The bo-Adams spectral sequence

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Joint with Agnes Beaudry (Uchicago), Prasit Bhattacharya (Notre Dame), Dominic Culver (Notre Dame), Zhouli Xu (Uchicago)

Generalized Adams spectral sequences

- R = ring spectrum (cohomology theory with cup products)
  - $R_* := R_*(pt) = \pi_* R$

- $R_* R$ = ring of "R-cooperations".
  - Dual to $R^* R$ = ring of R-operations (natural transformations $R^*(-) \rightarrow R^*$)
  - E.g. $R = H\mathbb{F}_p$ (ordinary mod 2 cohomology):

$$R^* R = A = \text{Steenrod algebra}$$
$$R_* R = A_* = \text{dual Steenrod algebra}$$

- R-Adams-spectral sequence (ASS):

$$^{R}E_1^{s,t} = R_t\Big(\underbrace{R \wedge \cdots \wedge R}_{s} \wedge X\Big) \Rightarrow \pi_{t-s}(X_R)$$

Generalized Adams spectral sequences

R-ASS:

$$ {}^{R}E_1^{s,t} = R_t\left(\underbrace{R \wedge \cdots \wedge R}_{s} \wedge X\right) \Rightarrow \pi_{t-s}(X_R) $$

- $X =$ space or spectrum

- $\pi_*(X_R) =$ stable homotopy groups of the ``R-localization'' of X
  E.g. $R = H\mathbb{F}_p$:
  $$ \pi_* X_R = (\pi_* X)_p^\wedge $$

- If R_*R is flat over R_*, then R_*R is a Hopf algebra, and

$$ {}^{R}E_2^{s,t} = Ext_{R_*R}(R_*, R_*X) $$

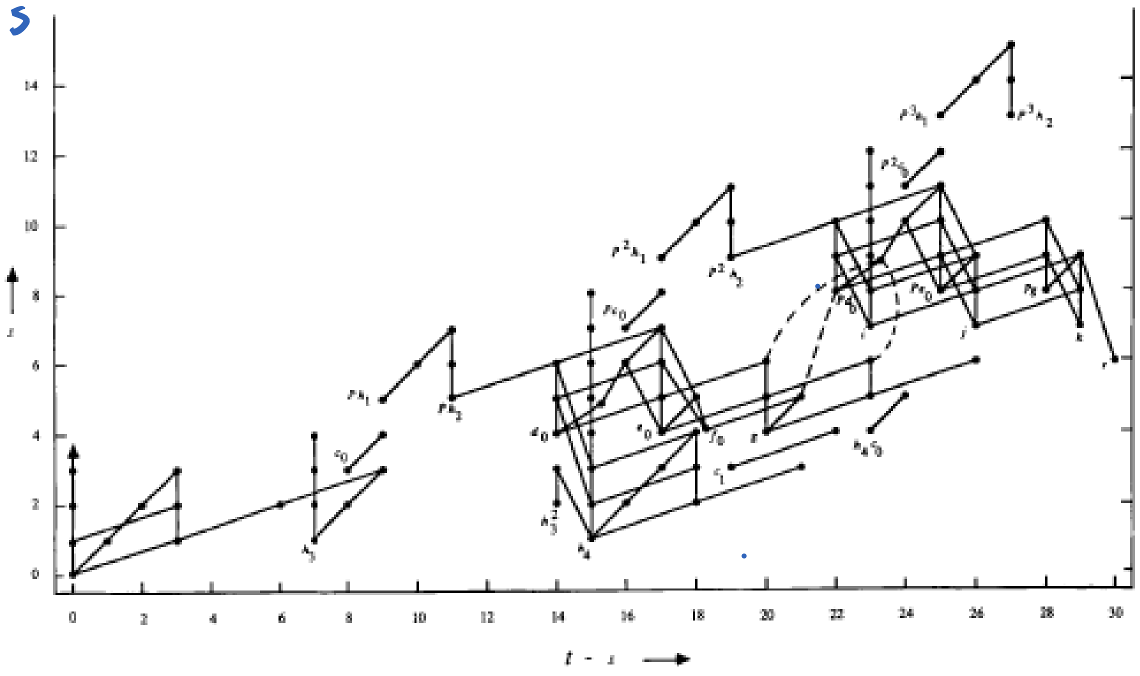
WARNING: this is Ext of comodules over the coalgebra R_*R

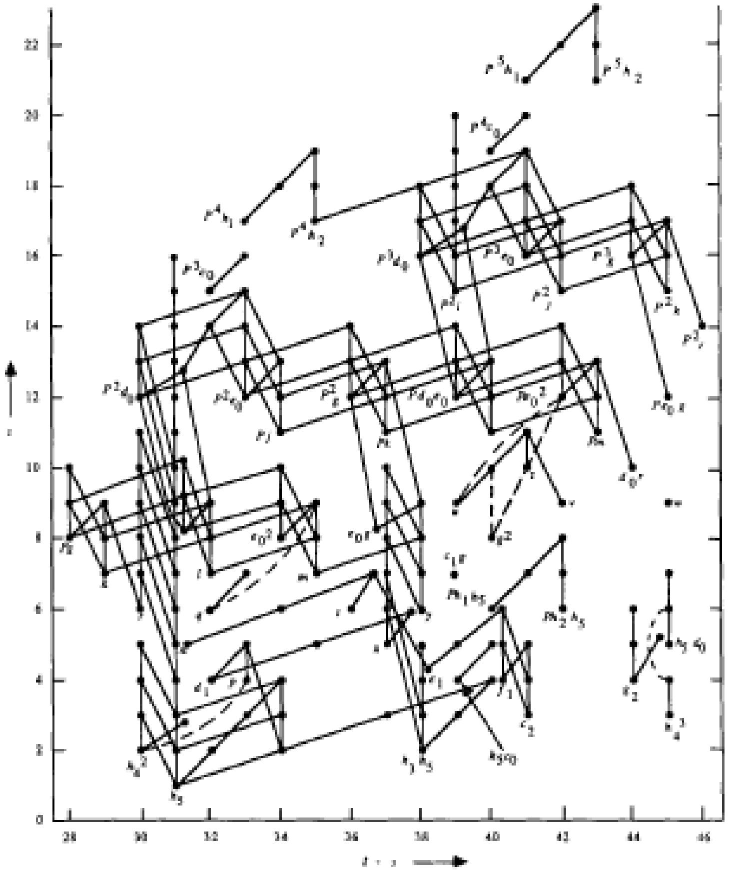
Adams spectral sequence

$$Ext_{A_*}^{s,t}(\mathbb{F}_p , \mathbb{F}_p) \Rightarrow (\pi_{t-s}^{S})_p$$

$\boxed{p=2}$

s





t - s

Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

Adams spectral sequence

$$Ext_{A_*}^{s,t}(\mathbb{F}_p\,,\mathbb{F}_p) \Rightarrow (\pi_{t-s}^{S})_p$$

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s

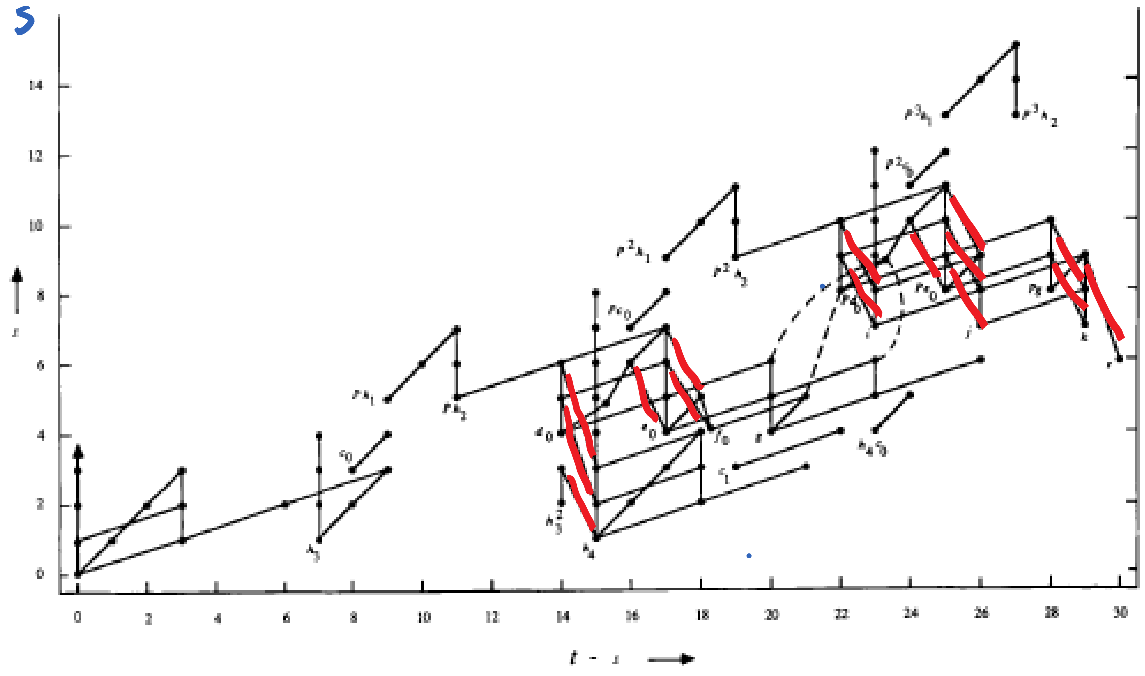
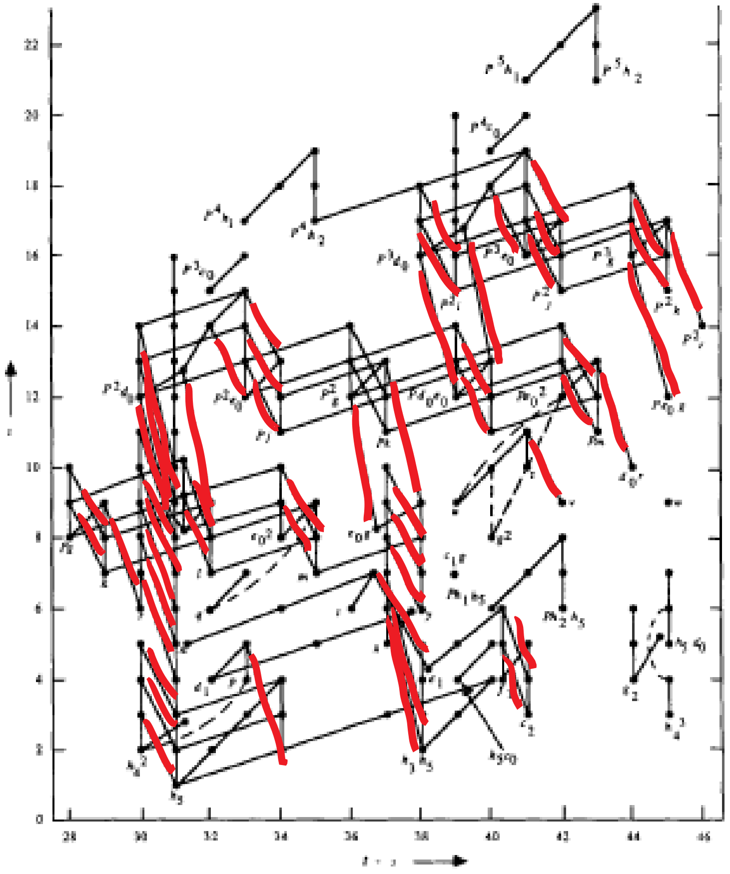


Figure A3.1a The Adams spectral sequence for $p=2$, $t-s \leq 29$.



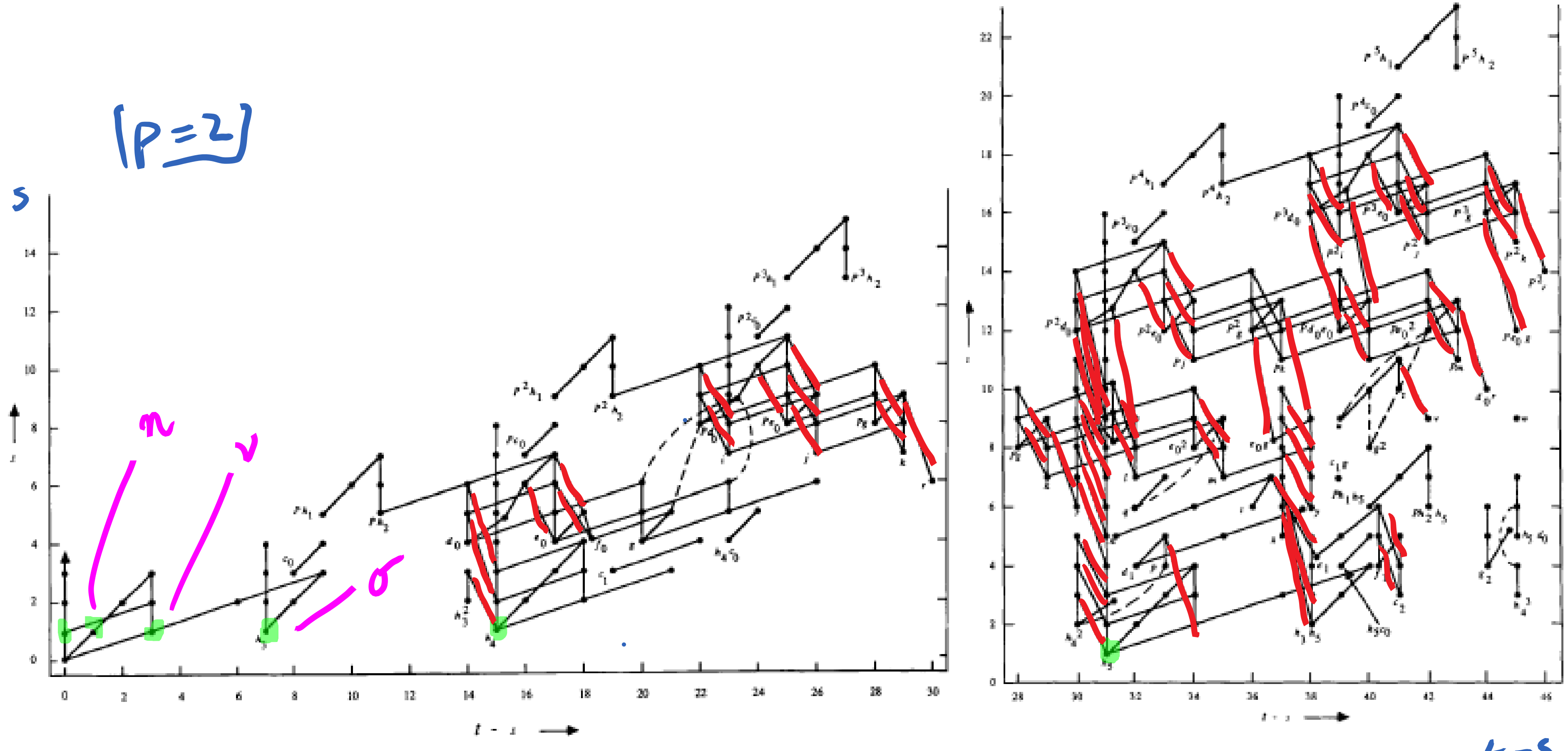
$t-s$

-Many differentials

-d_r differentials go back by 1 and up by r

Adams spectral sequence

$$Ext_{A_*}^{s,t}(\mathbb{F}_p\,, \mathbb{F}_p) \Rightarrow (\pi_{t-s}^{s})_p$$



: HI 1

Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

-Many differentials

-d_r differentials go back by 1 and up by *r* .

