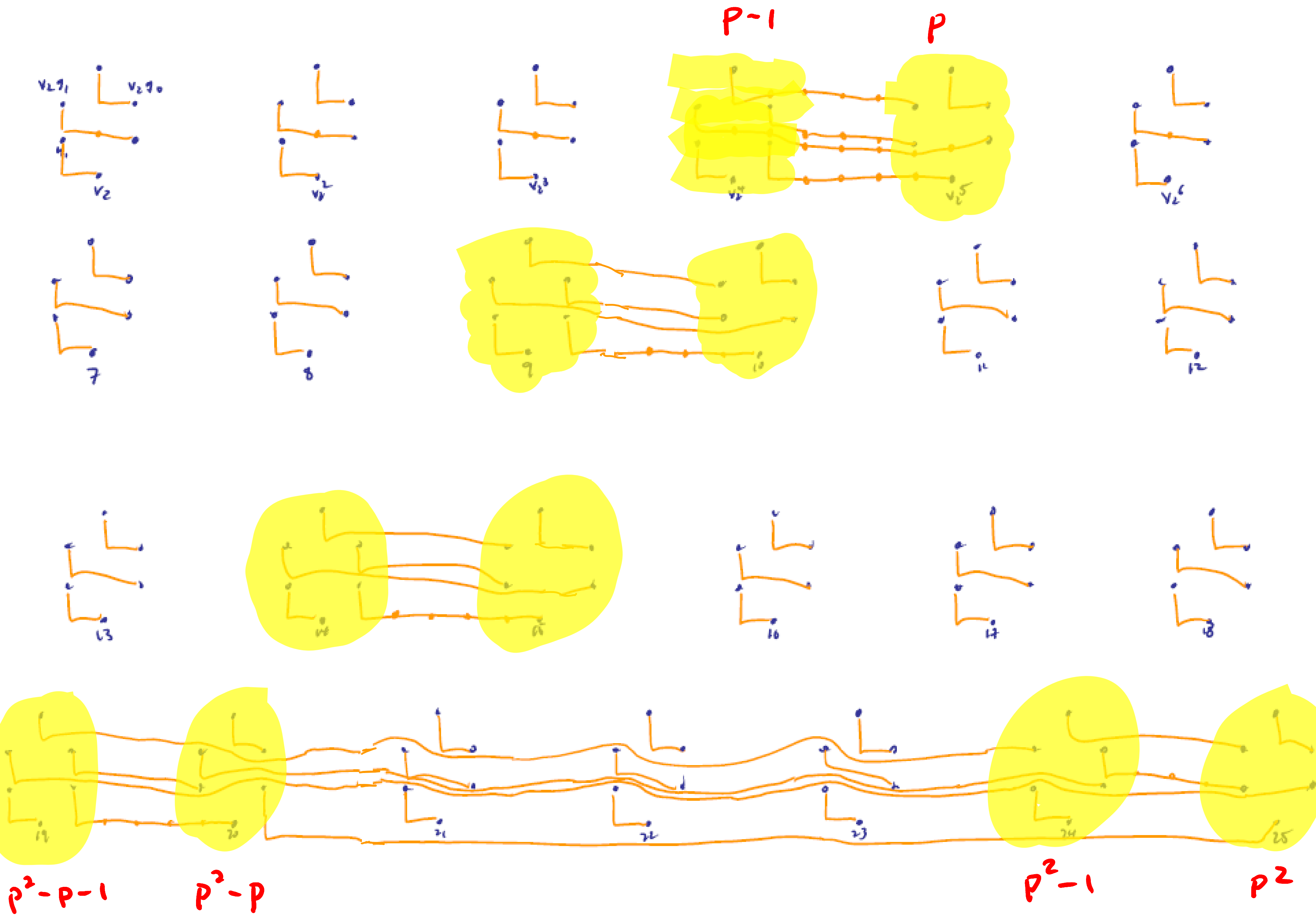
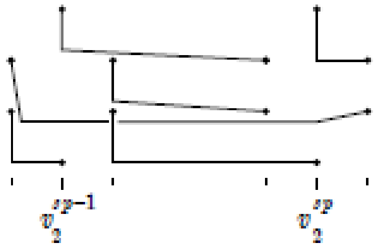
$(p=5)$



INDUCTIVE CONSTRUCTION

Step 1. Start with the pattern in the vicinity of $v_2^{sp^{n-1}}$.



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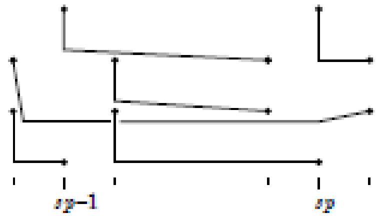
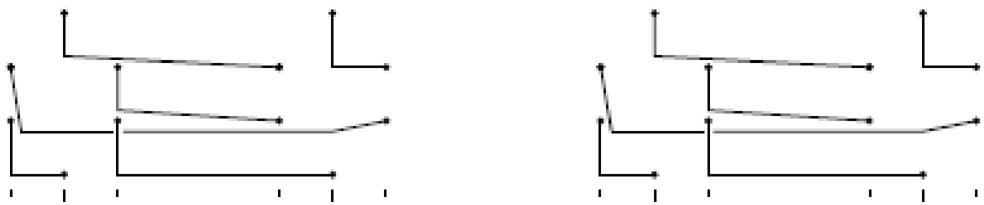


Step 2. Double the pattern.



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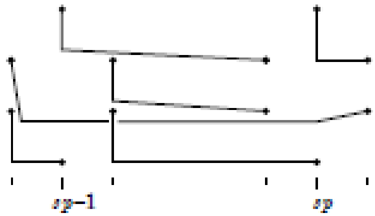
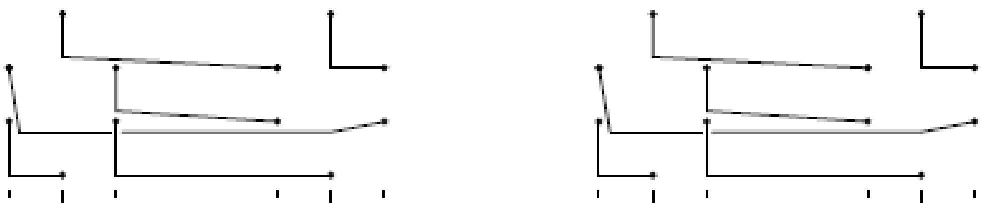


Step 2. Double the pattern.



Step 3. Delete the following differentials:

- the rightmost longest differential on the 0-line,

INDUCTIVE CONSTRUCTION

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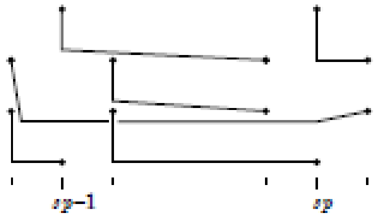

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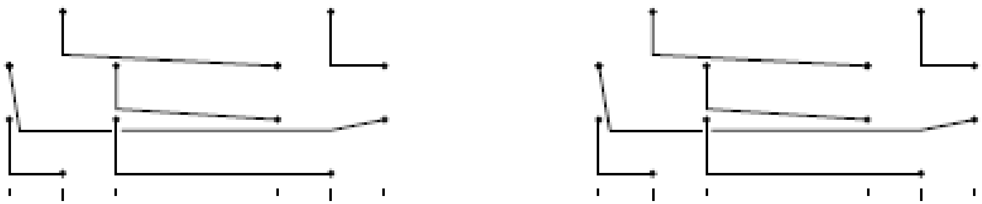

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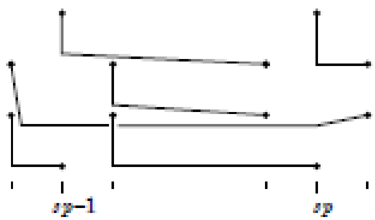

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INDUCTIVE CONSTRUCTION

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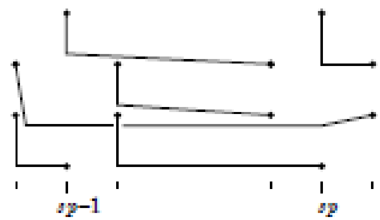