Adams spectral sequence

$$Ext_{A_*}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow (\pi_{t-s}^s)_p$$

$\boxed{p=2}$

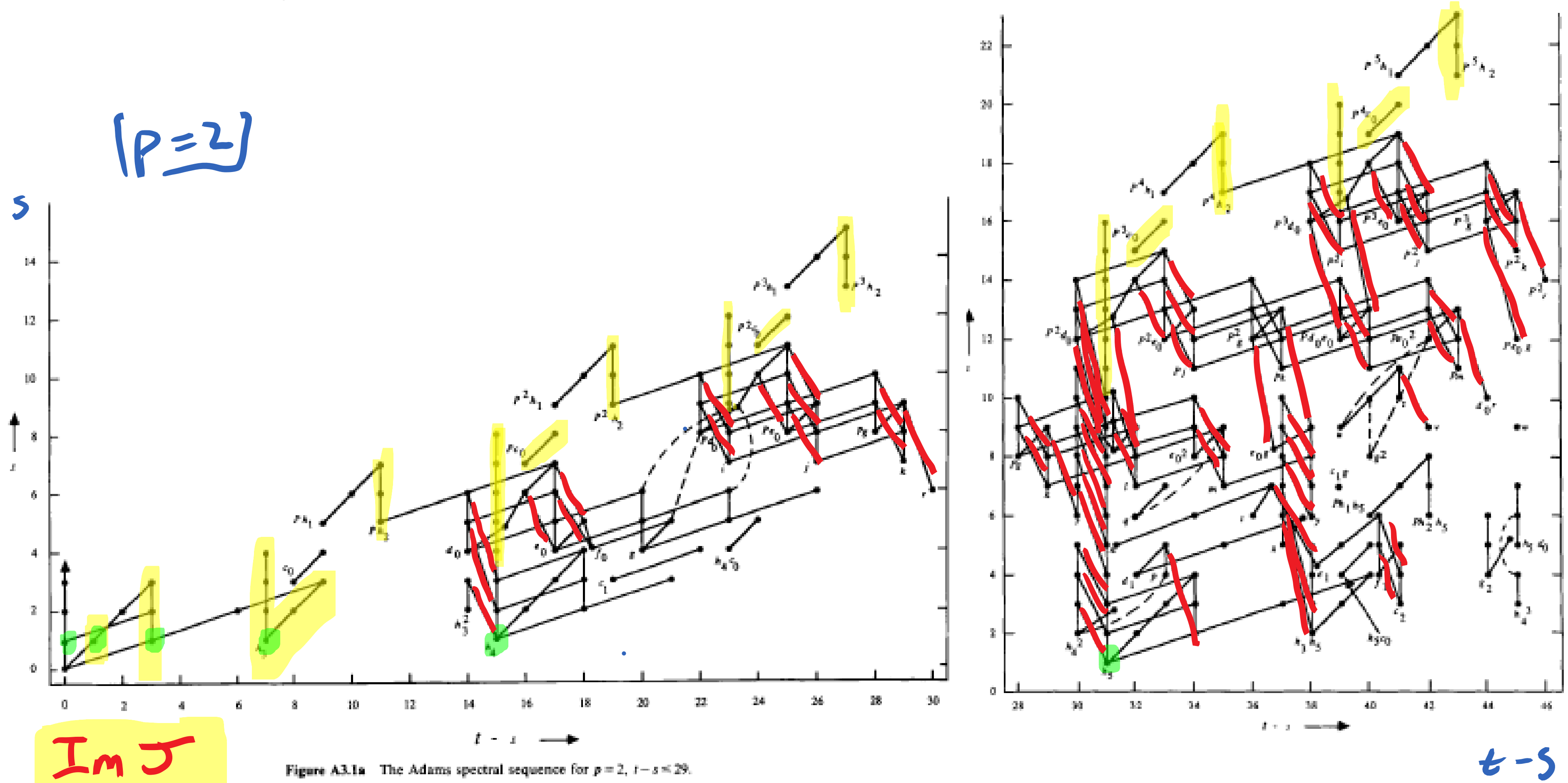


Figure A3.1a The Adams spectral sequence for $p=2$, $t-s\le 29$.

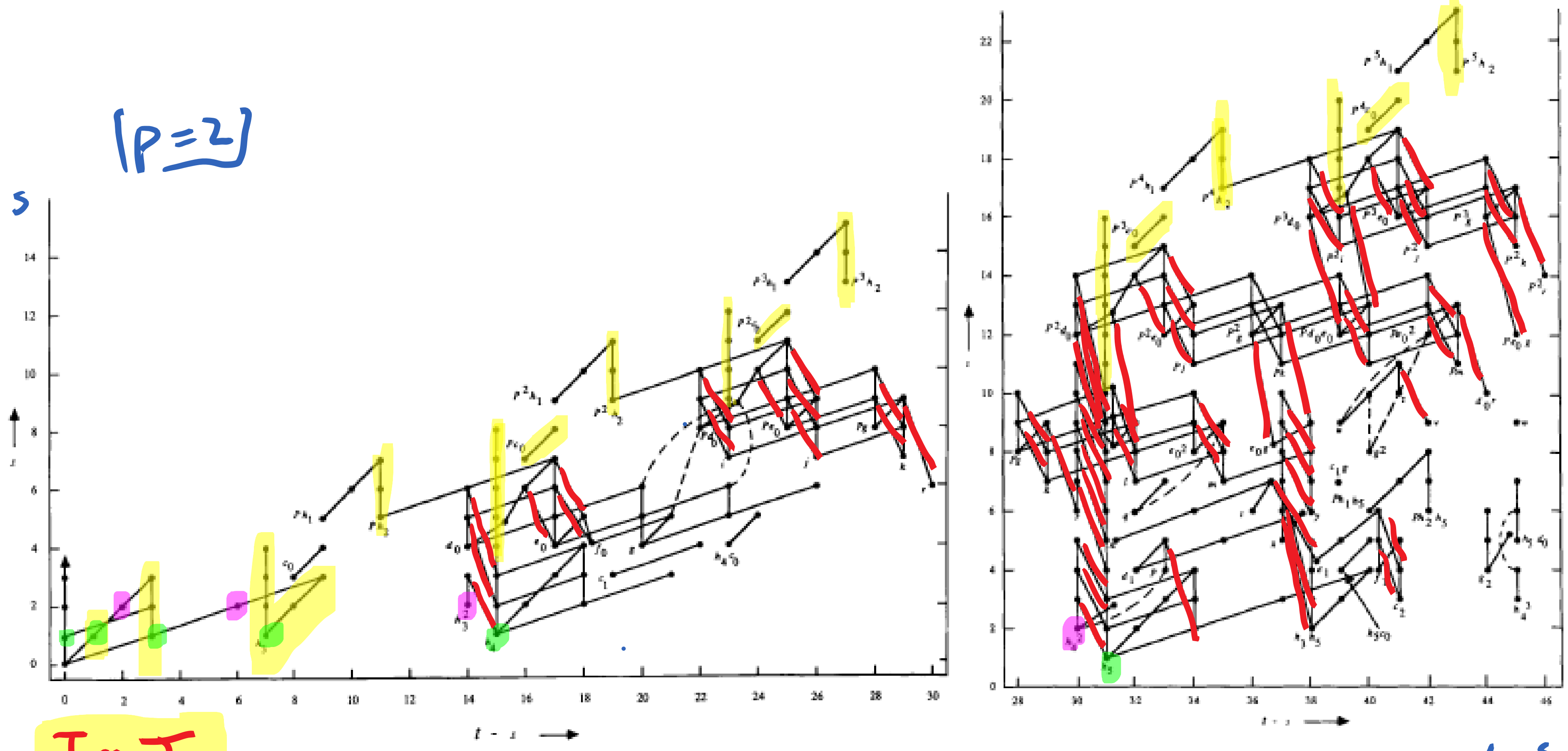
: HI 1   Im J

$J: \pi_* O \to \pi_*^s$ "v_1-periodic"

Adams spectral sequence

$$Ext_{A_*}^{s,t}(\mathbb{F}_p\,,\mathbb{F}_p) \Rightarrow (\pi_{t-s}^{s})_p$$

$\boxed{p=2}$



t-s

Figure A3.1a The Adams spectral sequence for $p=2$, $t-s\leq 29$.

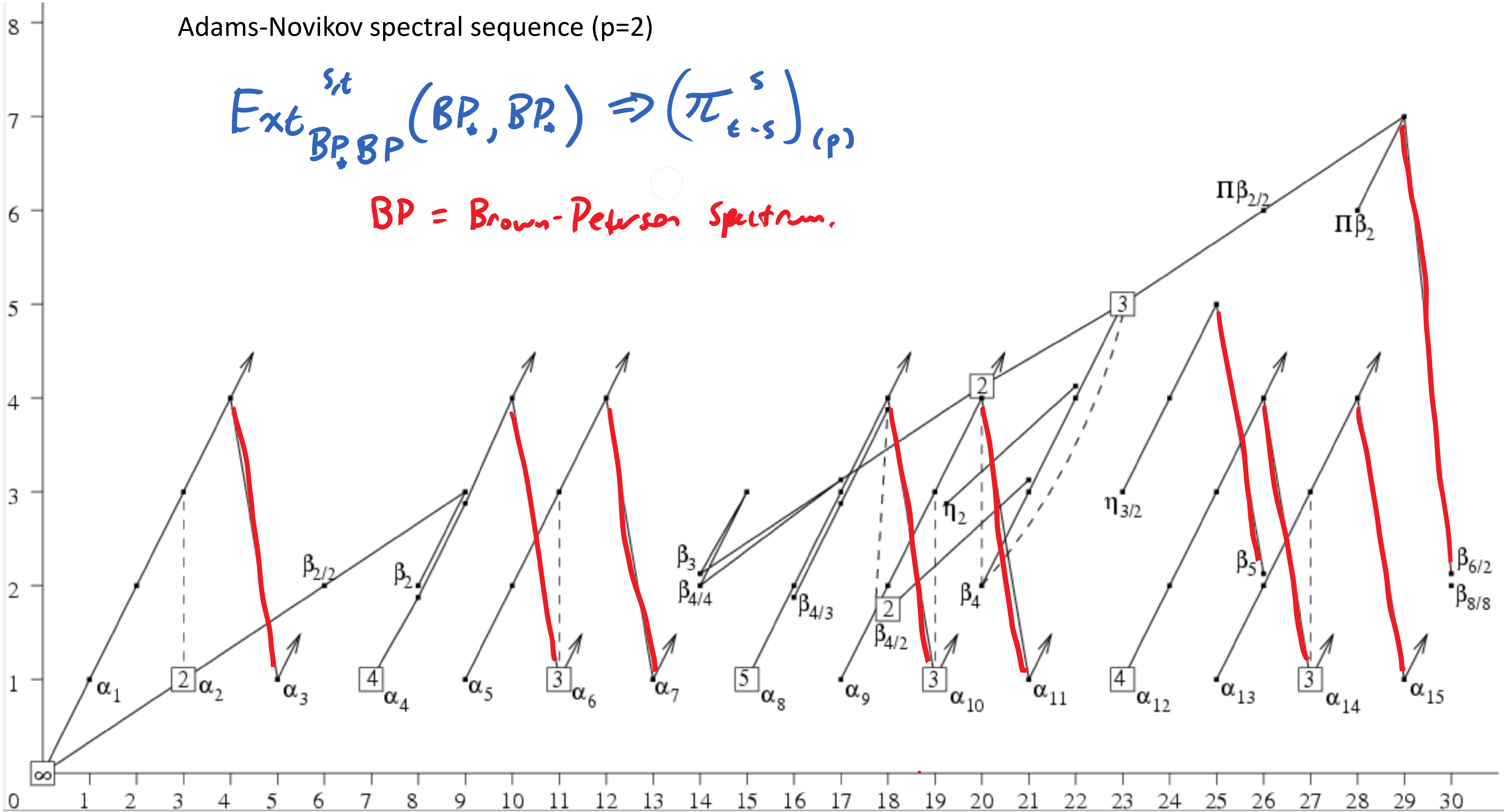
: HI 1 Im J

= Kervaire Invariant 1 (θ_j)

Adams-Novikov spectral sequence (p=2)

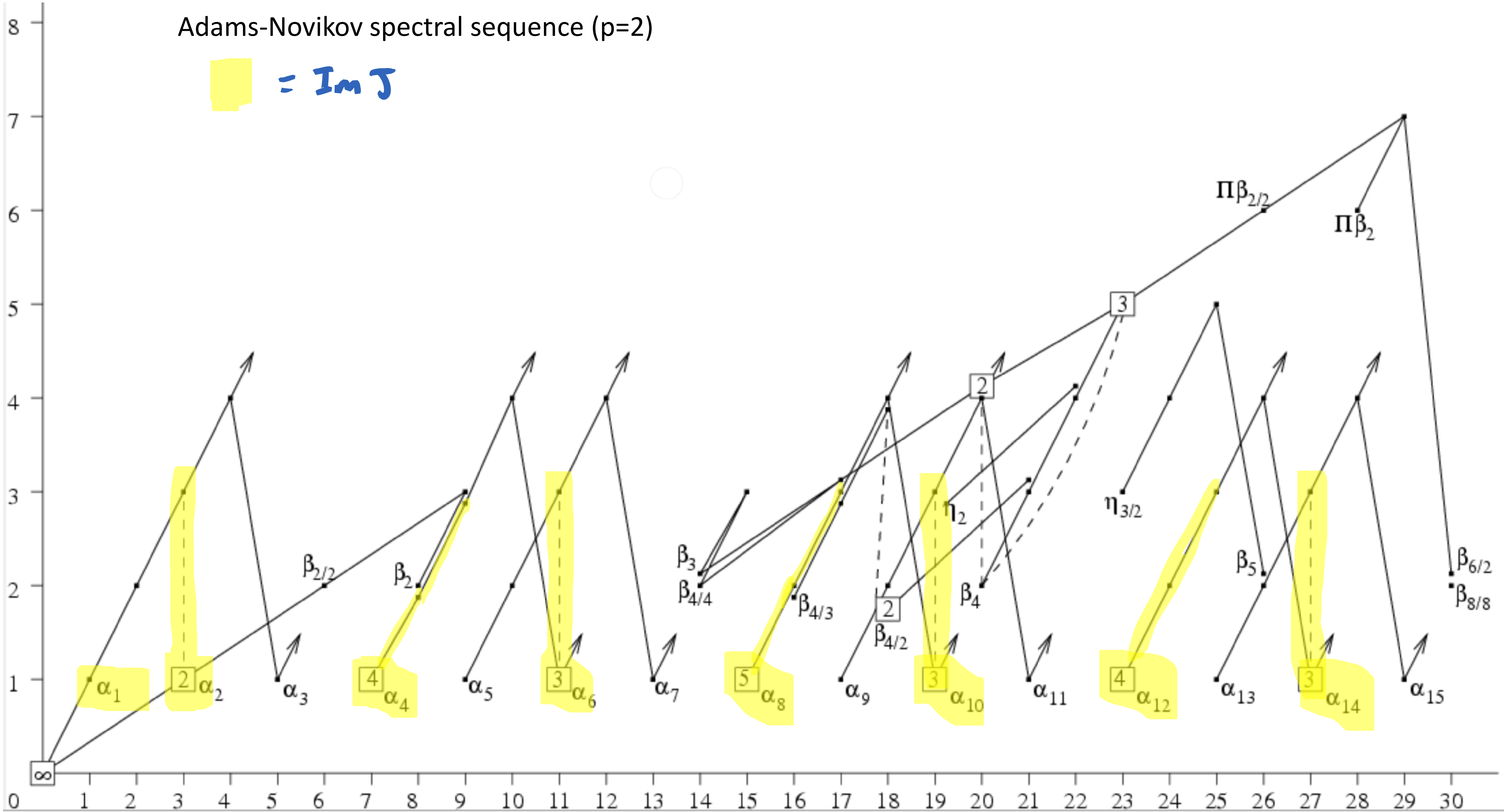
$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow (\pi^s_{t-s})_{(p)}$$

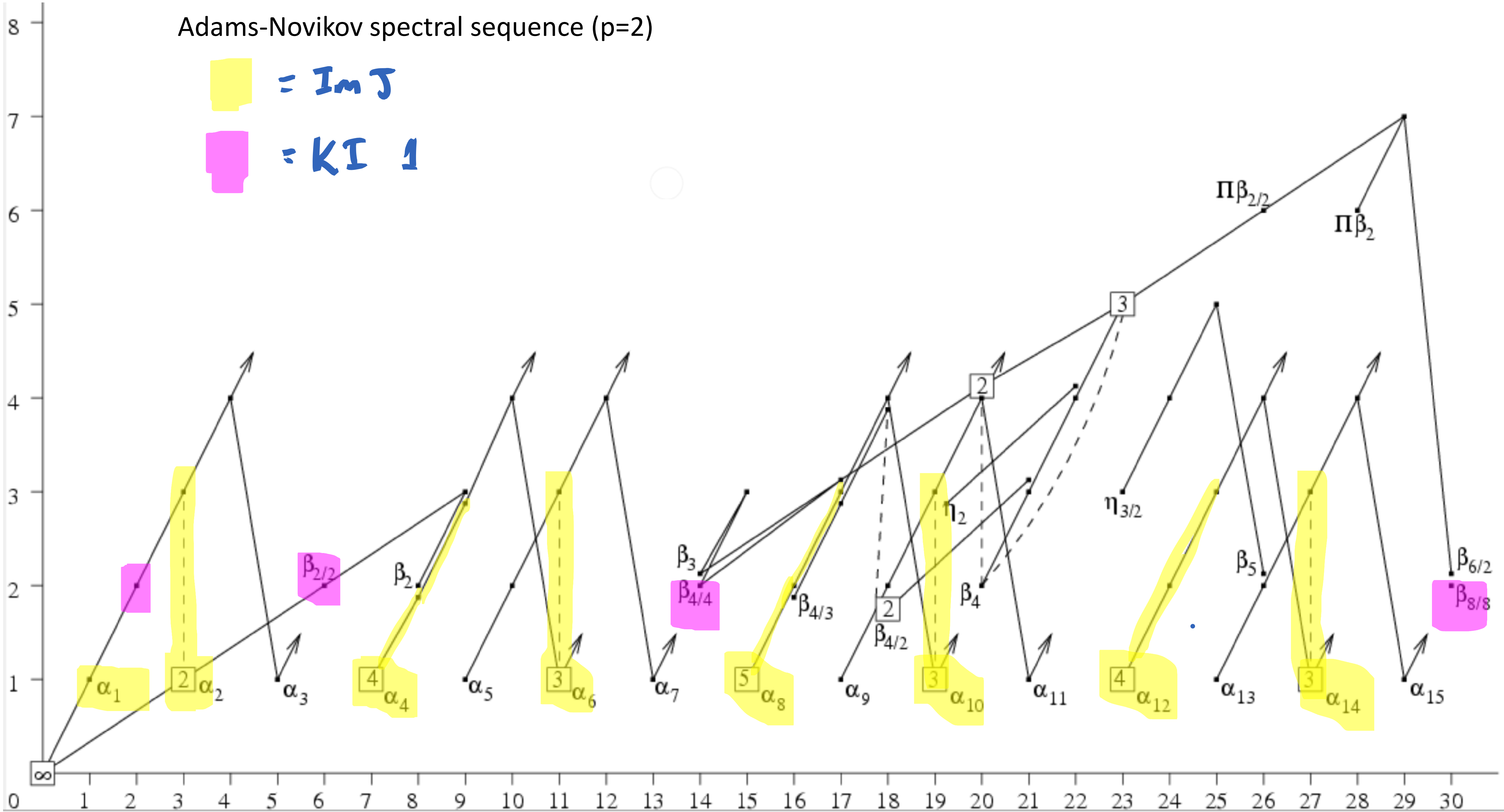
BP = Brown-Peterson spectrum.



Adams-Novikov spectral sequence (p=2)

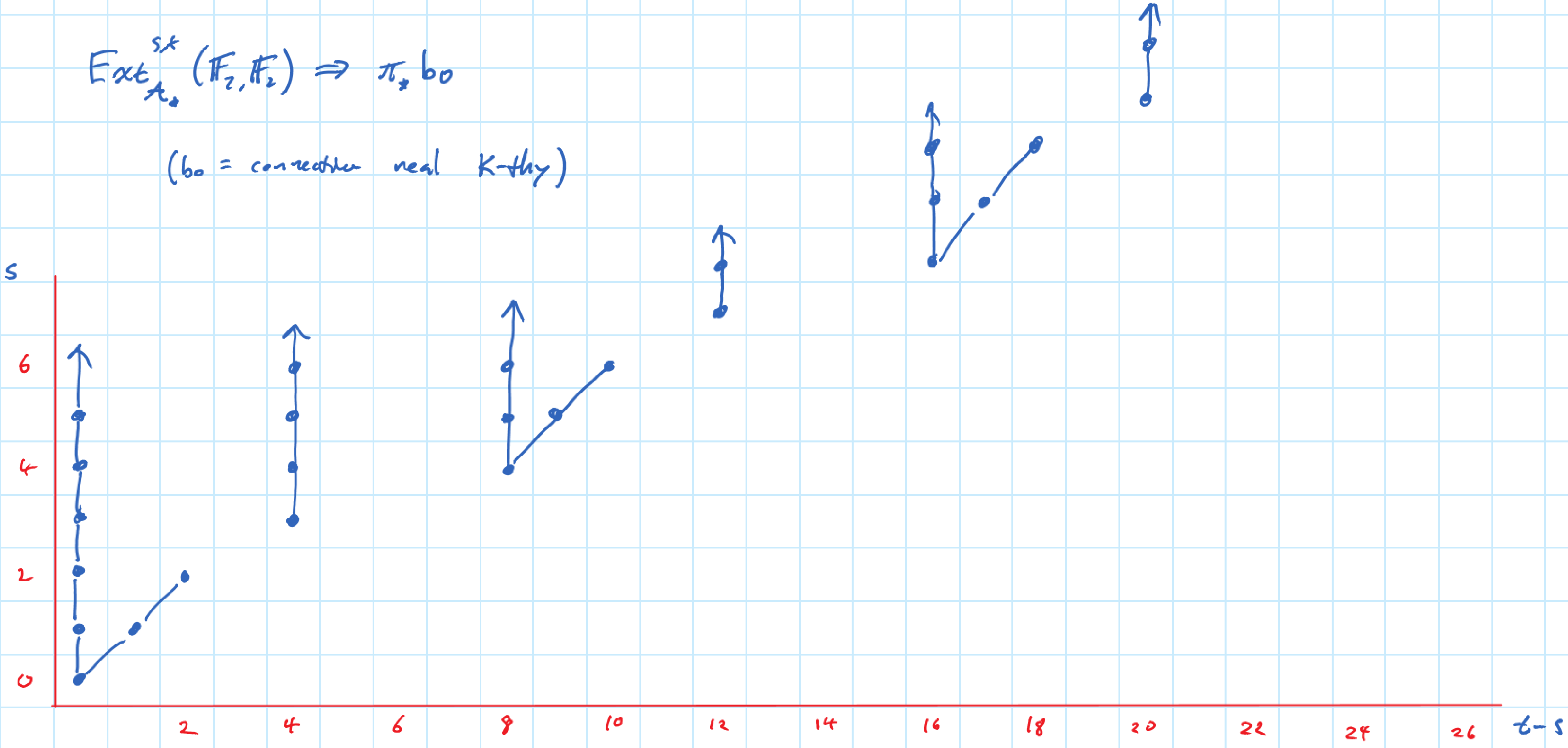
= Im J




Adams-Novikov spectral sequence (p=2)
= Im J
= KI 1
α_1
α_2
α_3
α_4
α_5
α_6
α_7
α_8
α_9
α_{10}
α_{11}
α_{12}
α_{13}
α_{14}
α_{15}
$\beta_{2/2}$
β_2
β_3
$\beta_{4/4}$
$\beta_{4/3}$
$\beta_{4/2}$
β_4
η_2
$\eta_{3/2}$
β_5
$\beta_{6/2}$
$\beta_{8/8}$
$\Pi\beta_{2/2}$
$\Pi\beta_2$

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2,\mathbb{F}_2) \Rightarrow \pi_* bo$$

(bo = connective real K-thy)



$\pi_* bo$: $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$ - - -

From now on all spaces/spectra are implicitly localized at 2!

bo-ASS: ${}^{bo}E_1^{s,t} = bo_t(bo^{\wedge s}) \Rightarrow \pi_{t-s}(S)$

- Motivation: bo is v_1-periodic (aka Bott periodic) – good at detecting v_1-periodic homotopy.
- Used by Mahowald to compute $v_1^{-1}\pi_* S$ at $p = 2$.
  (proving the 2-primary v_1-periodic telescope conjecture)
- Lellmann-Mahowald computed bo-ASS for sphere through dimension 20
- GOOD NEWS: no differentials through this range!
- BAD NEWS: $bo_* bo$ is NOT flat over bo_* --- thus hard to compute ${}^{bo}E_2^{s,t}$
- TODAY: method to compute ${}^{bo}E_2^{s,t}$, computes bo-ASS through dimension 40 “with ease”

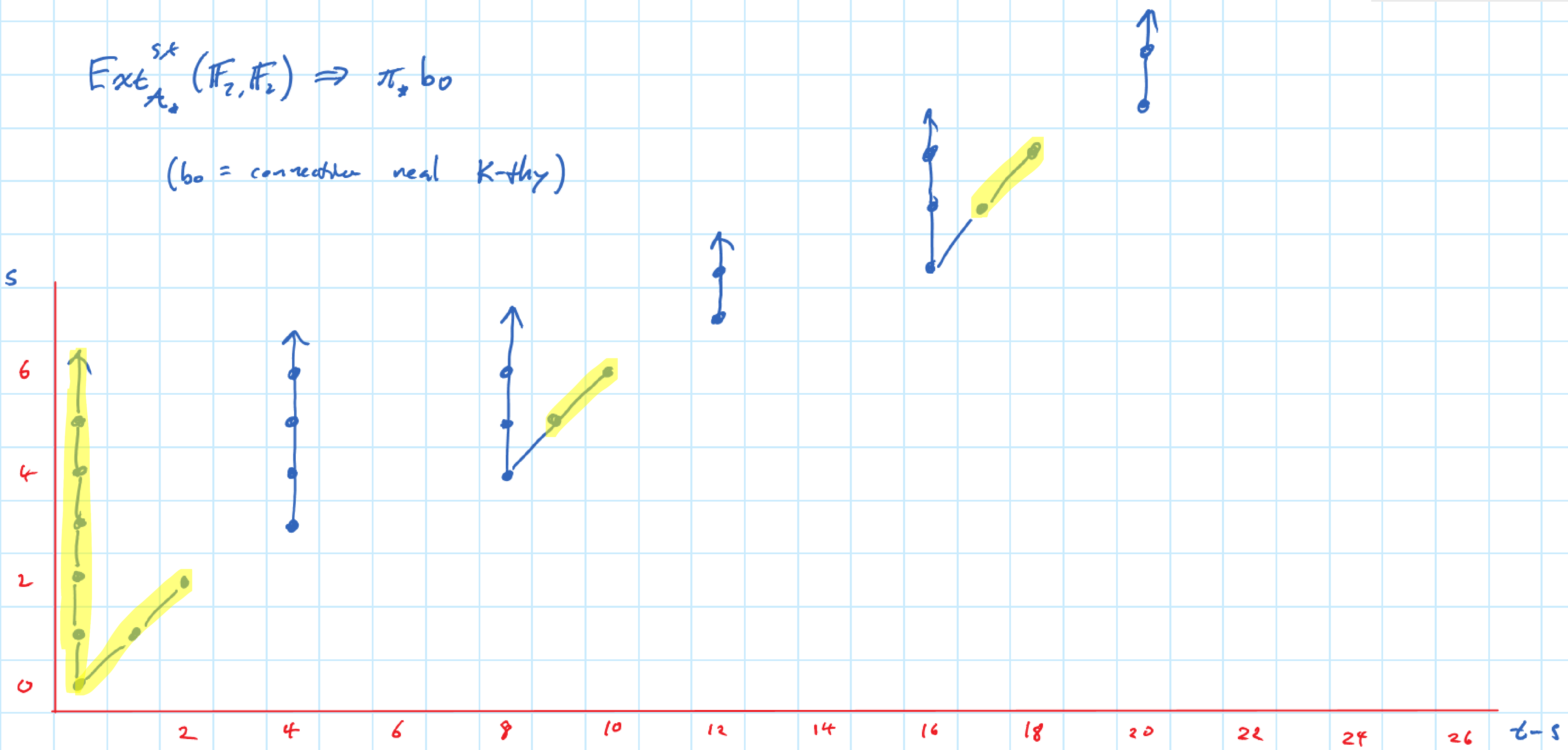
The real motivation: tmf-ASS

- tmf = topological modular forms (sees v_1 and v_2-periodic homotopy)

- Much more powerful than bo

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2,\mathbb{F}_2) \Rightarrow \pi_* bo$$

(bo = connective real K-thy)



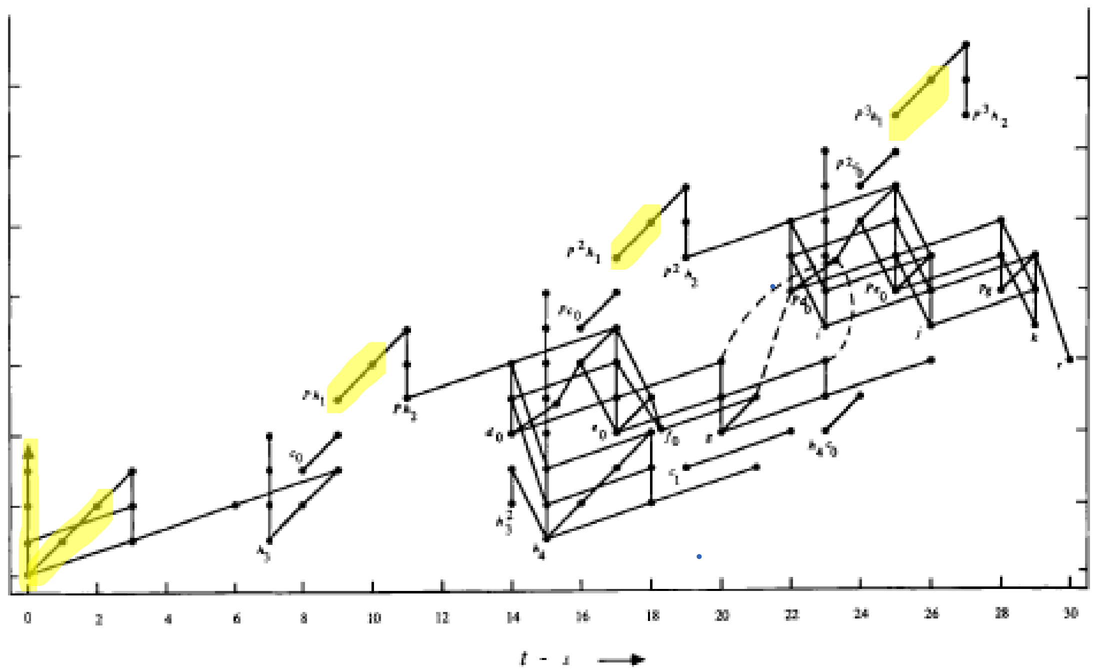
$\pi_* bo$ $\mathbb{Z}$ $\mathbb{Z}/2$ $\mathbb{Z}/2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ $\mathbb{Z}/2$ $\mathbb{Z}/2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ $\mathbb{Z}/2$ $\mathbb{Z}/2$ 0 $\mathbb{Z}$ - - -