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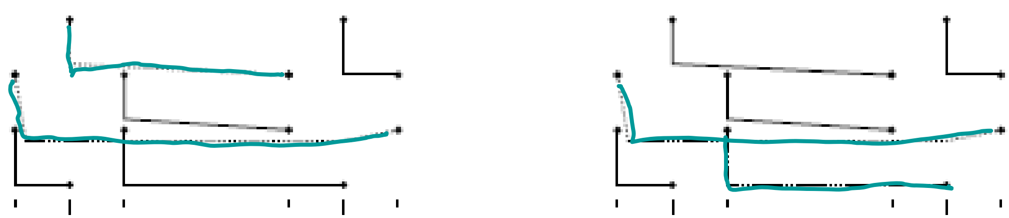
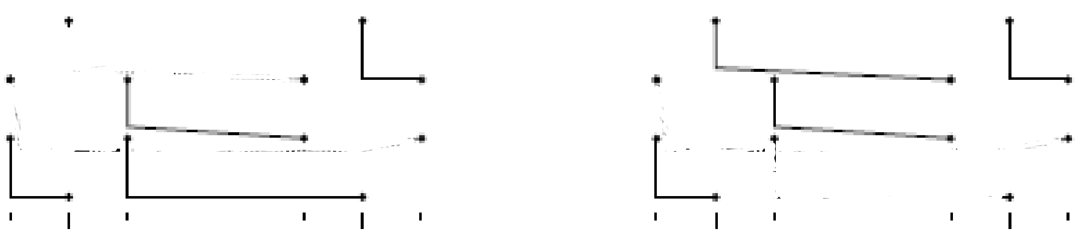


Step 4. Add the following differentials:



INDUCTIVE CONSTRUCTION

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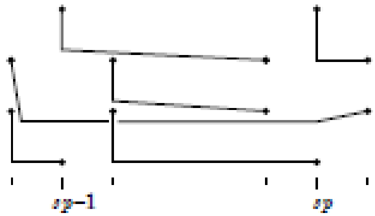
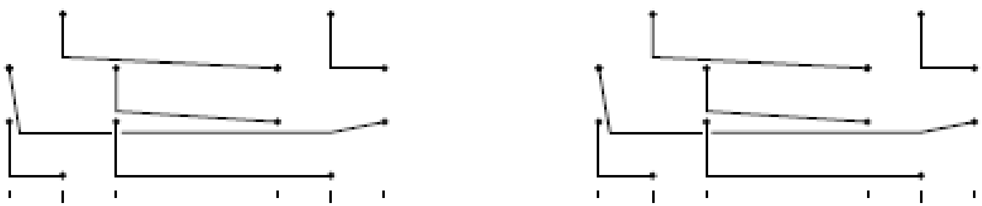


Step 2. Double the pattern.



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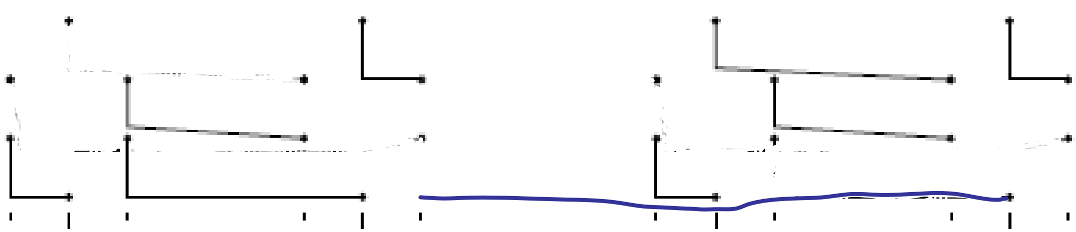

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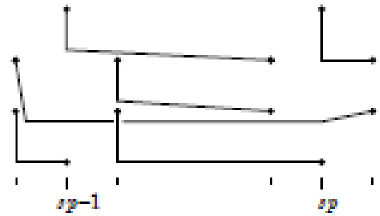

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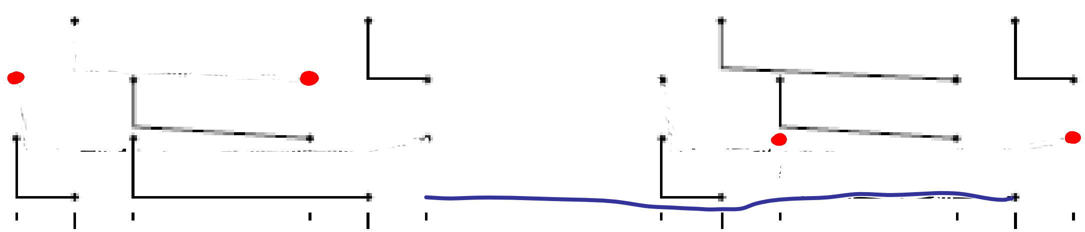

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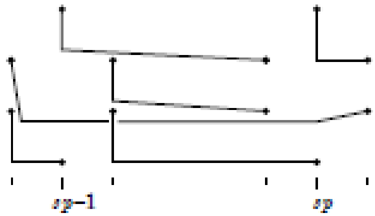
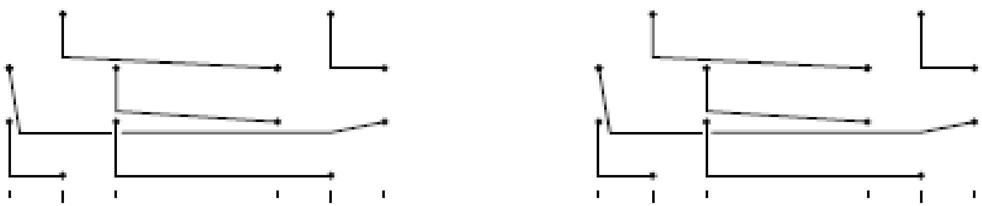


Step 2. Double the pattern.



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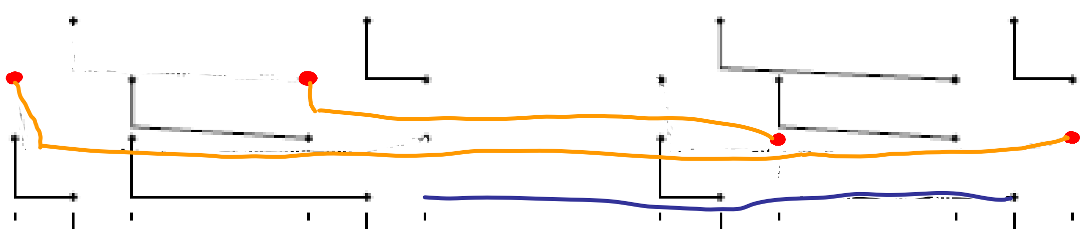
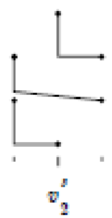
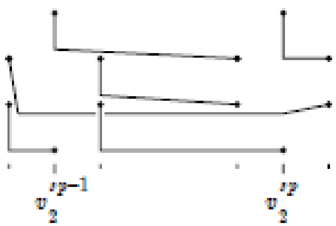
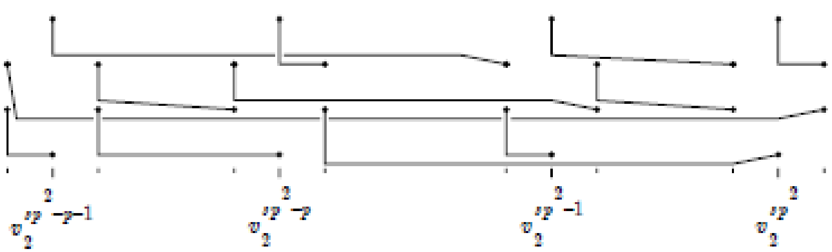
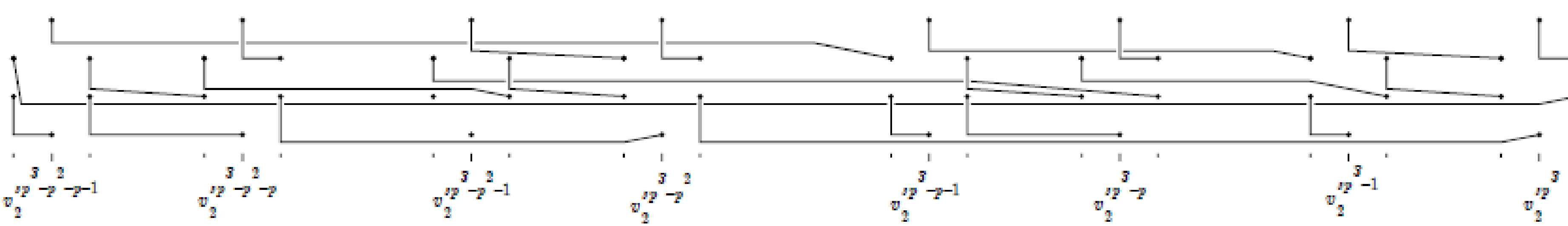


v_1-BSS: first few patterns:

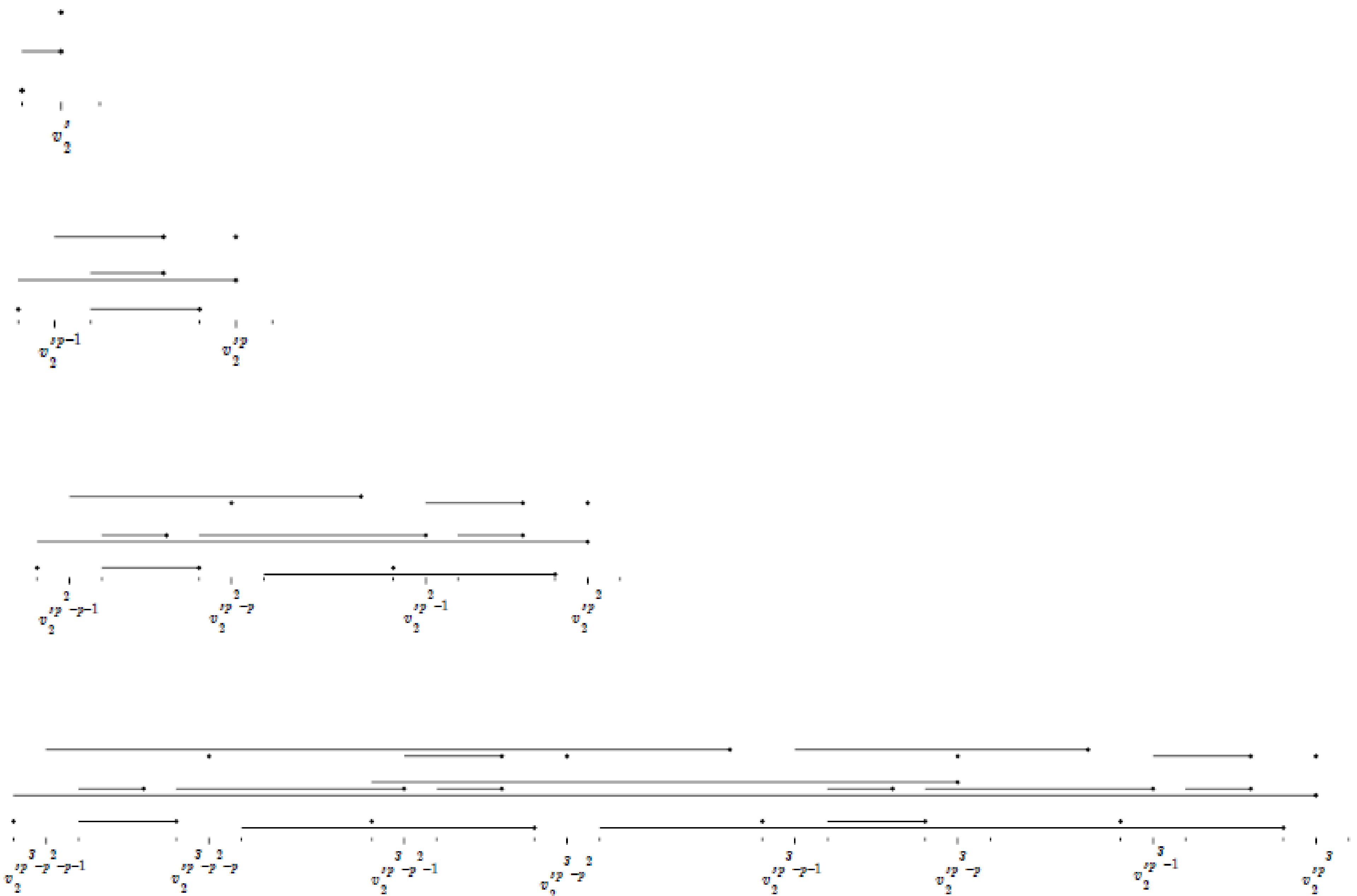




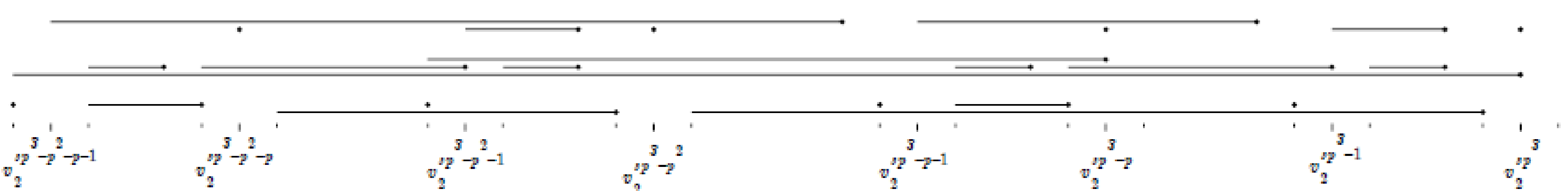




$H^*(M_1^1)$: first few patterns



GROSS-HOPKINS DUALITY:

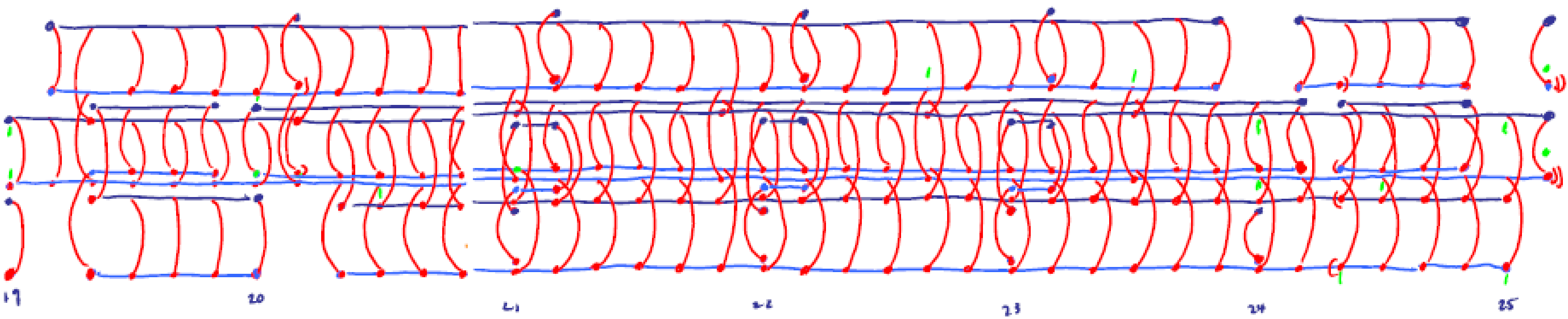


$\underline{v_0\text{-BSS}}$:

$$H^{\bullet\bullet}(M^1_1) \otimes \frac{\mathbb{F}_p[v_0]}{(v_0^\infty)} \Rightarrow H^{\bullet\bullet}(M^2_0)$$

v_0-BSS:

$$H^{**}(M^1_1) \otimes \frac{\mathbb{F}_p[v_0]}{(v_0^\infty)} \Rightarrow H^{**}(M^2_0)$$



= Non-ζ-factor

= ζ-factor

Problem: ζ and Non-ζ interact in a complicated manner

SOLUTION: PROJECTIVE v_0-BSS

[Goerss-Henn-Karamanov-Mahowald $p=3$]

SOLUTION: PROJECTIVE v_0-BSS

[Goerss - Henn - Karamanov - Mahowald $p=3$]

Morava change of rings:

$$H^\bullet(M_0^2) \cong H^\bullet\left(\mathbb{S}_2; \frac{(E_2)_*}{(p^\infty, v_1^\infty)}\right)$$

SOLUTION: PROJECTIVE v_0-BSS

[Goerss-Henn-Karamanov-Mahowald $p=3$]

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Projective Morava Stabilizer gp:

$$1 \longrightarrow \mathbb{Z}_p^\times \longrightarrow \mathbb{S}_2 \longrightarrow P\mathbb{S}_2 \longrightarrow 1$$

SOLUTION: PROJECTIVE v_0-BSS

[Goerss-Henn-Karamanov-Mahowald $p=3$]

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Projective Morava Stabilizer gp.:

$$1 \longrightarrow \mathbb{Z}_p^\times \longrightarrow \mathbb{S}_2 \longrightarrow P\mathbb{S}_2 \longrightarrow 1$$

LHSSS:

$$H^\bullet\left(P\mathbb{S}_2; H^\bullet\left(\mathbb{Z}_p^\times; \frac{(E_2)_*}{p^\infty, v_1^\infty}\right)\right) \Rightarrow H^\bullet\left(\mathbb{S}_2; \frac{(E_2)_*}{p^\infty, v_1^\infty}\right)$$

SOLUTION: PROJECTIVE v_0-BSS

[Goerss-Henn-Karamanov-Mahowald $p=3$]

Key fact:

$$H^{s,t}\left(\mathbb{Z}_p^\times; \frac{(E_2)_*}{p^\infty, v_1^\infty}\right) = \begin{cases} \left[\dfrac{(E_2)_*}{p^k, v_1^\infty}\right]_t, & s=0,\ t=(p-1)p^{k-1}t' \\ 0, & \text{o/w} \end{cases}$$