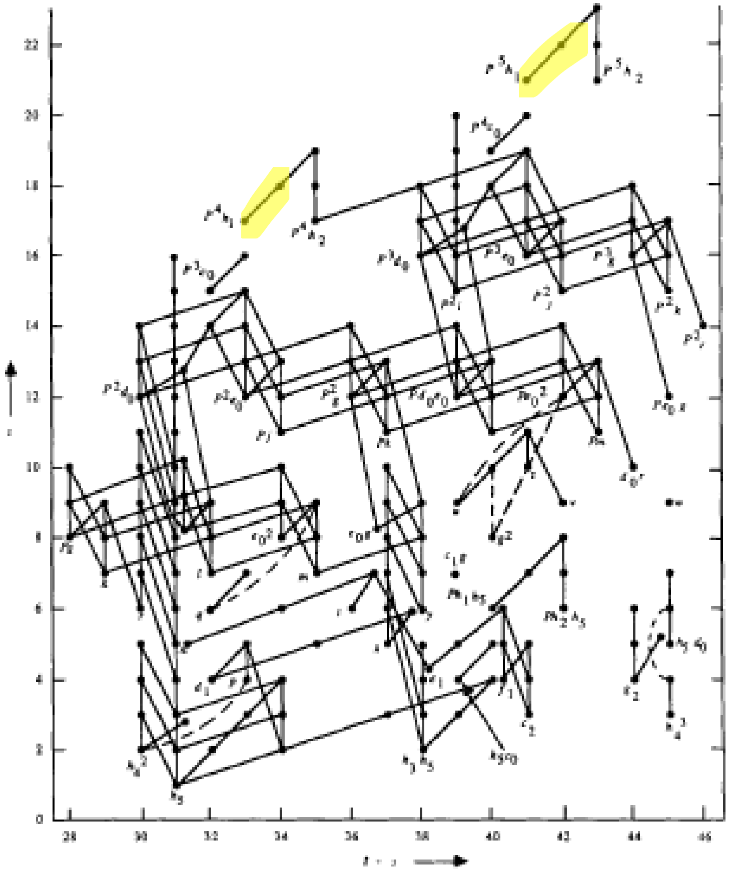
Adams spectral sequence

$$Ext_{A_*}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow (\pi_{t-s}^S)_p$$

[p=2]

s





t-s

Figure A3.1a The Adams spectral sequence for $p=2$, $t-s \leq 29$.

= detected by bo-Hurewicz

Adams spectral sequence

$$Ext_{A_*}^{s,t}(\mathbb{F}_p\,,\mathbb{F}_p) \Rightarrow (\pi_{t-s}^{s})_p$$

$\boxed{p=2}$

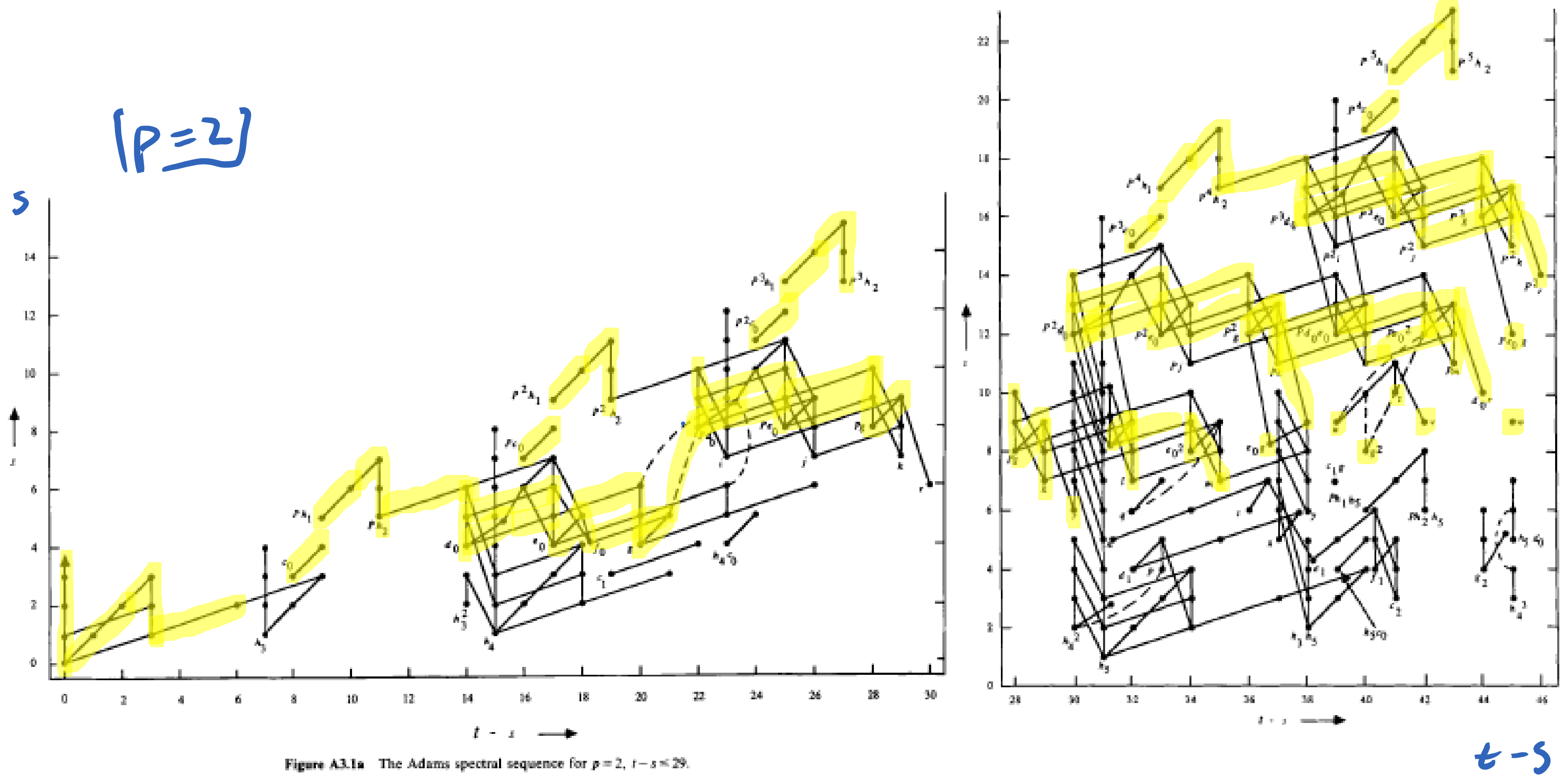


Figure A3.1a The Adams spectral sequence for $p=2$, $t-s\leq 29$.

= detected by tmf-Hurewicz

The real motivation: tmf-ASS

- tmf = topological modular forms (sees v_1 and v_2-periodic homotopy)
- Much more powerful than bo
- Bad news: $tmf_* tmf$ not flat over tmf_*
- Good news: getting a fairly good understanding of $tmf_* tmf$

  (B-Ormsby-Stapleton-Stojanoska --- aka B.O.S.S.)

  [Current collaboration does not have same ring to it: "BBBCX"]

The real motivation: tmf-ASS

- Hope: tmf-ASS should get up through dimension 60 or 70 “with ease”.

- Wang-Xu recently showed $\pi_{61}S$ has no 2-torsion

  CONSEQUENCE: no exotic spheres in dimension 61!

  (super-hard mod 2 ASS computation)

- If we could push our understanding of π_*S into the 90’s, we would have a shot at resolving the outstanding Kervaire invariant question in dimension 126

The real motivation: tmf-ASS

- Potential to understand $v_2^{-1}\pi_* S$ at $p = 2$ (we currently only have good understanding at odd primes)

- Telescope conjecture??? (don't know this for v_2-periodic homotopy at any prime – folks believe it's false…)

Connective K-theory cooperations

Brown-Gitler spectrum perspective

- $H\mathbb{Z} = \bigcup_i B_i$ ($B_i = i^{th}$ integral Brown-Gitler spectrum)

- $H_*(B_i) \subset H_*(H\mathbb{Z}) = \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots]$

subspace spanned by monomials of weight $\leq 2i$ (weight(ζ_i) = 2^{i-1})
$\zeta_i = c(\xi_i)$

- $bo \wedge bo \simeq \bigvee_i \Sigma^{4i}\, bo \wedge B_i$ [Mahowald-Milgram]

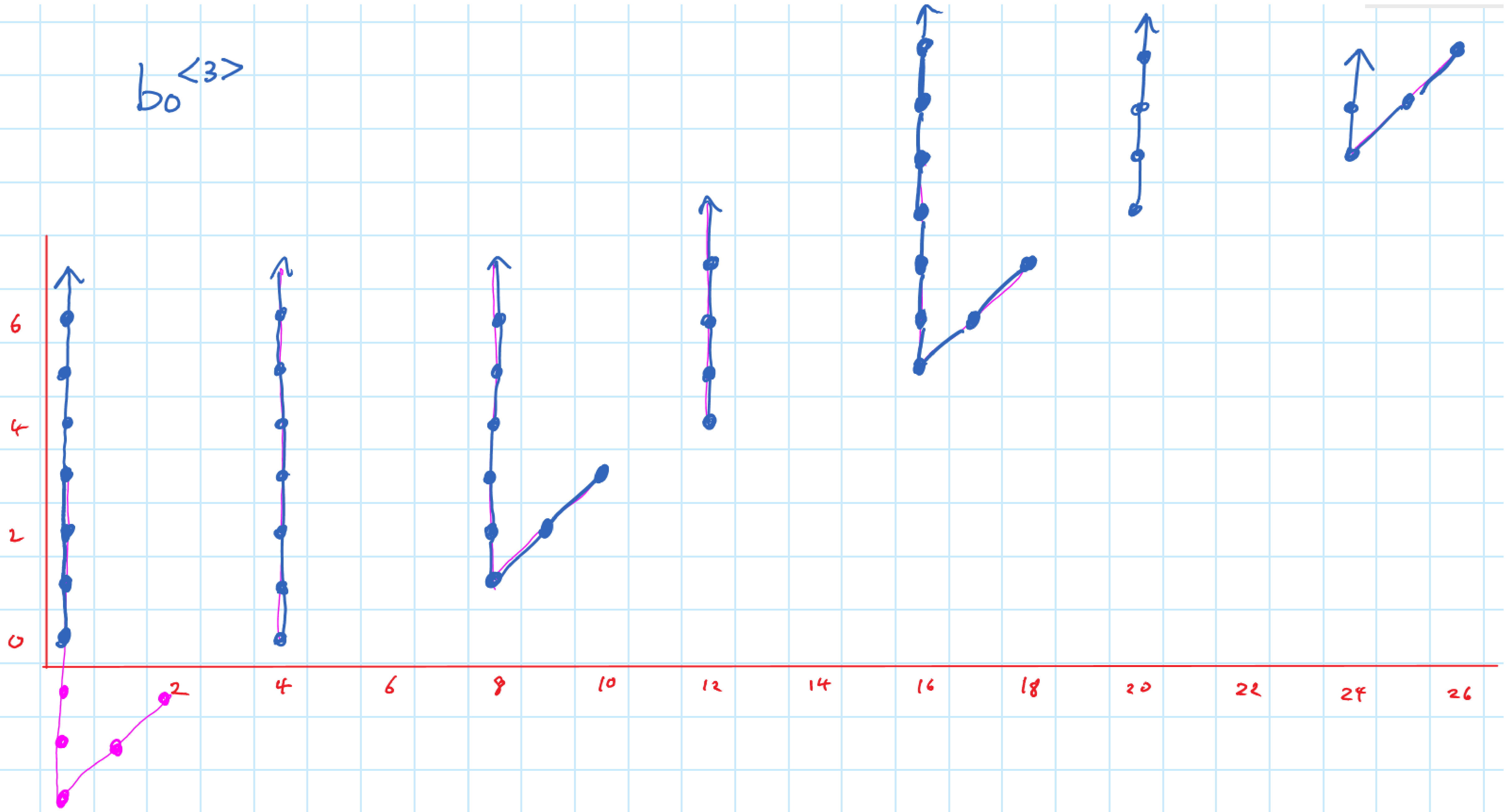
Connective K-theory cooperations

Brown-Gitler spectrum perspective

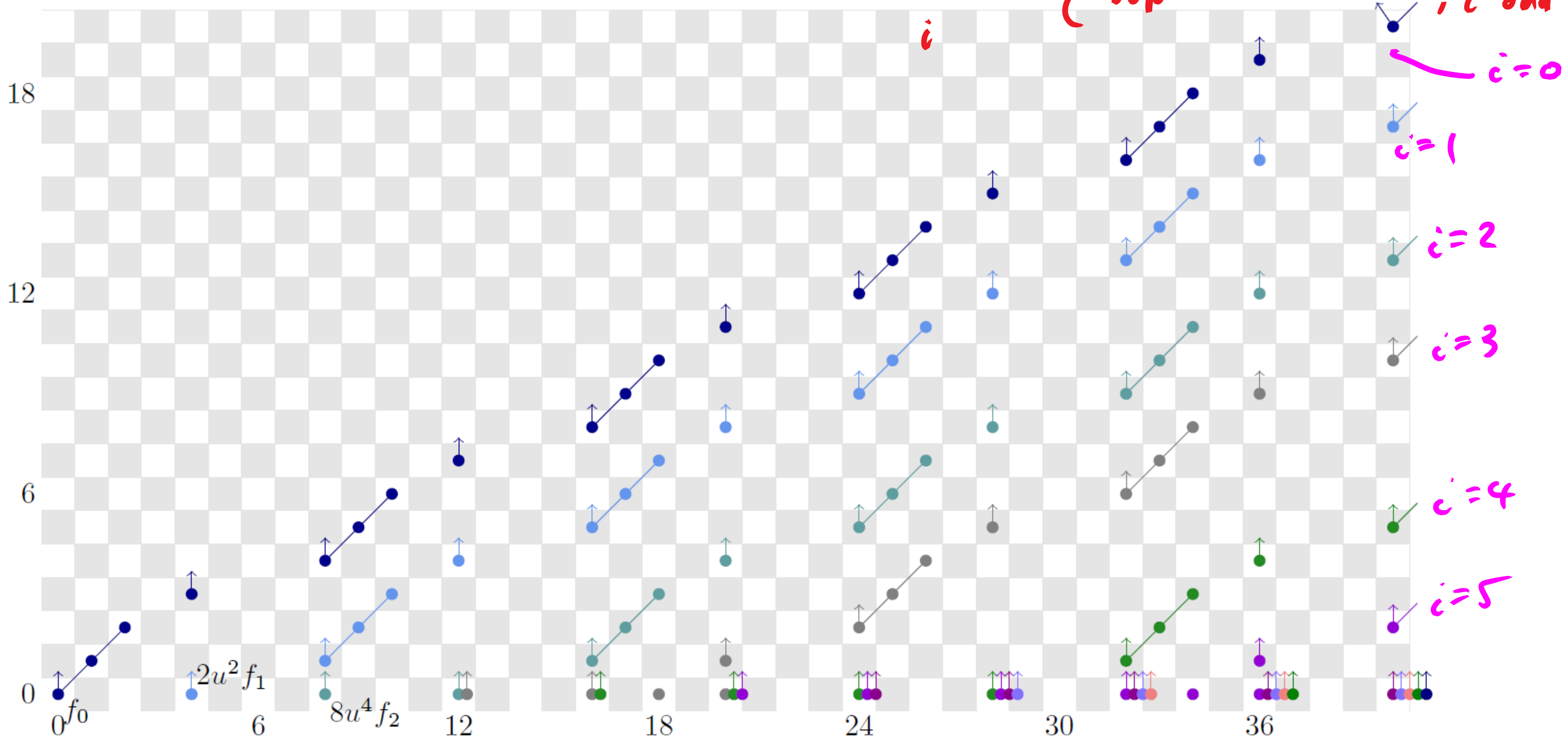
- $bo \wedge bo^s \simeq \bigvee_{I=(i_1,\ldots,i_s)} \Sigma^{4|I|}\, bo \wedge B_I$
  - $B_I = B_{i_1} \wedge \cdots \wedge B_{i_s}$
  - $|I| = i_1 + \cdots + i_s$

- $bo \wedge B_I \simeq HV_I \vee \begin{cases} bo^{<2|I|-\alpha(I)>}, & |I|\ even \\ bsp^{<2|I|-\alpha(I)-1>}, & |I|\ odd \end{cases}$
  - V_I = graded $\mathbb{F}_2$-vector space
  - $\alpha(i)$ = number of 1's in dyatic expansion of i
  - $\alpha(I) = \alpha(i_1) + \cdots + \alpha(i_s)$
  - $bo^{<n>}$ = nth Adams cover of bo (bsp = symplectic K-theory)

$b_0^{\langle 3 \rangle}$



Adams spectral sequence for “good” part of $bo \wedge bo$ $\longleftarrow$ $\bigvee_i \Sigma^{4i} \begin{cases} bo^{\langle 2i-\alpha(i)\rangle}, & i \text{ even} \\ bsp^{\langle 2i-\alpha(i)-1\rangle}, & i \text{ odd} \end{cases}$



bo-ASS: $E_1 - term$

$${}^{bo}E_1^{s,t} = bo_t bo^s \simeq \bigoplus_{I=(i_1,\ldots,i_s)} V_I \oplus \begin{cases} \pi_t \Sigma^{4|I|} bo^{<2|I|-\alpha(I)>}, & |I|\ even \\ \pi_t \Sigma^{4|I|} bsp^{<2|I|-\alpha(I)-1>}, & |I|\ odd \end{cases}$$

bo-ASS: $E_1 - term$

$$ {}^{bo}E_1^{s,t} = bo_t bo^s \simeq \bigoplus_{I=(i_1,\dots,i_s)} V_I \oplus \boxed{\begin{cases} \pi_t \Sigma^{4|I|} bo^{<2|I|-\alpha(I)>}, & |I|\ even \\ \pi_t \Sigma^{4|I|} bsp^{<2|I|-\alpha(I)-1>}, & |I|\ odd \end{cases}} $$

good $E_1^{s,t}$

evil $E_1^{s,t}$

"big, ugly, and incomputable"

bo-ass: E_2-term

$$0 \longrightarrow {}^{evil}E_1^{s,t} \longrightarrow {}^{bo}E_1^{s,t} \longrightarrow {}^{good}E_1^{s,t} \longrightarrow 0$$

$\Longrightarrow$ LES

$$\cdots \longrightarrow {}^{evil}E_2^{s,t} \longrightarrow {}^{bo}E_2^{s,t} \longrightarrow {}^{good}E_2^{s,t} \longrightarrow \cdots$$

bo-ass: E_2-term

$$0 \to {}^{evil}E_1^{s,t} \to {}^{bo}E_1^{s,t} \to {}^{good}E_1^{s,t} \to 0$$

$\Longrightarrow$ LES

$$\cdots \to {}^{evil}E_2^{s,t} \to {}^{bo}E_2^{s,t} \to {}^{good}E_2^{s,t} \to \cdots$$

- d_1^{good} completely computed
- ${}^{good}E_2$ completely computed [weight SS]
- d_1^{evil} not computable! 

Lellmann-Mahowald

Our contribution: can use $Ext_{A_*}(\mathbb{F}_2, \mathbb{F}_2)$ to compute ${}^{evil}E_2^{*,*}$ directly!

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"Algebraic bo-resolution"

$${}^{bo}_{alg}E_1^{s,t,\ell} = Ext_{A(1)}^{s,t}(bo^{\wedge \ell}) \Rightarrow Ext_A^{s+\ell,t}(\mathbb{F}_2, \mathbb{F}_2)$$

Our contribution: can use $Ext_{A_*}(\mathbb{F}_2, \mathbb{F}_2)$ to compute ${}^{evil}E_2^{*,*}$ directly!

"Algebraic bo-resolution"

$$ {}^{bo}_{alg}E_1^{s,t,\ell} = Ext_{A(1)}^{s,t}(\mathrm{bo}^{\wedge \ell}) \Rightarrow Ext_A^{s+\ell,t}(\mathbb{F}_2, \mathbb{F}_2) $$

$$ 0 \longrightarrow {}^{evil}_{alg}E_1^{\cdots} \longrightarrow {}^{bo}_{alg}E_1^{\cdots} \longrightarrow {}^{good}_{alg}E_1^{\cdots} \longrightarrow 0 $$

$\Rightarrow$ LES on E_2-terms

Main Thm (BBBCX)

$$ {}^{\text{evil}}_{\text{alg}}E_2^{s,t,\ell} = \begin{cases} {}^{\text{evil}}E_2^{\ell,t} , & s=0 \\ 0 , & \text{o/w} \end{cases} $$

Main Thm (BBBCX)

$$ {}^{\text{evil}}_{\text{alg}}E_2^{s,t,\ell} = \begin{cases} {}^{\text{evil}}E_2^{\ell,t}, & s = 0 \\ 0, & \text{o/w} \end{cases} $$

Strategy use knowledge of $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ and ${}^{\text{good}}_{\text{alg}}E_2^{\cdots}$ to deduce ${}^{\text{evil}}E_2^{\cdots}$!

$\underline{\text{Thm}}$ (BBBCX)

good $E_2^{\cdots}$ only gets contributions from

$$\mathrm{Ext}_{A(1)}(B_I)$$

for $I = (\underbrace{1, \ldots, 1}_{i_0}, \underbrace{2, \ldots, 2}_{i_1}, \underbrace{4, \ldots, 4}_{i_2}, \ldots)$

" $h_0^{i_0} h_1^{i_1} h_2^{i_2} \cdots$ "

Thm (BBBCX)

${}^{\text{good}}E_2^{*,*}$ only gets contributions from

$$\mathrm{Ext}_{A(1)}(B_I)$$

for $I = (\underbrace{1,\dots,1}_{i_0}, \underbrace{2,\dots,2}_{i_1}, \underbrace{4,\dots,4}_{i_2}, \dots)$

" $h_0^{i_0} h_1^{i_1} h_2^{i_2} \dots$ "

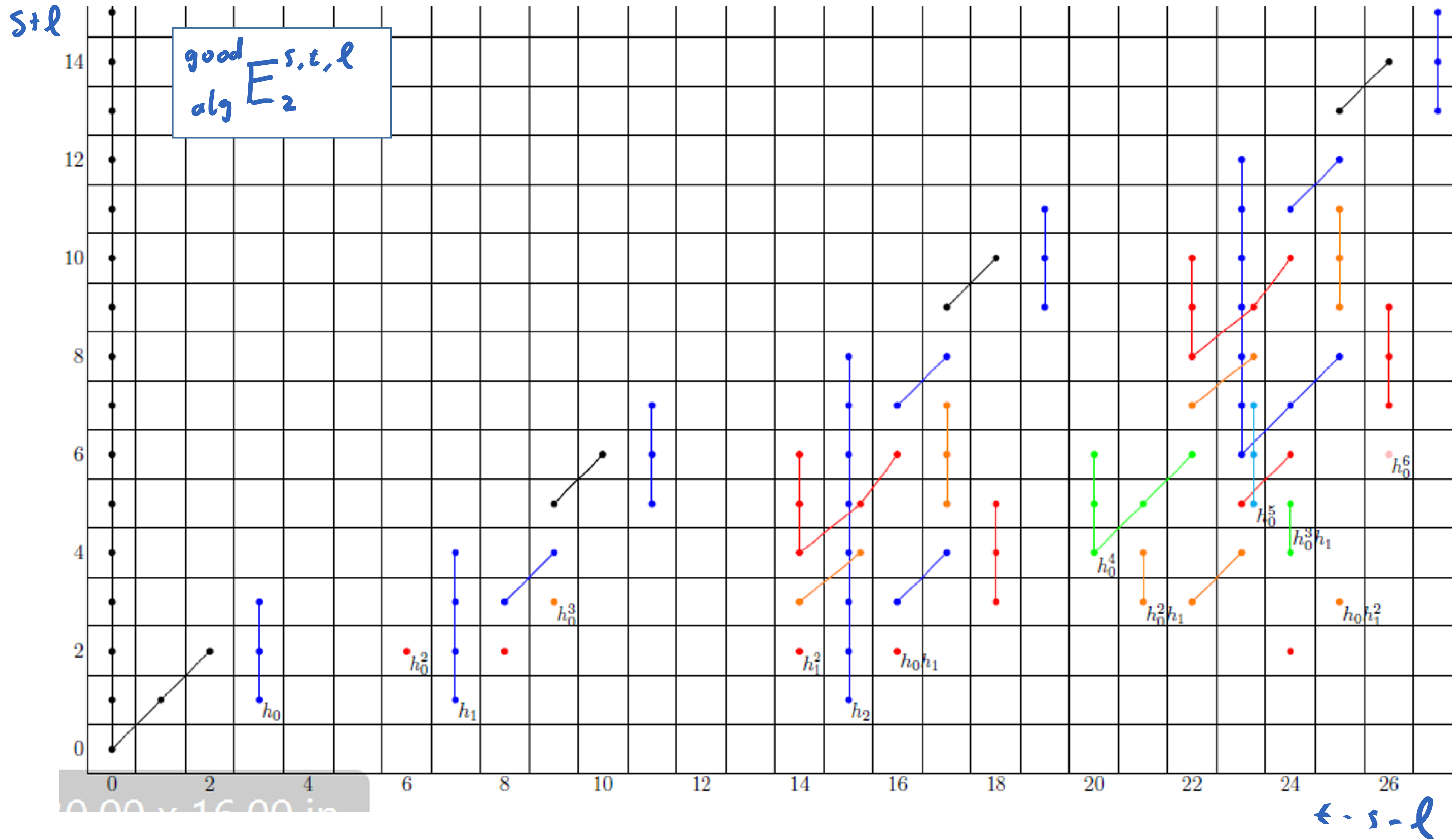
Contribution from

$h_k^{i_k} h_{k+1}^{i_{k+1}} \dots \qquad i_k \neq 0$

is a "$(2^{k+2}-1)$-truncated"

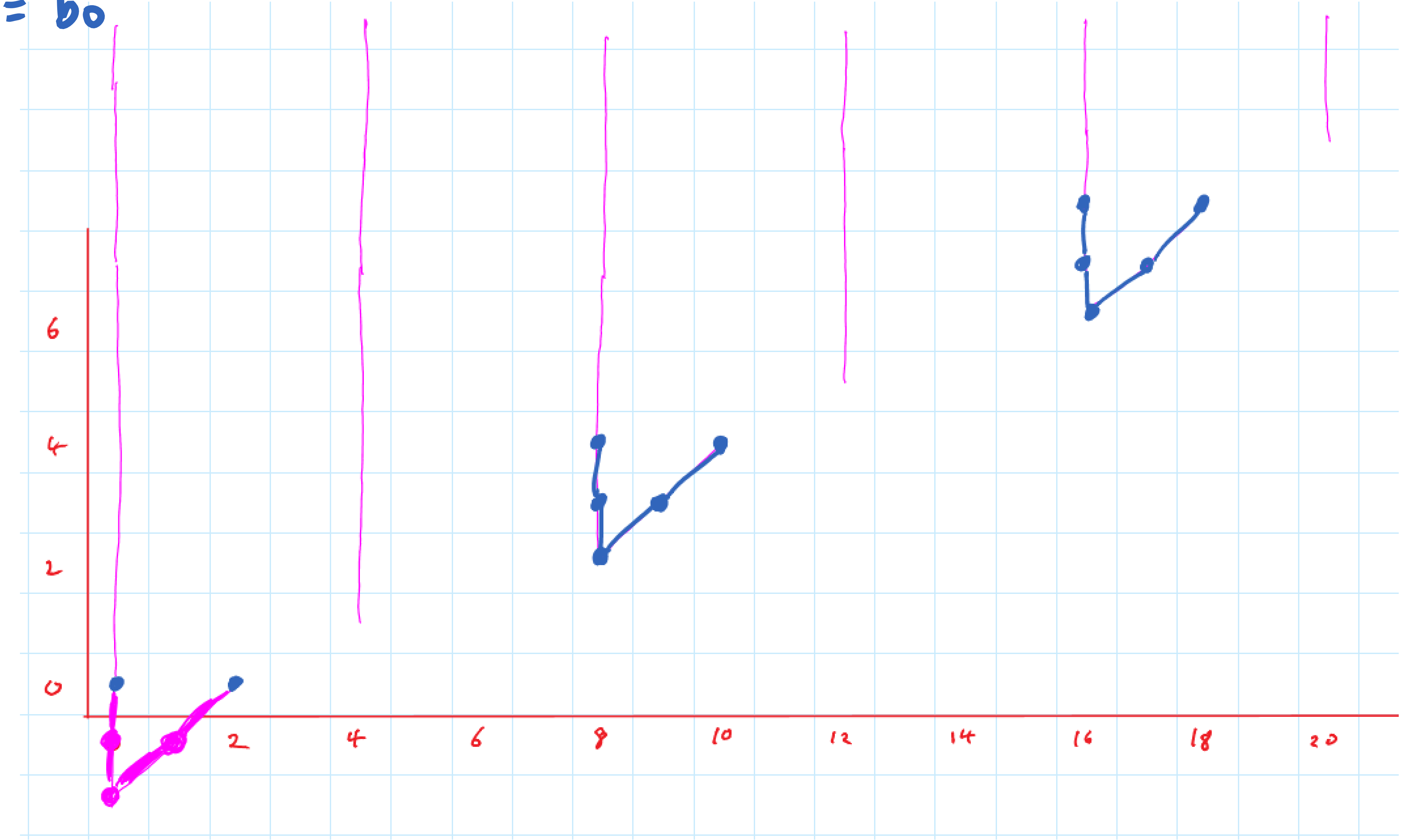
" 2^{k+3}-periodic "