LHSSS:

$$H^\cdot\left(P\mathbb{S}_2; H^\cdot\left(\mathbb{Z}_p^\times; \frac{(E_2)_*}{p^\infty, v_1^\infty}\right)\right) \Rightarrow H^\cdot\left(\mathbb{S}_2; \frac{(E_2)_*}{p^\infty, v_1^\infty}\right)$$

SOLUTION: PROJECTIVE v_0-BSS

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$\Rightarrow$ ss collapses:

$$H^{\bullet\bullet}\left(P\mathbb{S}_2; \left[\frac{E_2}{p^\infty, v_1^\infty}\right]^{\mathbb{Z}_p^\times}\right) \cong H^{\bullet}(M_0^2)$$

LHSSS:

$$H^\bullet\left(P\mathbb{S}_2; H^\bullet\left(\mathbb{Z}_p^\times; \frac{(E_2)_*}{p^\infty, v_1^\infty}\right)\right) \Rightarrow H^\bullet\left(\mathbb{S}_2; \frac{(E_2)_*}{p^\infty, v_1^\infty}\right)$$

SOLUTION: PROJECTIVE v_0-BSS

[Goerss-Henn-Karamanov-Mahowald $p=3$]

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p-adic filtration gives a spectral sequence ....

SOLUTION: PROJECTIVE v_0-BSS

[Goerss - Henn - Karamanov - Mahowald $p=3$]

$$E_1^{\bullet\bullet} \Longrightarrow H^{\bullet\bullet}(M_0^2)$$

$$E_1^{s,t} = \begin{cases} H^{s,t'}\left(\mathbb{PS}_2 ; \frac{E_2}{(p, v_1^\infty)}\right) \otimes \mathbb{F}_p[v_0]/(v_0^k), & t = (p-1)p^{k-1}t' \\ 0, & \text{o/w} \end{cases}$$

SOLUTION: PROJECTIVE v_0-BSS

[Goerss-Henn-Karamanov-Mahowald $p=3$]

$$E_1^{\bullet\bullet} \Longrightarrow H^{\bullet\bullet}(M_0^2)$$

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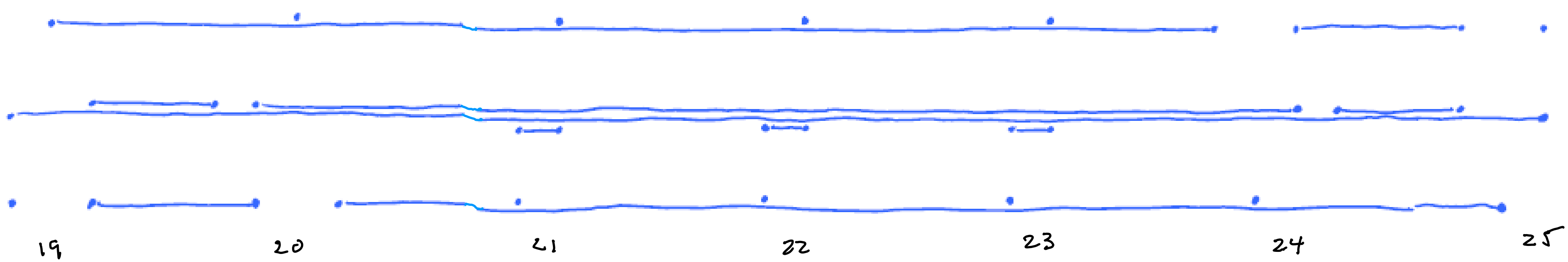
Key fact:

$$H^{\bullet\bullet}\left(P\mathbb{S}_2; \frac{E_2}{p, v_1^\infty}\right) = H^{\bullet\bullet}(M_1^{\bullet})/(\zeta)$$

Projective v_0-BSS: $p=5$, near v_2^{25}

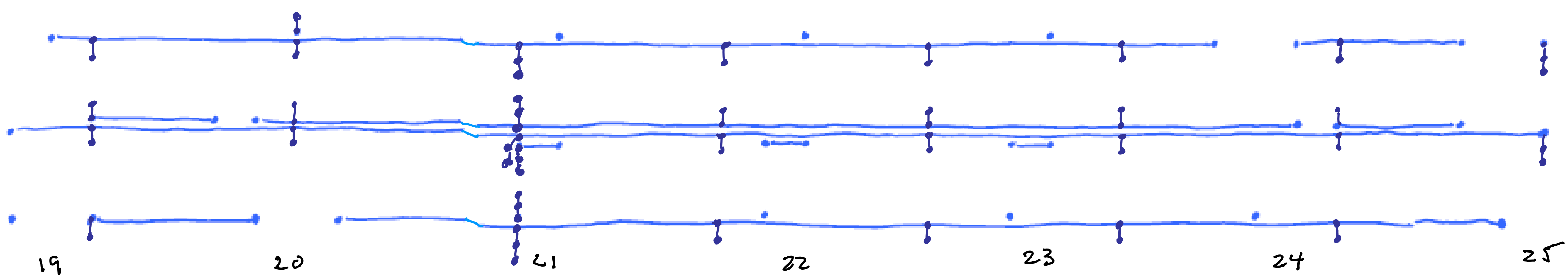
$H^{**}(M_*^!)$



$\otimes E(5)$

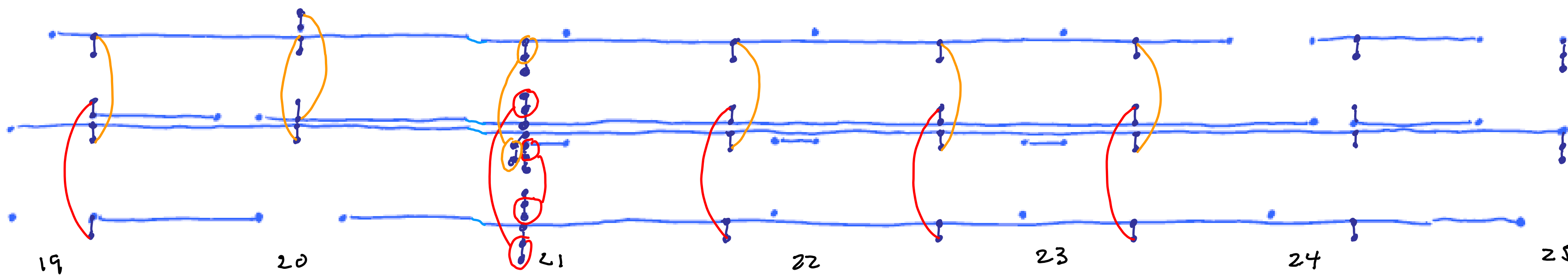
Projective v_0-BSS: $p=5$, near v_2^{25}

E_1^{st}: decorate w/ ImJ pattern



Projective v_0-BSS: $p=5$, near v_2^{25}

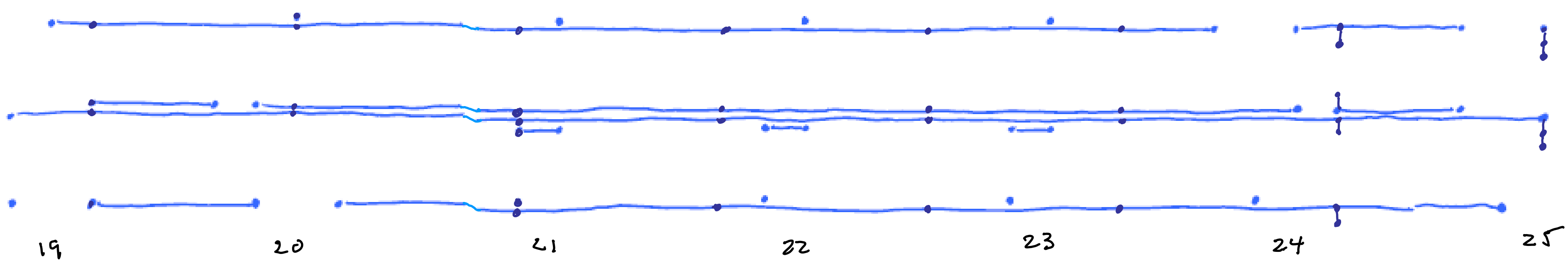
diff'ls:



"Shimomura's v_0-BSS diff'ls mod 5"