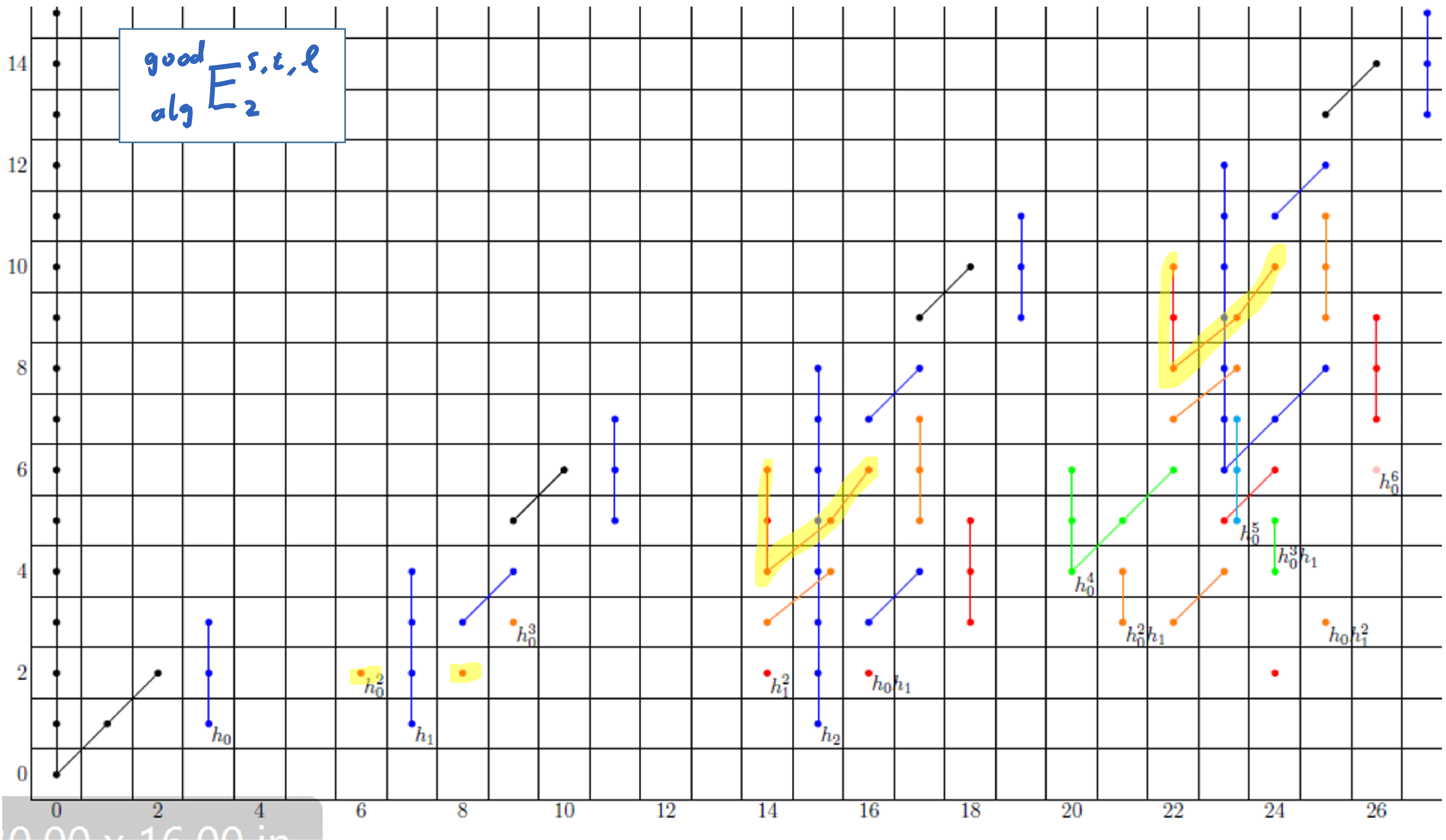
$$\begin{cases} bo^{\langle - \rangle} \\ bsp^{\langle - \rangle} \end{cases}$$



s+l
good
alg $E_2^{s,t,l}$
14
12
10
8
6
4
2
0
0
2
4
6
8
10
12
14
16
18
20
22
24
26
h_0
h_1
h_2
h_0^2
h_0^3
h_1^2
h_0h_1
h_0^4
$h_0^2h_1$
h_0^5
$h_0^3h_1$
$h_0h_1^2$
h_0^6
t-s-l

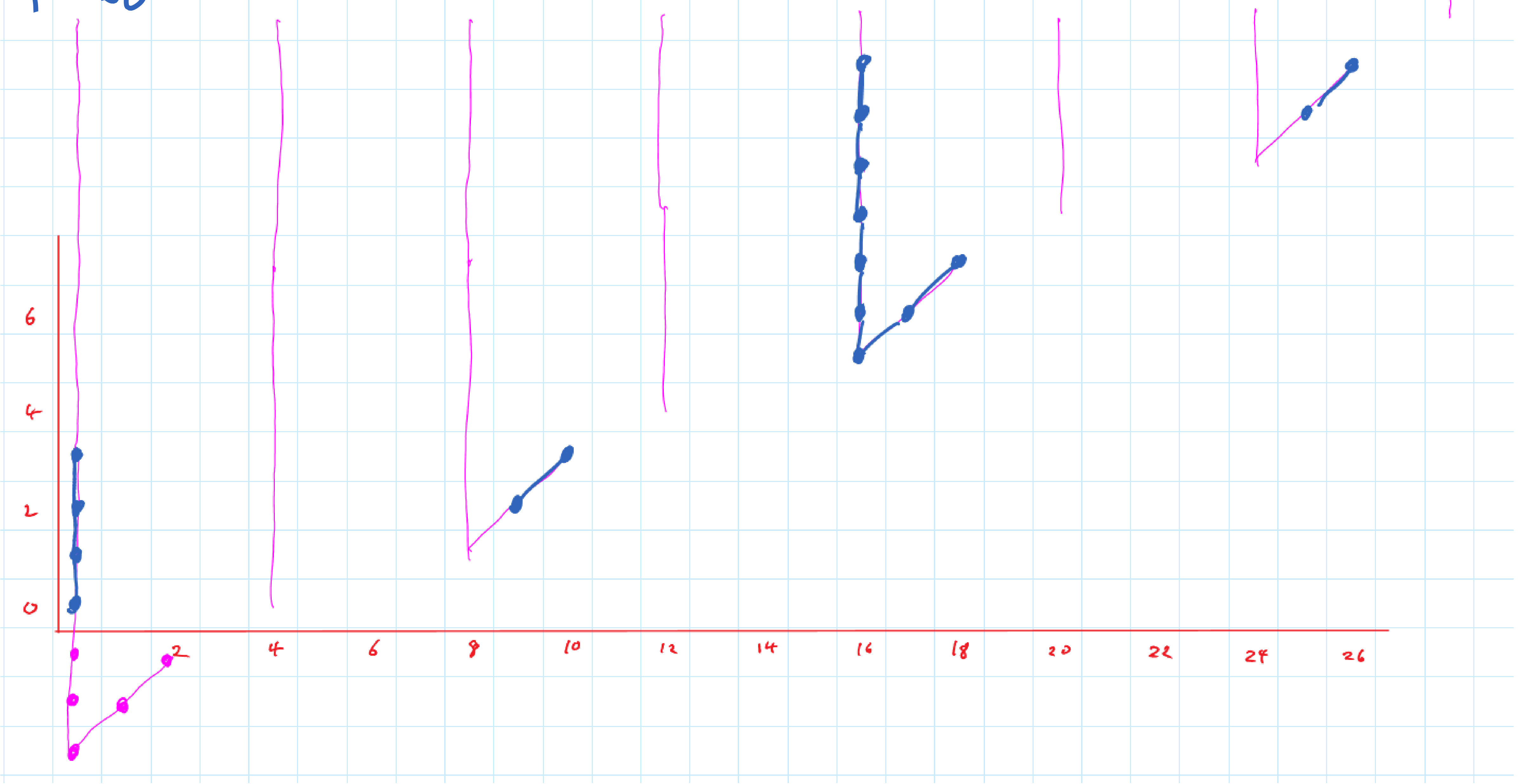
$$h_0^2 = b_0^{\langle 2 \rangle}$$

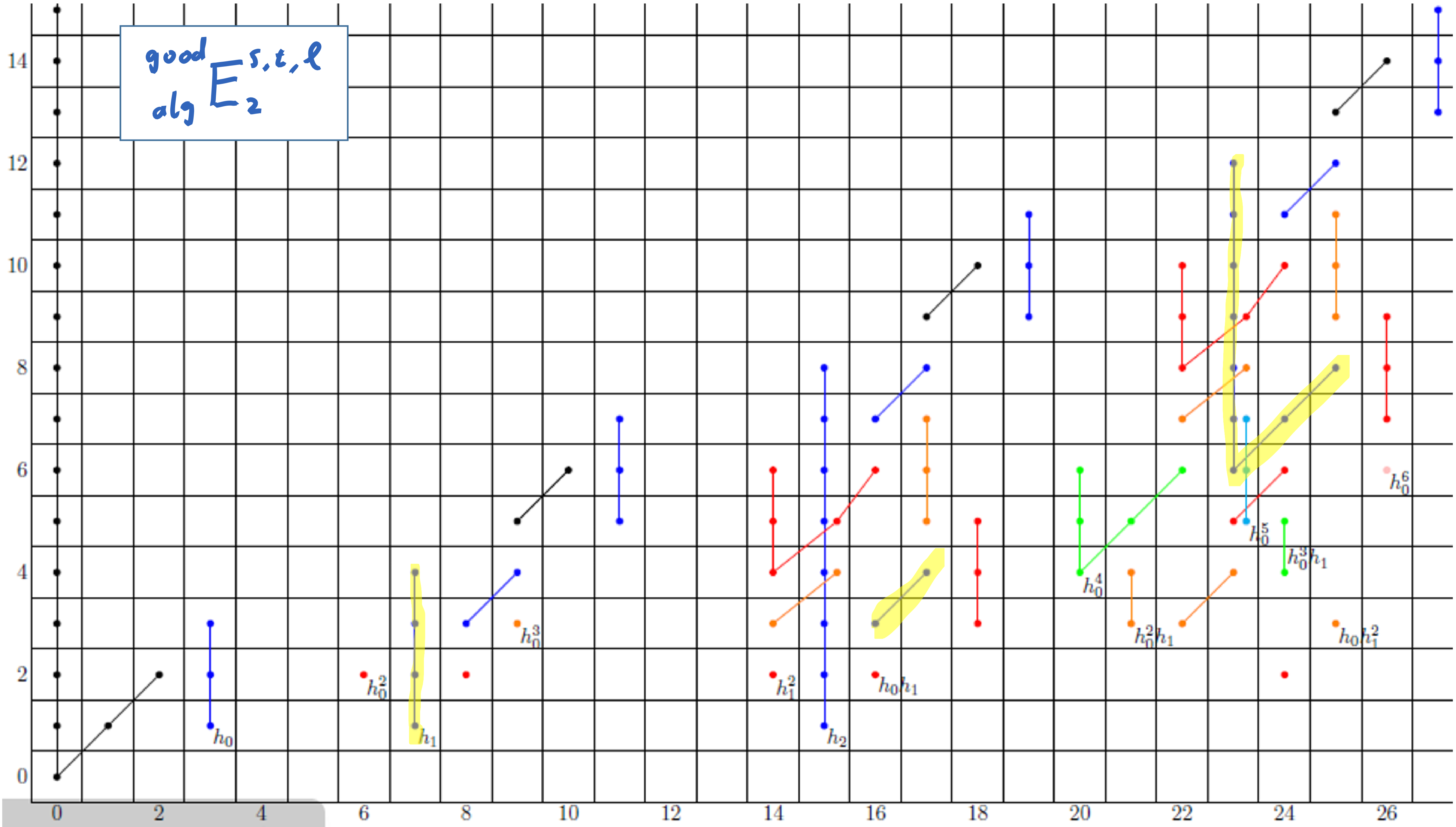




${}^{\mathrm{good}}_{\mathrm{alg}}E_2^{s,t,\ell}$
h_0
h_1
h_2
h_0^2
h_0^3
h_1^2
h_0h_1
h_0^4
$h_0^2h_1$
h_0^5
$h_0^3h_1$
$h_0h_1^2$
h_0^6

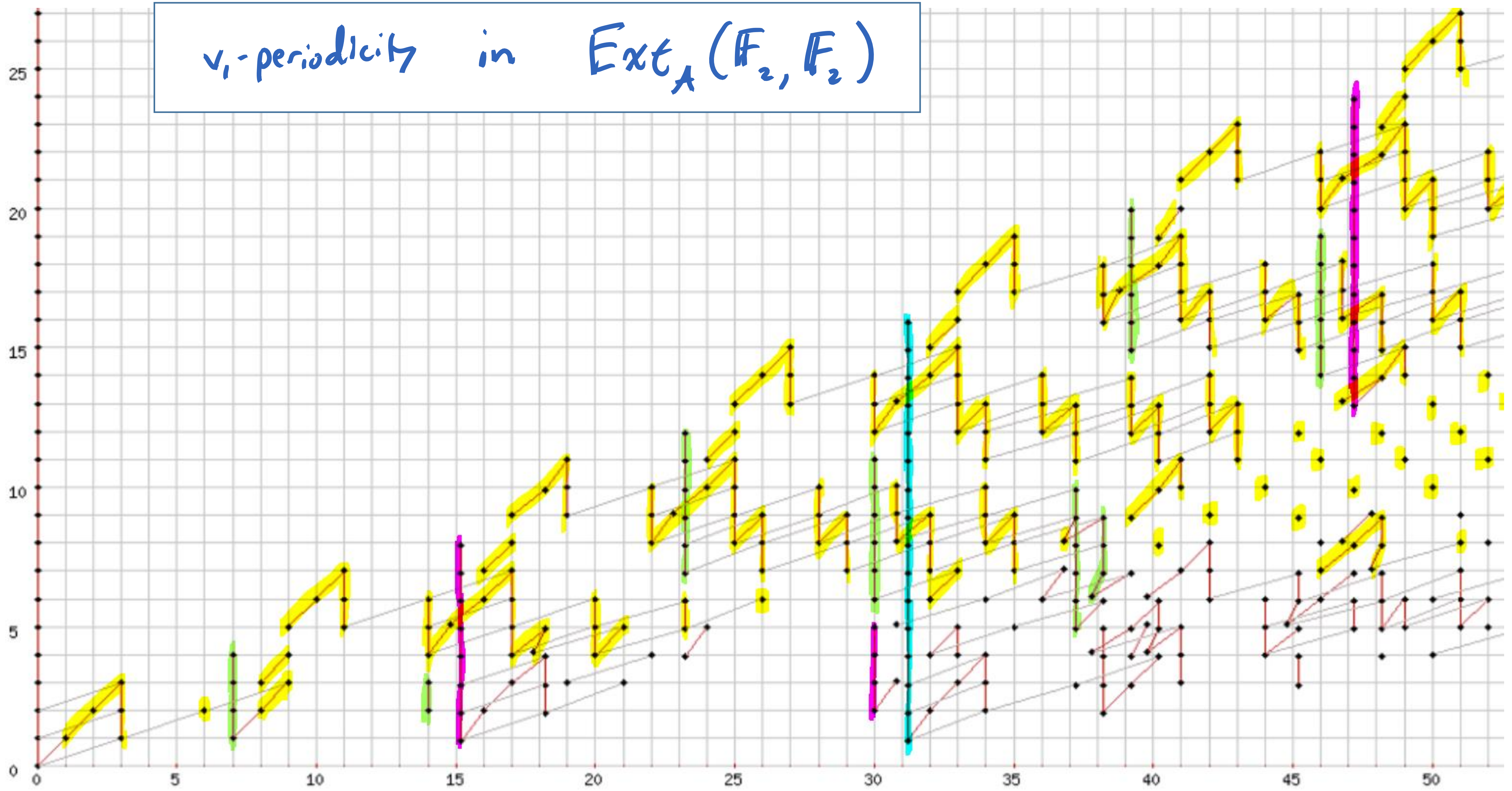
$h_1 = b_0^{<3>}$





good alg $E_2^{s,t,\ell}$
0 2 4 6 8 10 12 14
0 2 4 6 8 10 12 14 16 18 20 22 24 26
h_0
h_1
h_2
h_0^2
h_0^3
h_1^2
h_0h_1
h_0^4
$h_0^2h_1$
h_0^5
$h_0^3h_1$
$h_0h_1^2$
h_0^6

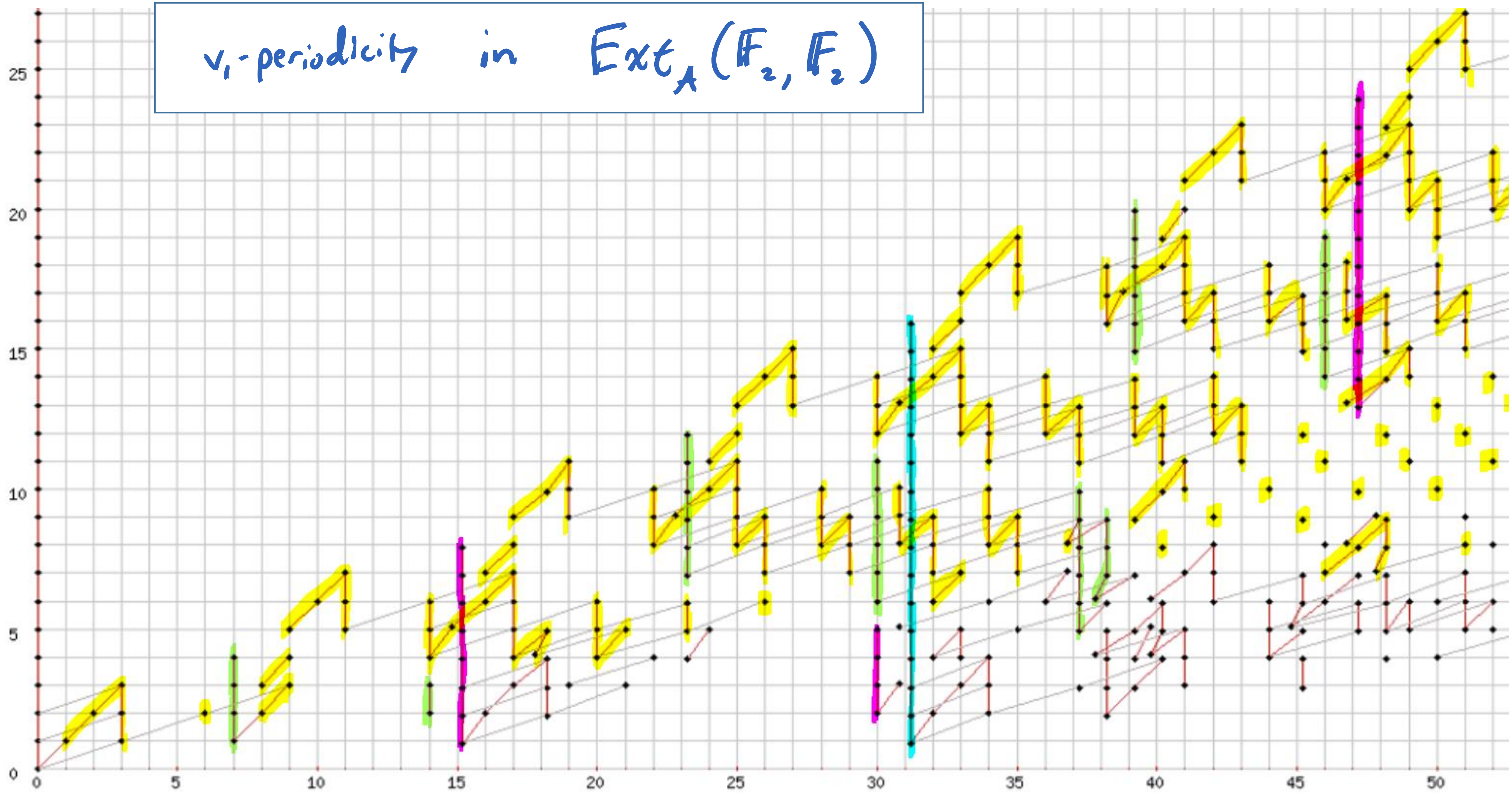
v_1-periodicity in $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$



Connecting map: ${}^{\mathrm{bo}}_{\mathrm{alg}}E_2^{\bullet\bullet\bullet} \longrightarrow {}^{\mathrm{good}}_{\mathrm{alg}}E_2^{\bullet\bullet\bullet}$

Use v_1-periodic Ext

v_1-periodicity in $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$



Connecting map: ${}^{bo\cdots}_{alg}E_2 \longrightarrow {}^{good\cdots}_{alg}E_2$

Thm [BBBCX]

"Algebraic telescope Conj"

$$\begin{array}{ccc} v_1^{-1}\, {}^{bo\cdots}_{alg}E_2 & \xrightarrow{\cong} & v_1^{-1}\, {}^{good\cdots}_{alg}E_2 \\ \Downarrow & & \\ v_1^{-1}\operatorname{Ext}_A(\mathbb{F}_2,\mathbb{F}_2) & & \end{array}$$

Connecting map: ${}^{\text{bo}\cdots}_{\text{alg}}E_2 \longrightarrow {}^{\text{good}\cdots}_{\text{alg}}E_2$

Thm [BBBCX]

"Algebraic telescope Conj"

$$v_1^{-1}\,{}^{\text{bo}\cdots}_{\text{alg}}E_2 \xrightarrow{\cong} v_1^{-1}\,{}^{\text{good}\cdots}_{\text{alg}}E_2$$

$$\Downarrow$$

$$v_1^{-1}\operatorname{Ext}_A(\mathbb{F}_2,\mathbb{F}_2)$$

Consequence:

$x \in {}^{\text{bo}\cdots}_{\text{alg}}E_2$ v_1-periodic

$\Rightarrow$ $v_1^k x \longmapsto$ nonzero $\in {}^{\text{good}\cdots}_{\text{alg}}E_2$

$k \gg 0$

Connecting map:

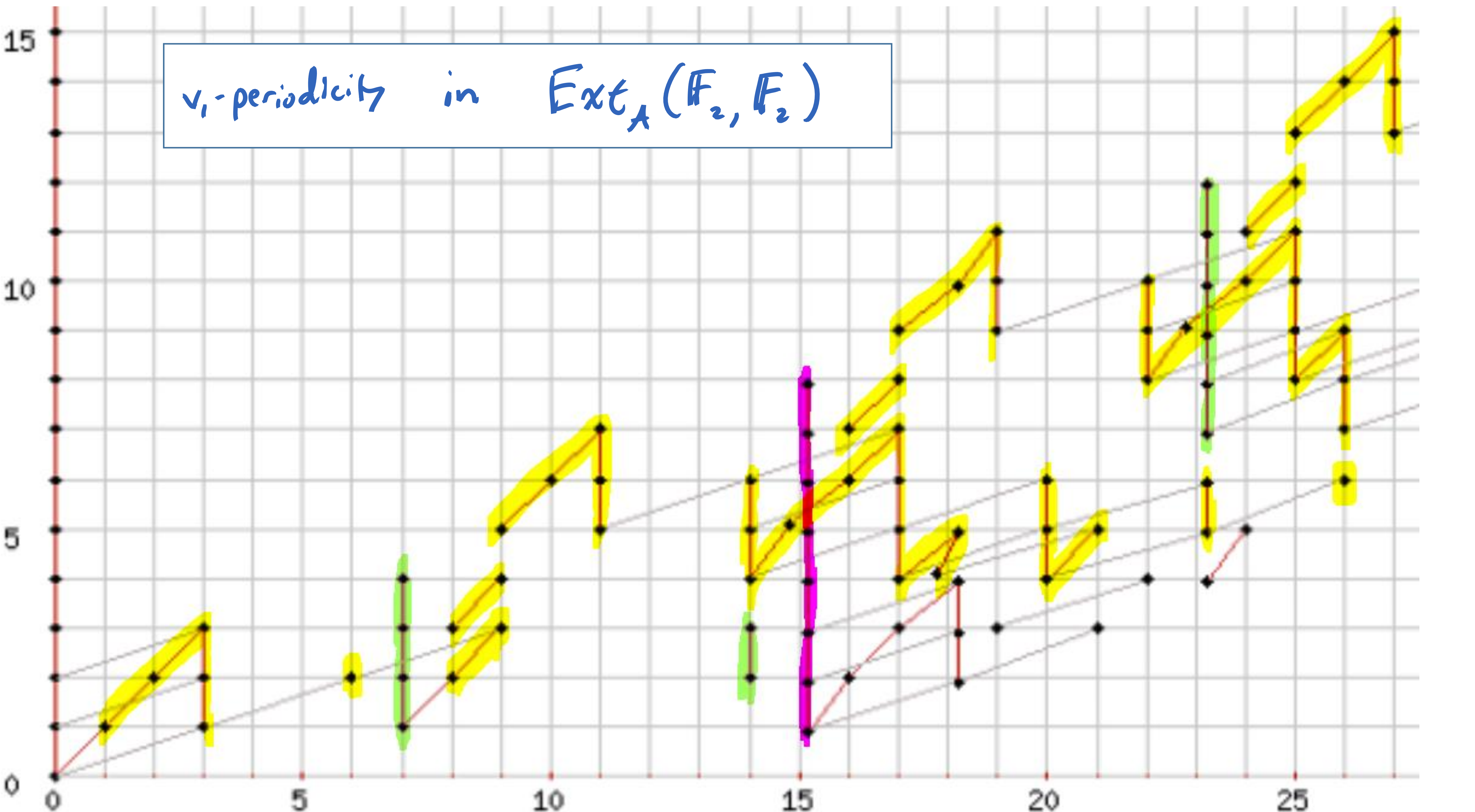
$$ {}^{bo}_{alg}E_2^{\cdots} \longrightarrow {}^{good}_{alg}E_2^{\cdots} $$

Control thm [BBBCX]

$$ v_1^k x \longmapsto v_1^\ell y $$
$$ \psi \qquad\qquad \psi $$
$$ {}^{bo}_{alg}E_2 \qquad {}^{sul}_{alg}E_2 $$

$$ \Rightarrow \quad v_1^{k-\ell} x \longmapsto y $$

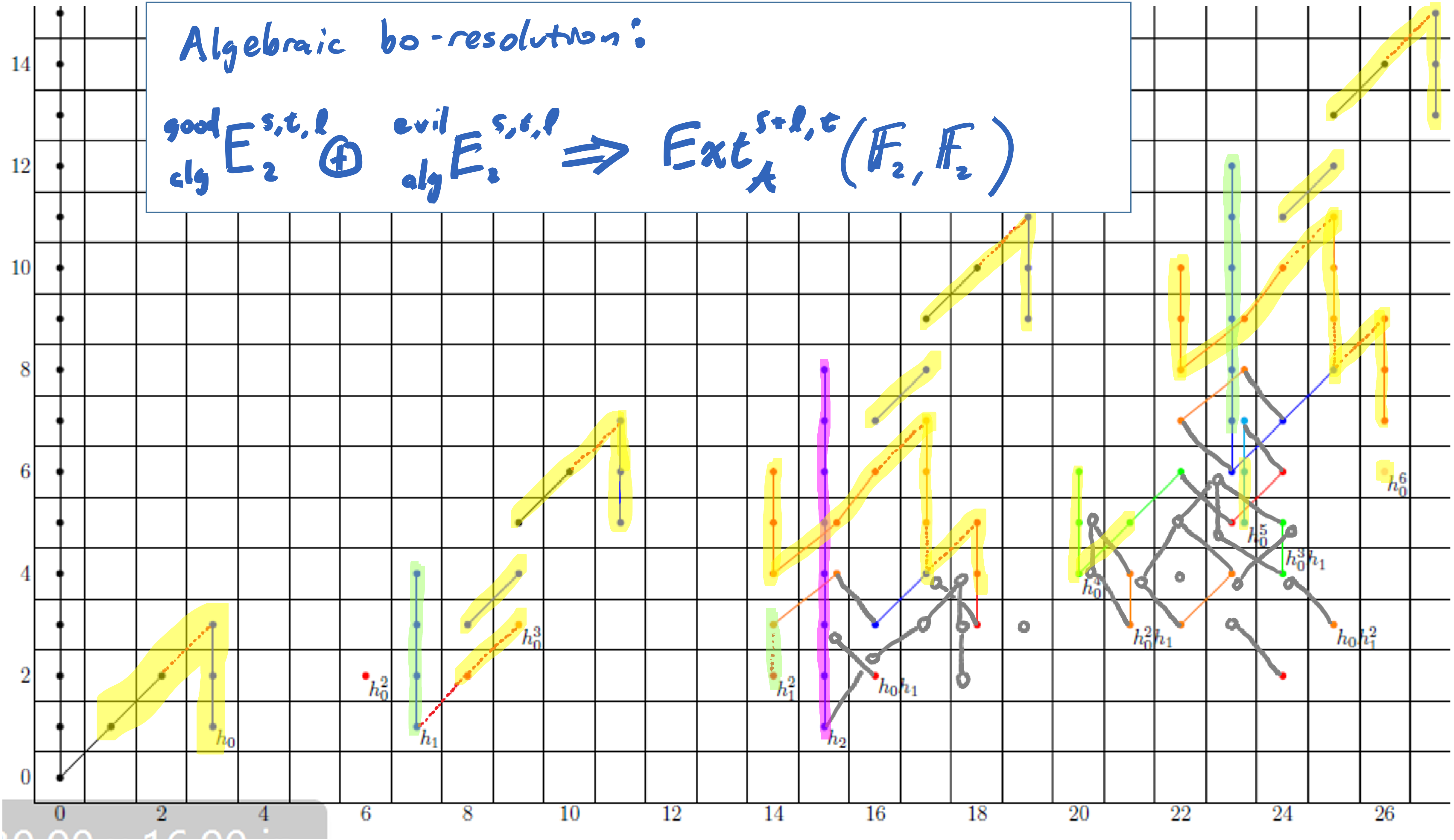
$$ v_1^{-1}\, {}^{bo}_{alg}E_2^{\cdots} \xrightarrow{\cong} v_1^{-1}\, {}^{good}_{alg}E_2^{\cdots} $$
$$ \Downarrow $$
$$ v_1^{-1} \mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2) $$



v_1-periodicity in $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$
0
5
10
15
20
25

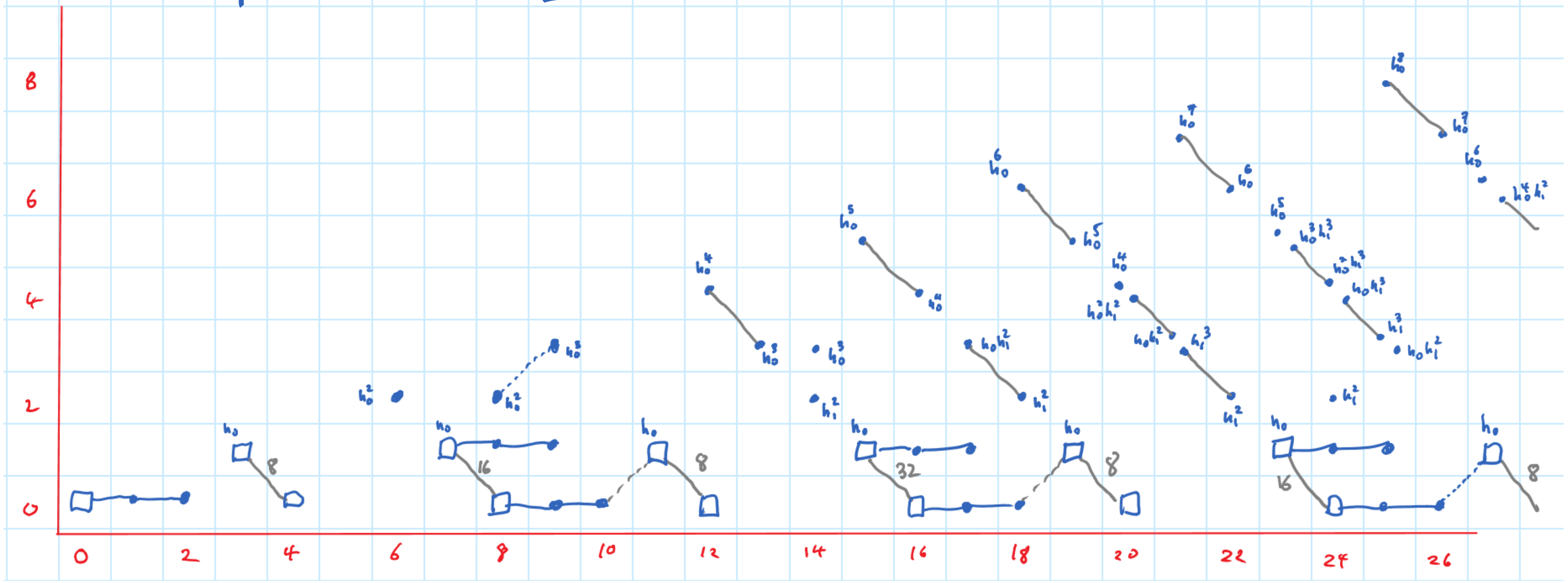
Algebraic bo-resolution:

$$ {}^{good}_{alg}E_2^{s,t,\ell} \oplus {}^{evil}_{alg}E_2^{s,t,\ell} \Rightarrow \mathrm{Ext}_A^{s+\ell,t}(\mathbb{F}_2,\mathbb{F}_2) $$