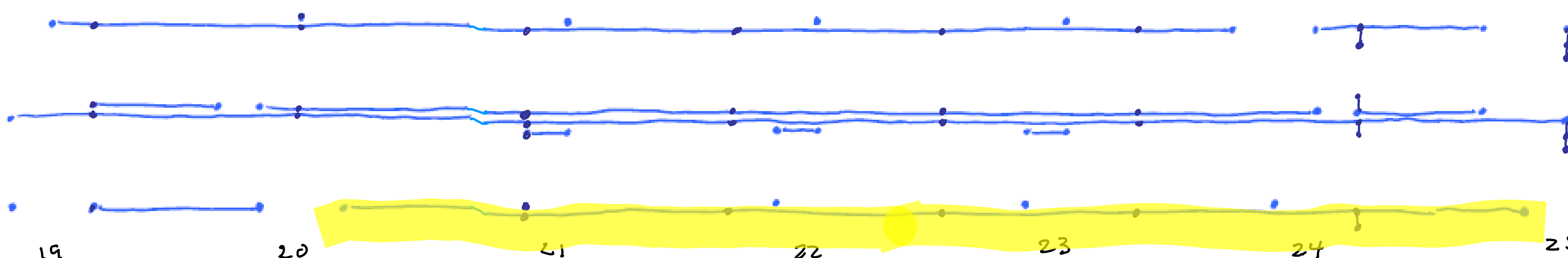
Projective v_0-BSS: $p=5$, near v_2^{25}

$E_\infty^{s,t}$

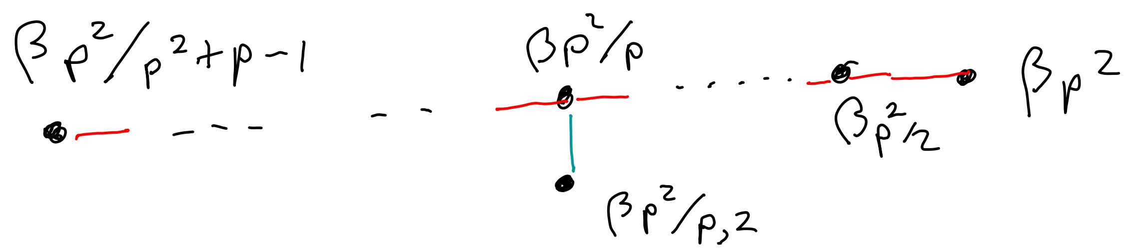


Projective v_0-BSS: $p=5$, near v_2^{25}

$E_\infty^{s,t}$

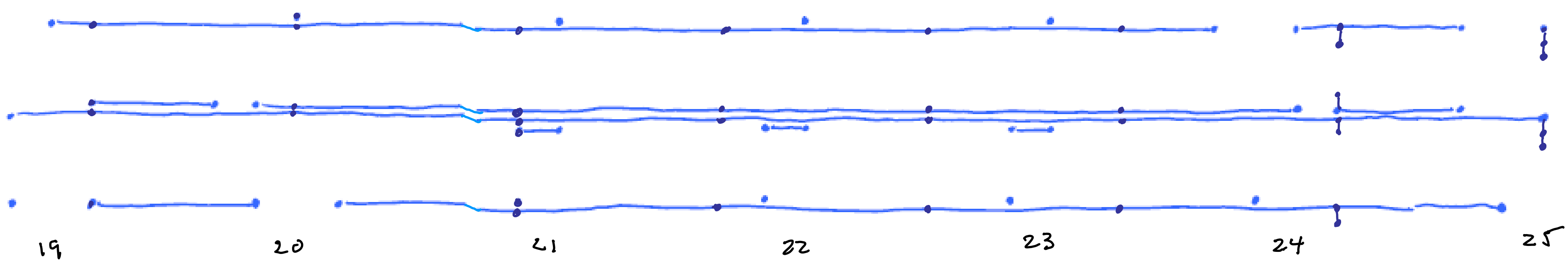


β-pattern

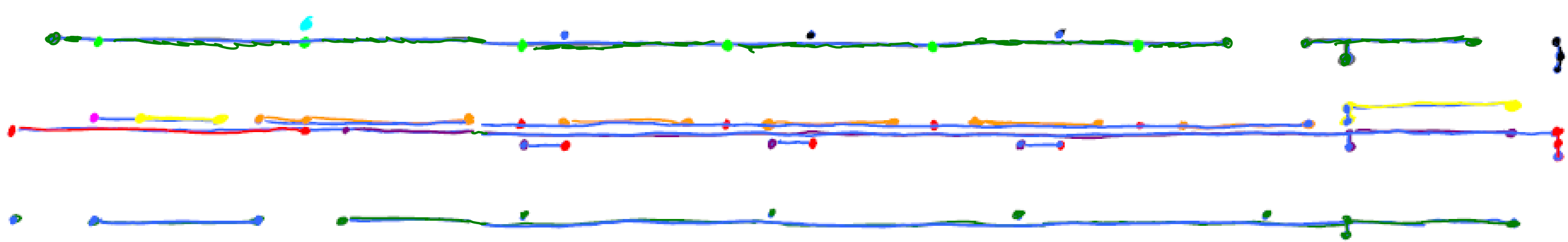


Projective v_0-BSS: $p=5$, near v_2^{25}

$E_\infty^{s,t}$



Classical v_0-BSS:



■ $= X^\infty$

■ $= XS_c^\infty$

■ $= XS_c^\infty$ $s = 6p^{k+1}-1$

■ $= Y_{0,c}^\infty$

■ $= Y_{1,c}^\infty$

■ Y_c^∞

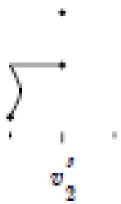
■ G_c^∞ g_0 part

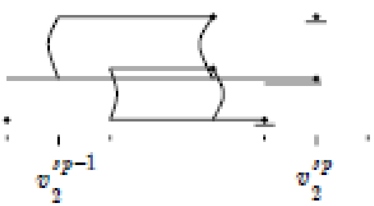
■ G_c^∞ g_1 part

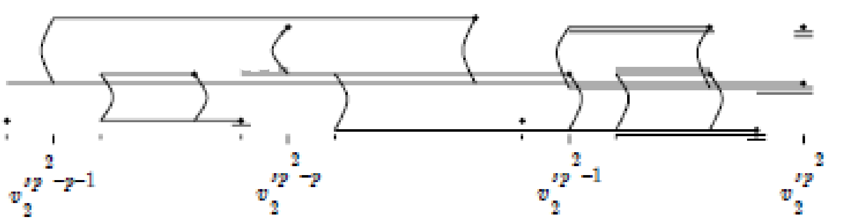
■ $Y_{0,c}^{\infty,G}\{S\}$

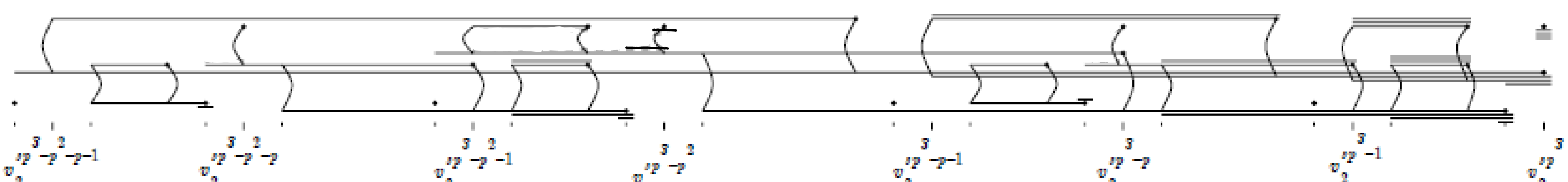
■ $Y_{1,c}^{\infty,G}\{S\}$

Projective v_0-BSS: patterns of diff'ls

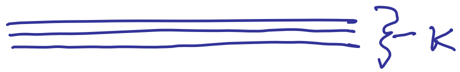






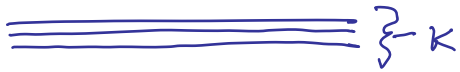


Notation



means ImJ-pattern
w/ p-torsion bounded by p^K

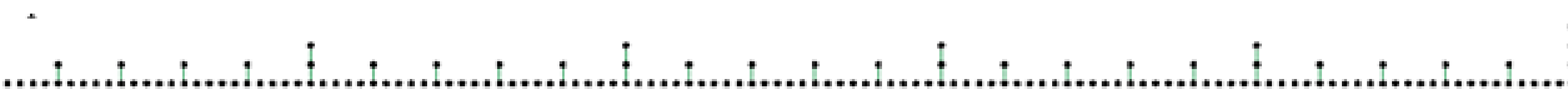
Notation



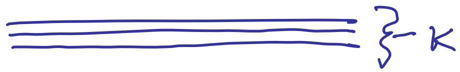
means ImJ-pattern
w/ p-torsion bounded by p^k

Example: $p = 5$

ImJ pattern:

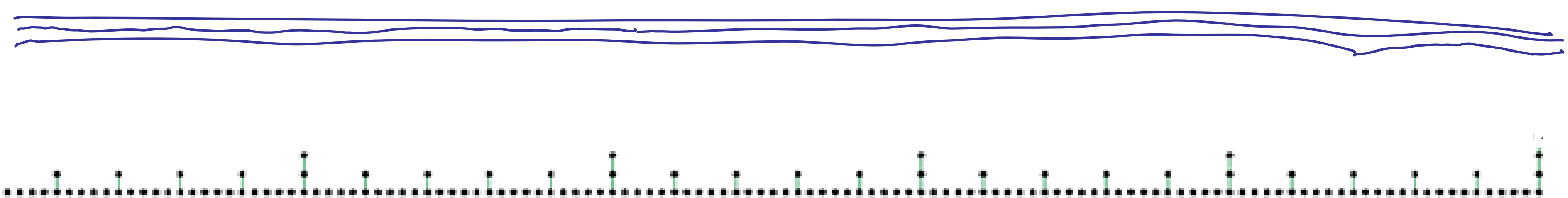


Notation

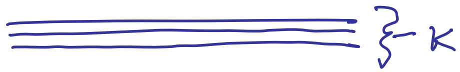


means Im J - pattern w/ p-torsion bounded by p^K

Example: $p = 5$



Notation



means ImJ-pattern
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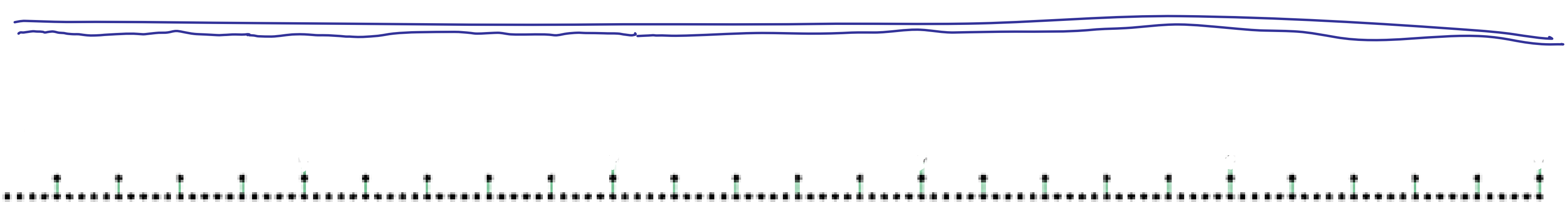
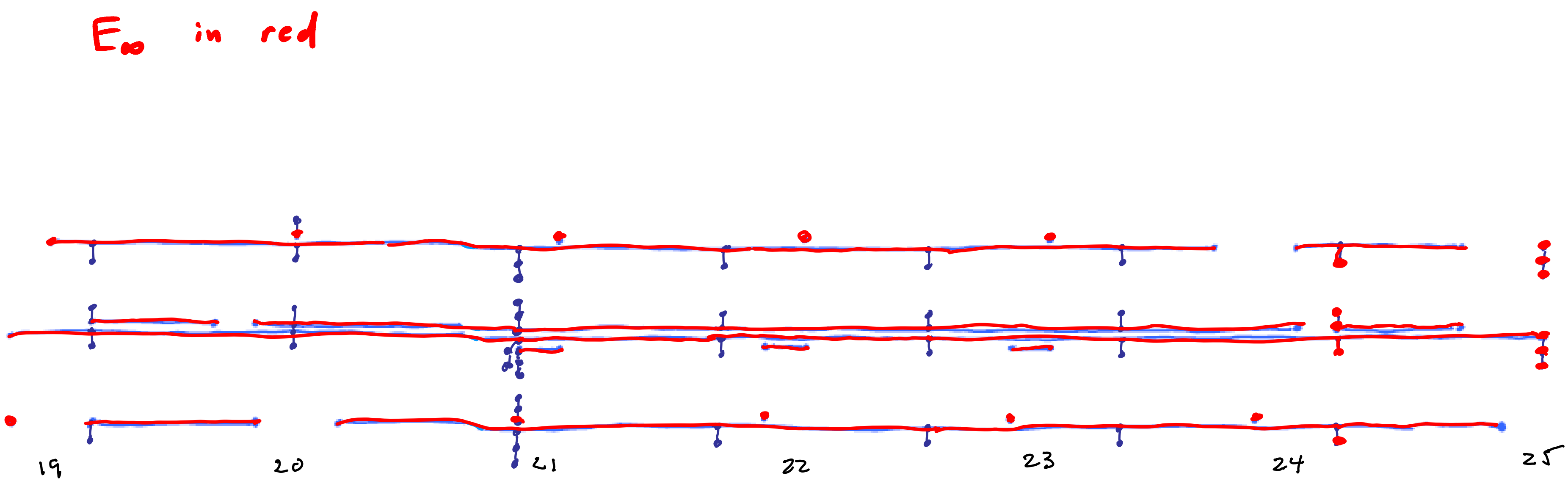
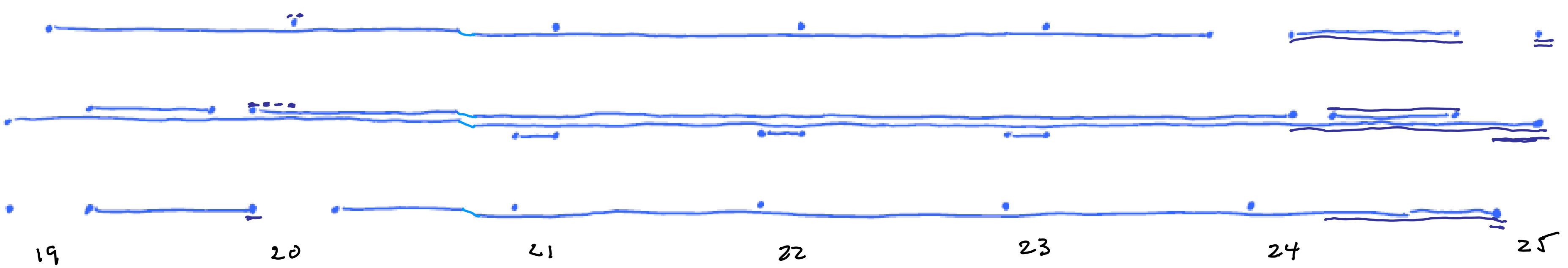
Example: $p = 5$



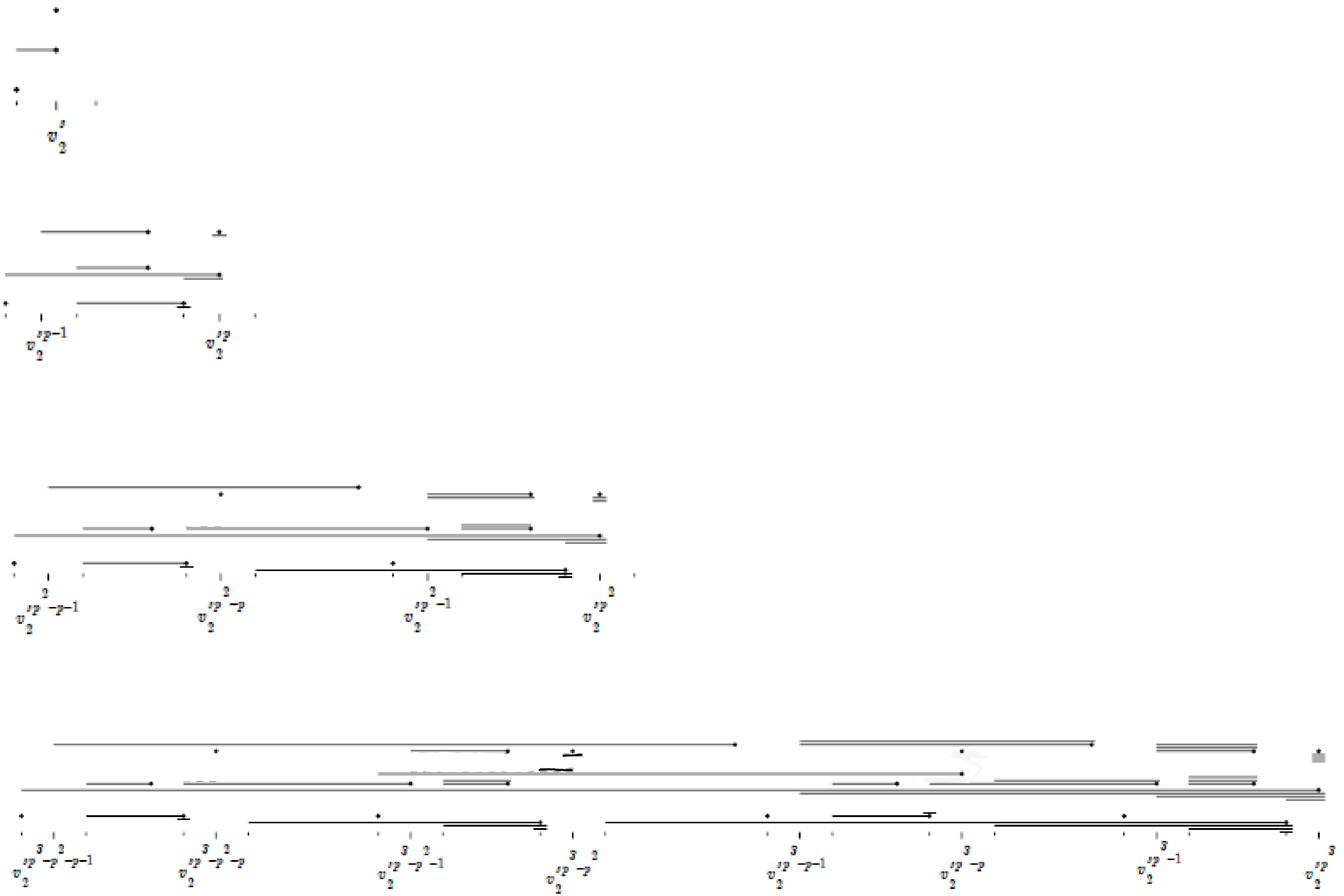
Projective v_0-BSS: $p=5$, near v_2^{25}