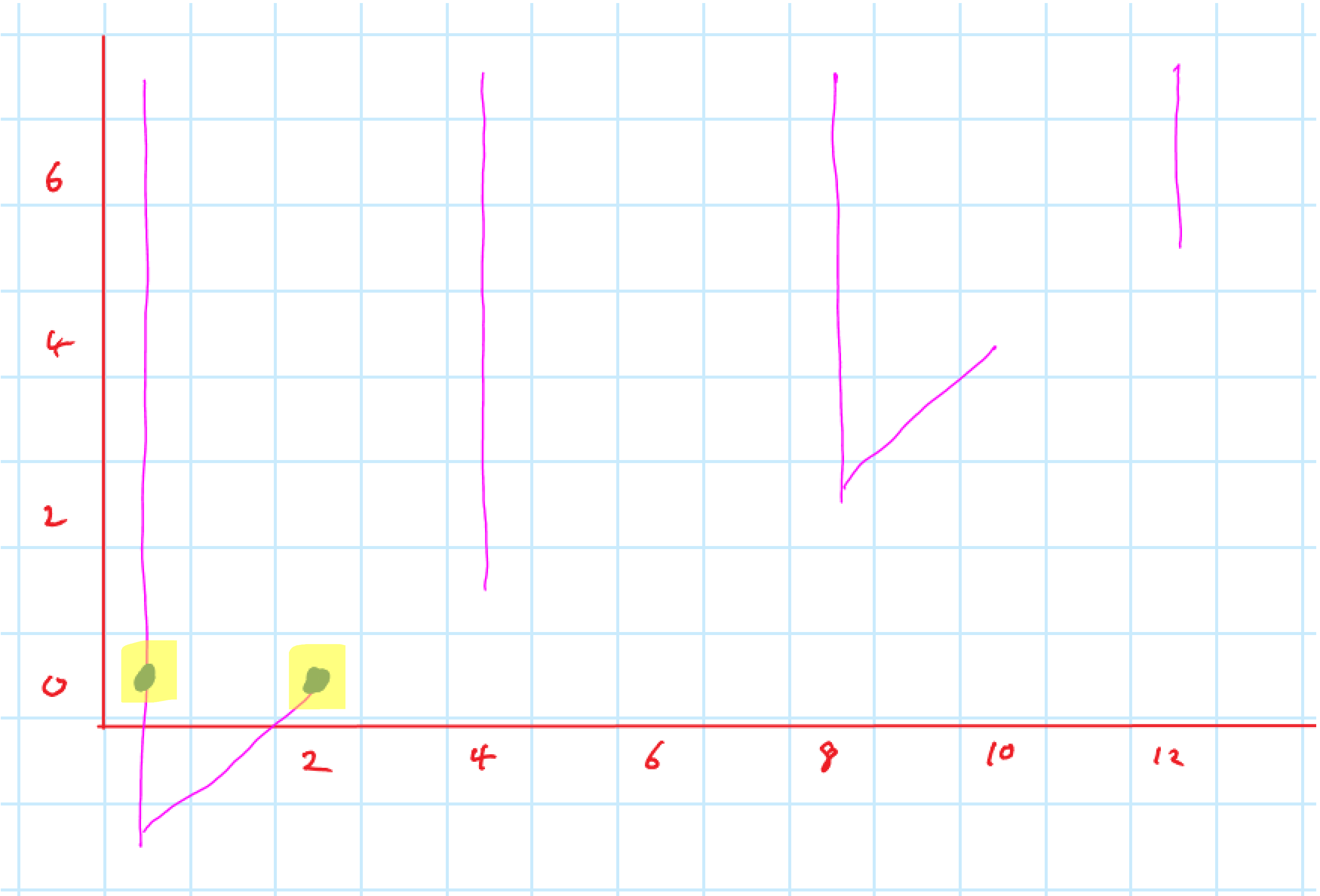


Topological Weight Spectral Sequence

$${}^{wss}E_1^{s,t,wt} \Rightarrow {}^{gaad}E_2^{s,t}$$

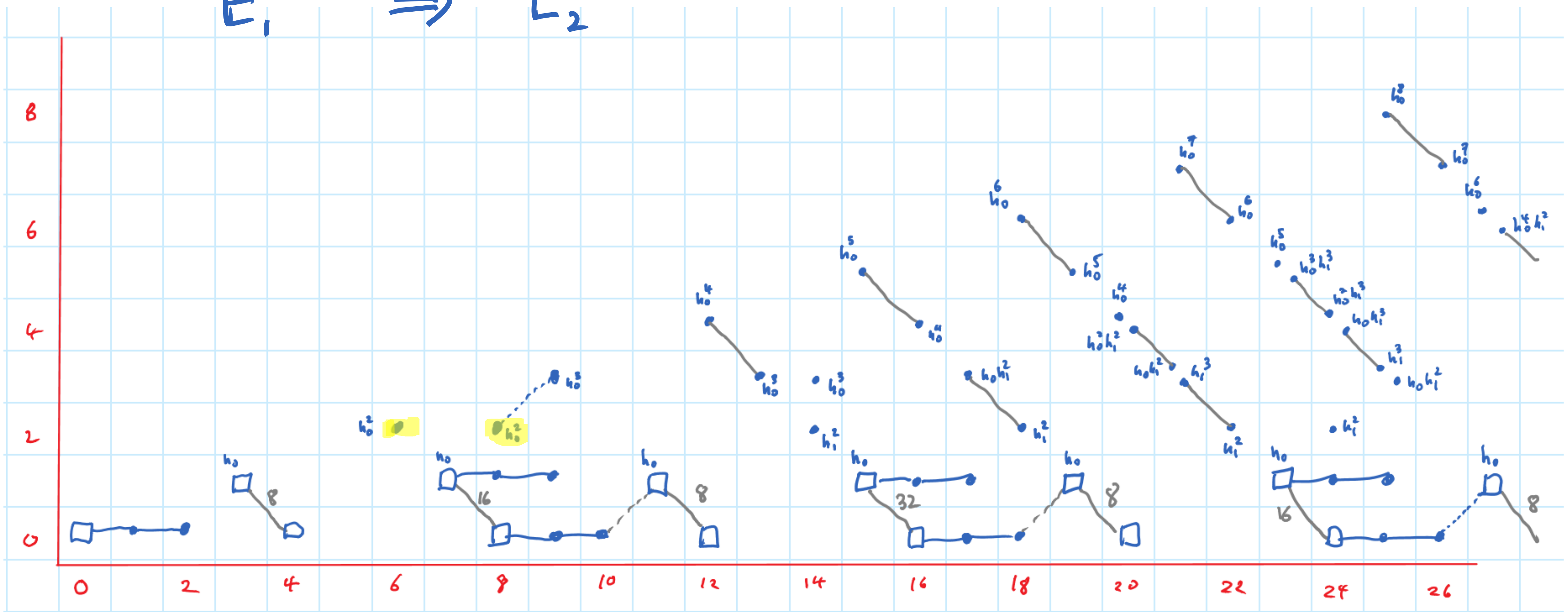


$$h_0^2 = b_0^{\langle 2 \rangle}$$

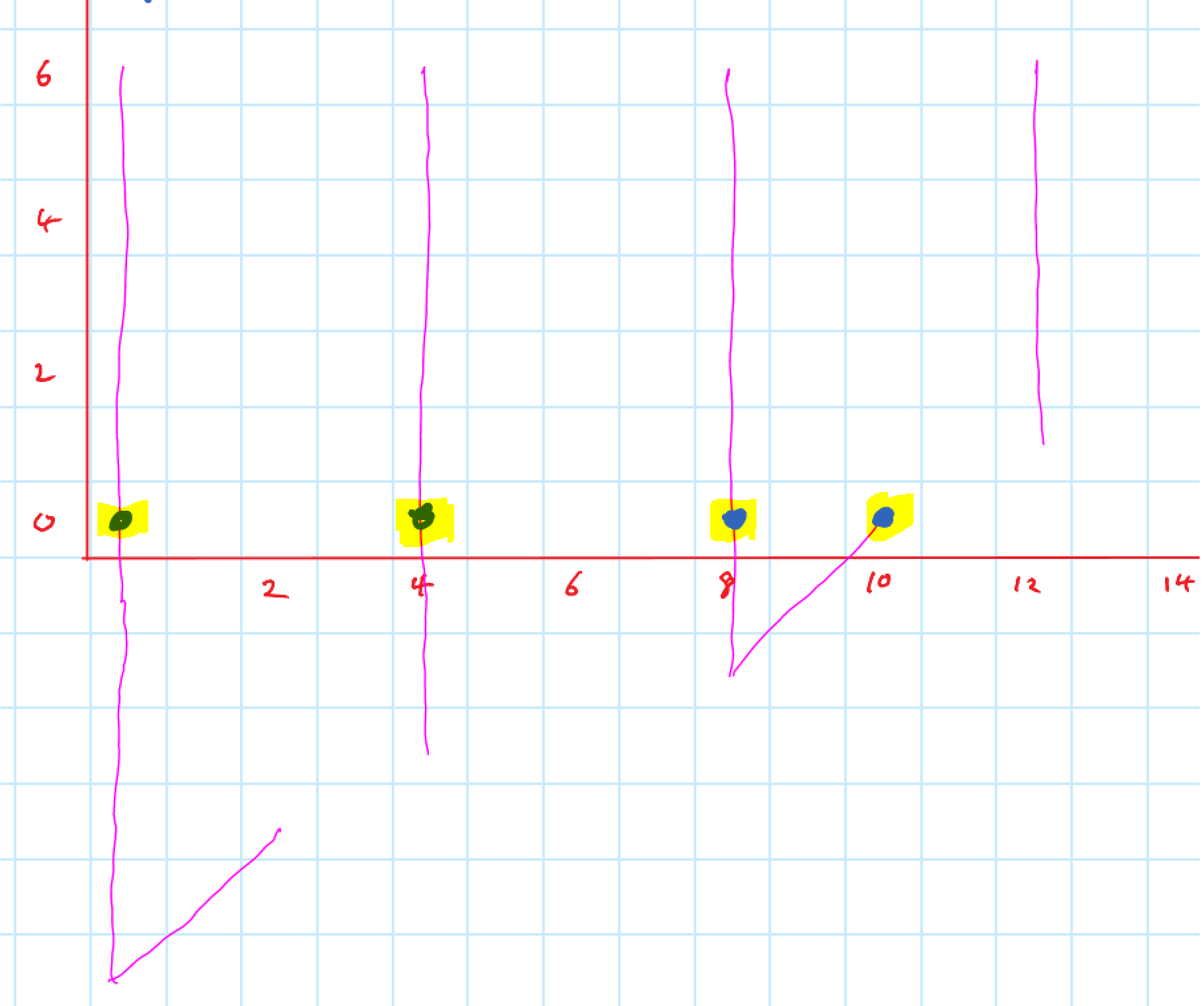


Topological Weight Spectral Sequence

$$ {}^{wss}E_1^{s,t,wt} \Longrightarrow {}^{gaod}E_2^{s,t} $$

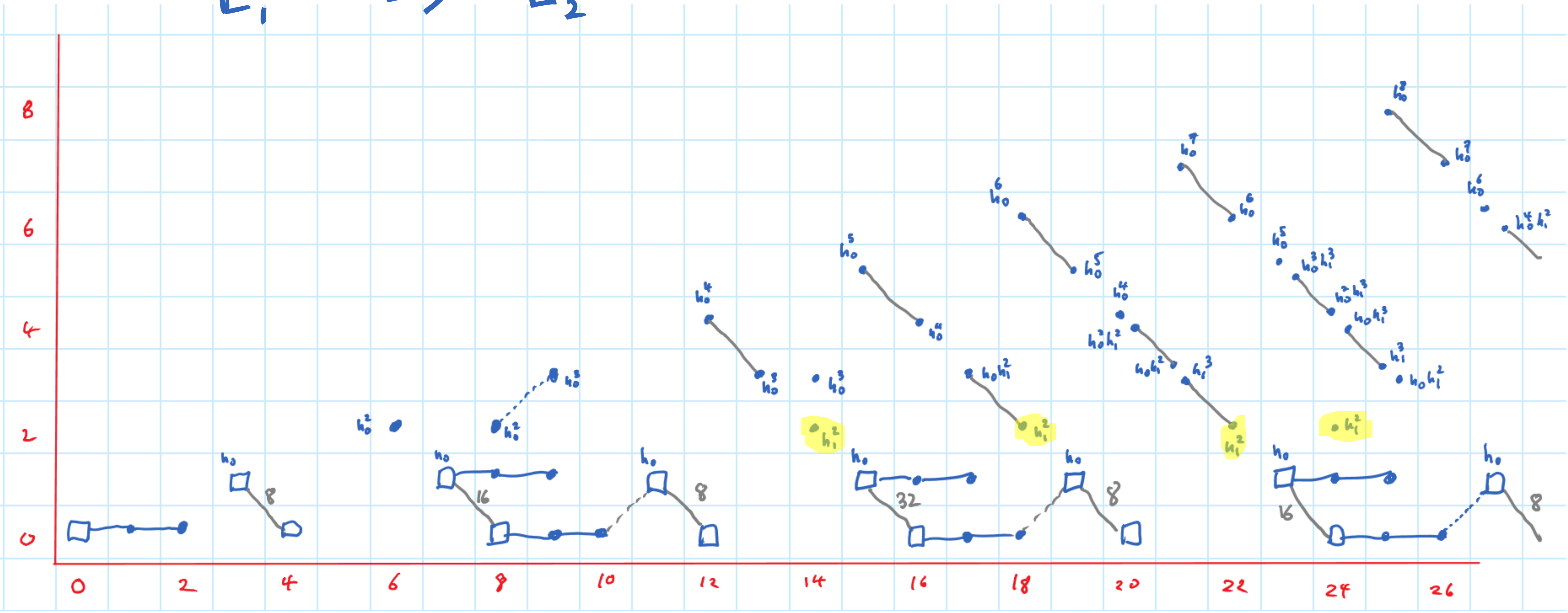


$h_1^2 = b_0^{<6>}$



Topological Weight Spectral Sequence

$$^{wss}E_1^{s,t,wt} \Rightarrow {}^{gad}E_2^{s,t}$$



bo-Adams spectral sequence

$$^{good}E_2^{s,t} \oplus {}^{evil}E_2^{s,t} \Rightarrow \pi_{t-s} S$$