Using this notation:



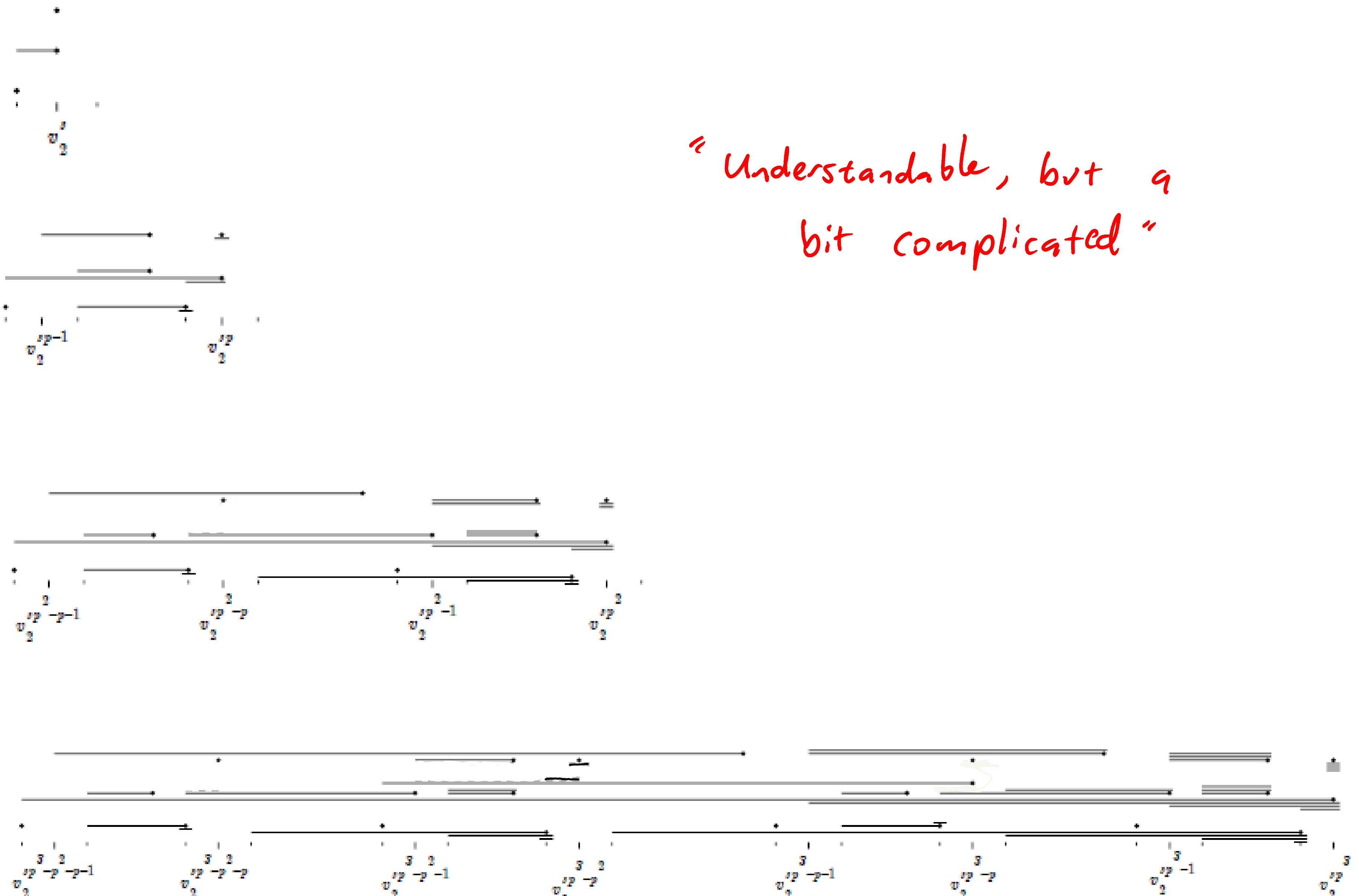


Projective v_0 - BSS $\quad E_\infty^{\cdot\cdot} = H^{\cdot\cdot} M_0^2$



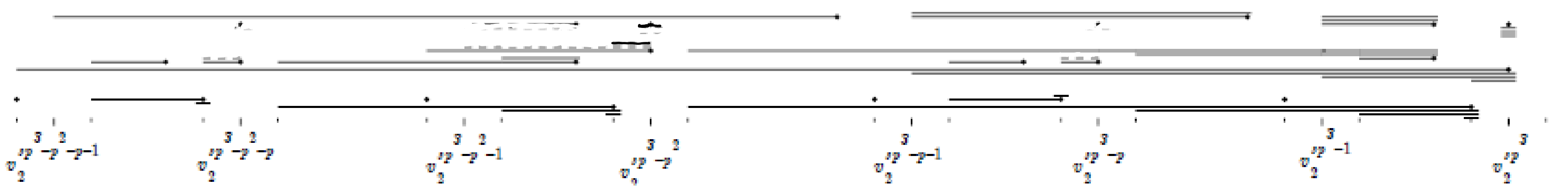
Projective v_0-BSS $\quad E_\infty^{**} = H^{**} M_0^2$

“Understandable, but a bit complicated”

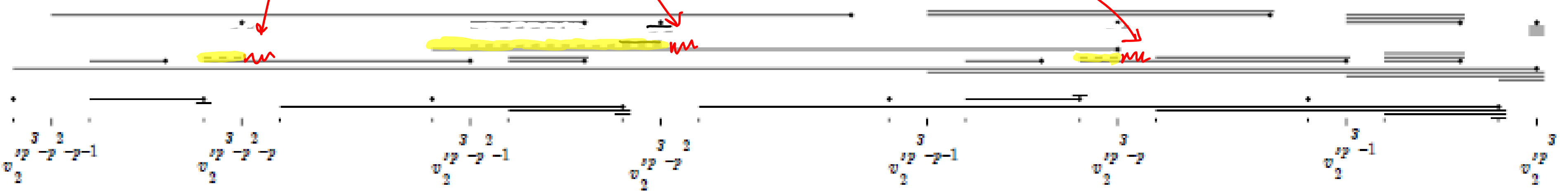



$v_2^{p^3-p^2-p-1}$
$v_2^{p^3-p^2-p}$
$v_2^{p^3-p^2-1}$
$v_2^{p^3-p^2}$
$v_2^{p^3-p-1}$
$v_2^{p^3-p}$
$v_2^{p^3-1}$
$v_2^{p^3}$

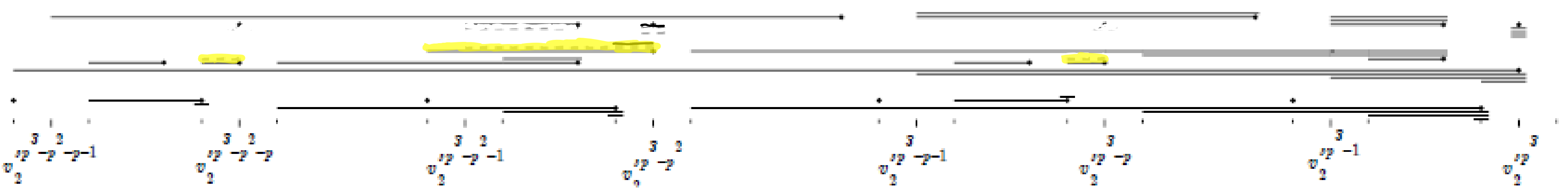
Reorganize


$v_2^{p^3-p^2-p-1}$
$v_2^{p^3-p^2-p}$
$v_2^{p^3-p^2-1}$
$v_2^{p^3-p^2}$
$v_2^{p^3-p-1}$
$v_2^{p^3-p}$
$v_2^{p^3-1}$
$v_2^{p^3}$

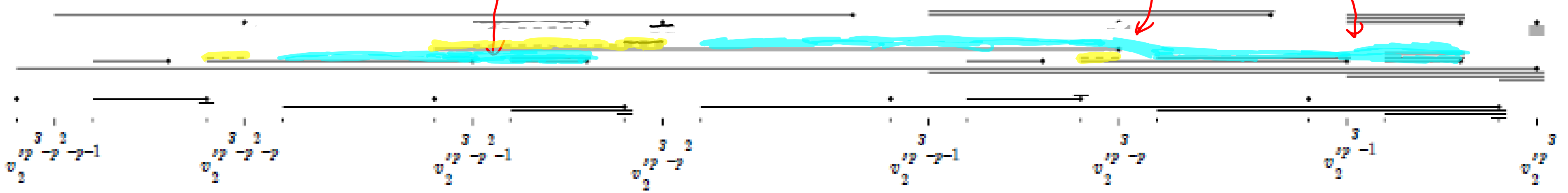
Remove these v_1-multiplications



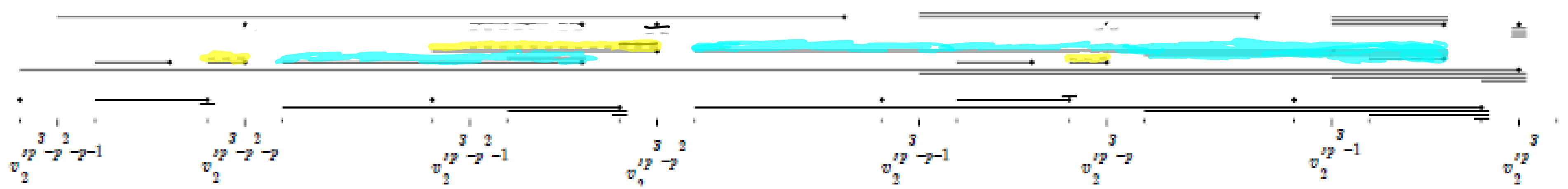
Reorganize



"join" these families

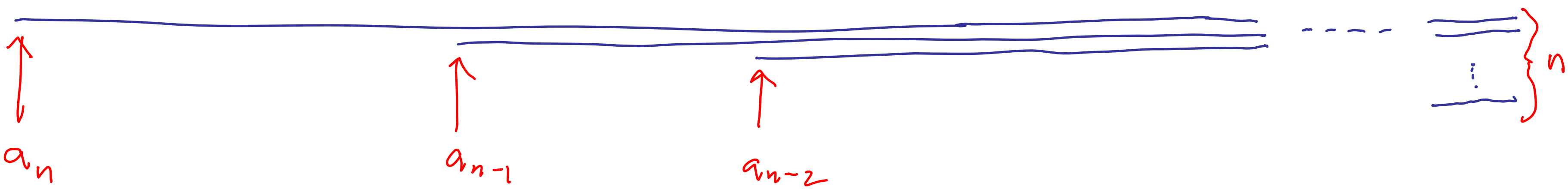


Reorganize

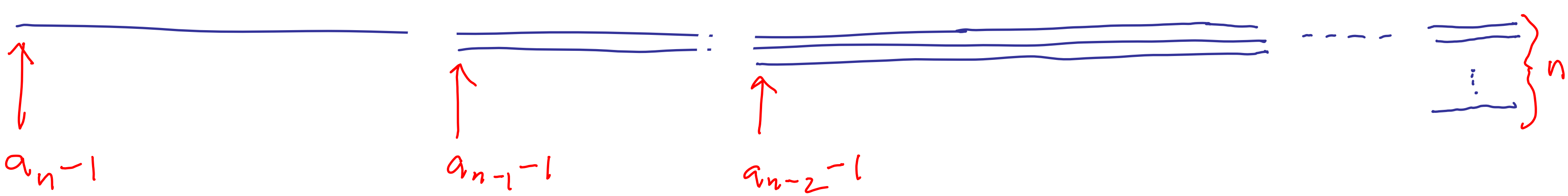


$H^*(M_0^2)$: Four families

$\frac{v_2^{sp^n}}{p^k v_1^j}$ = "β family" $\quad s \not\equiv 0 \ (p)$

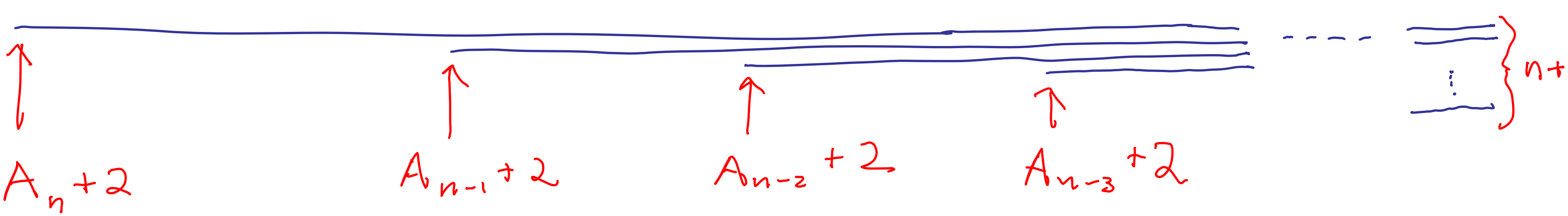


"h_1 family" $\quad h_1(j,k)_{sp^n} \quad s \not\equiv 0 \ (p)$

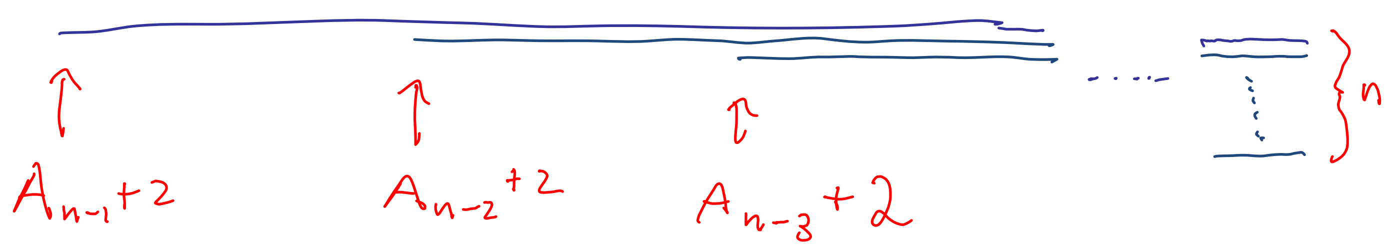


$H^\bullet(M_0^2)$: Four families

"h_0 family, generic case" $h_0(j,k)_{sp^n}$ $s \not\equiv 0,-1 \ (p)$

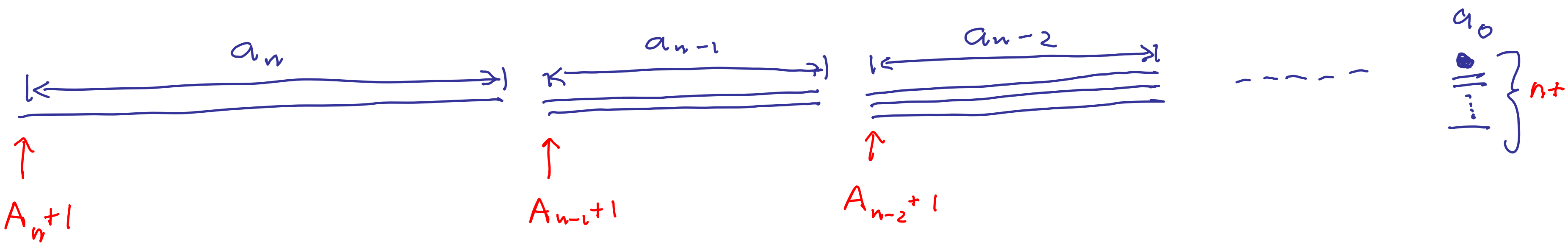


"h_0 family, special case" $h_0(j,k)_{sp^n}$ $s \equiv -1 \ (p)$



$H^\bullet(M^2_0)$: Four families

"G family, generic case" $\quad G_i(j,k)_{sp^n} \quad s \not\equiv 0,-1 \ (p)$



"G family, special case" $\quad G_i(j,k)_{sp^n} \quad s \equiv -1 \ (p)$