Wrapping spheres around spheres

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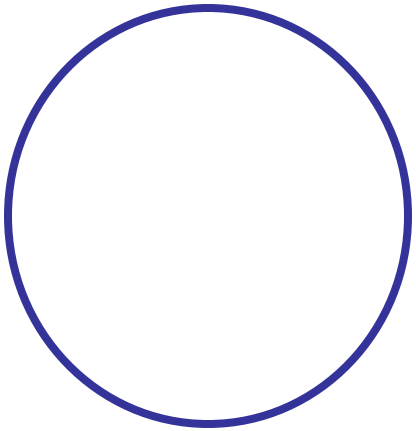
Spheres??

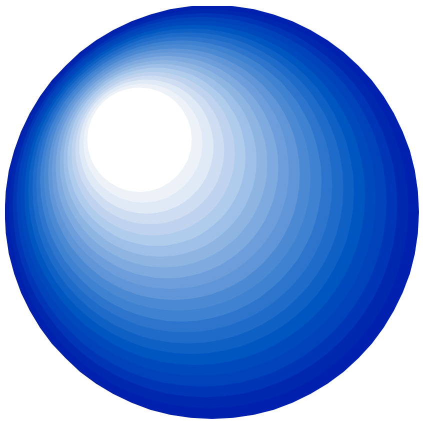
S^1



1-dimensional sphere

(Circle)

S^2



2-dimensional sphere

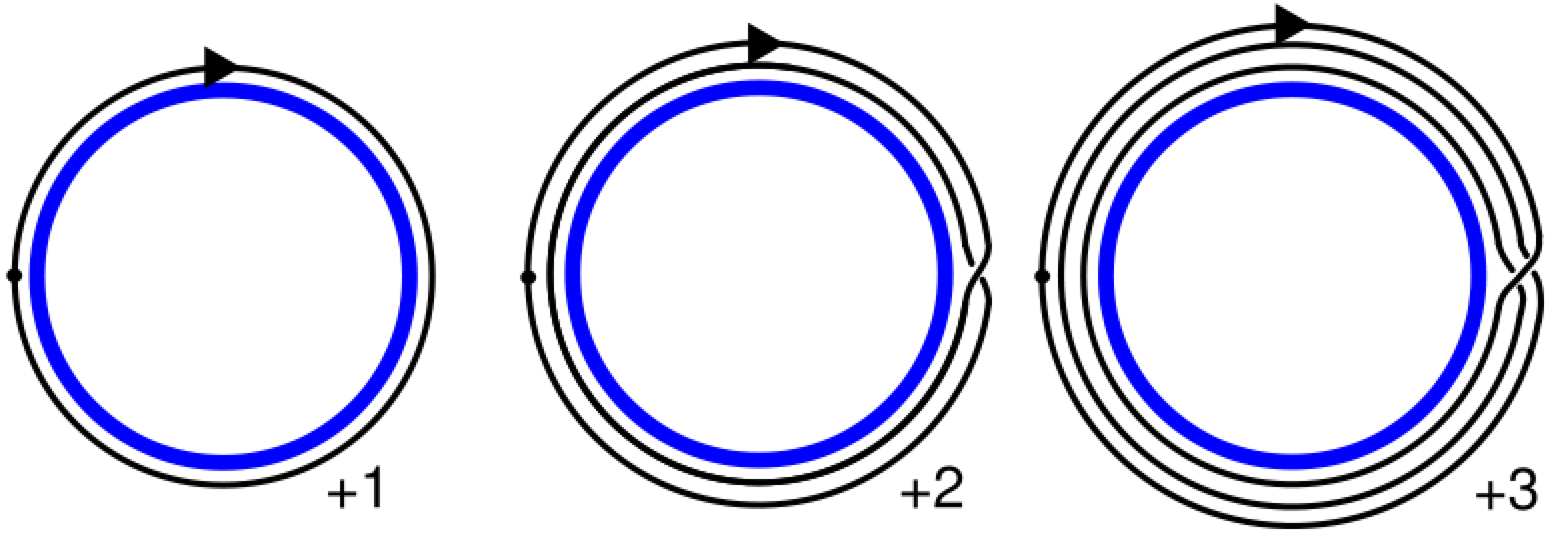
(Sphere)

S^n $(n > 2)$

3-dimensional sphere and higher…

(I'll explain later!)

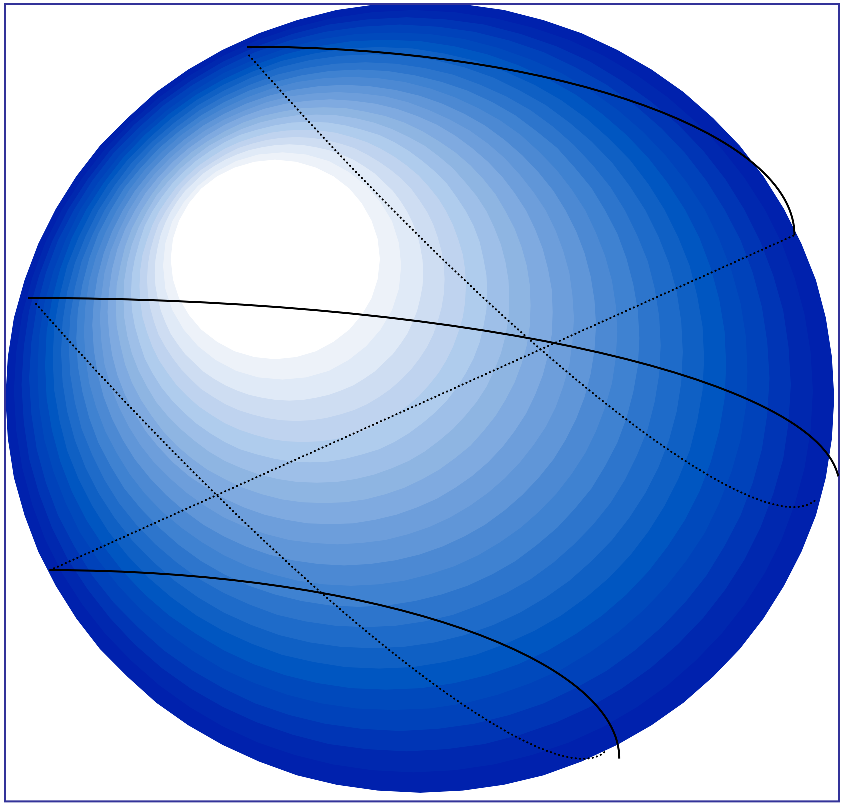
Wrapping spheres?



Wrapping S^1 around S^1

- Wrapping one circle around another circle
- Wrapping rubber band around your finger

…another example…



Wrapping S^1 around S^2

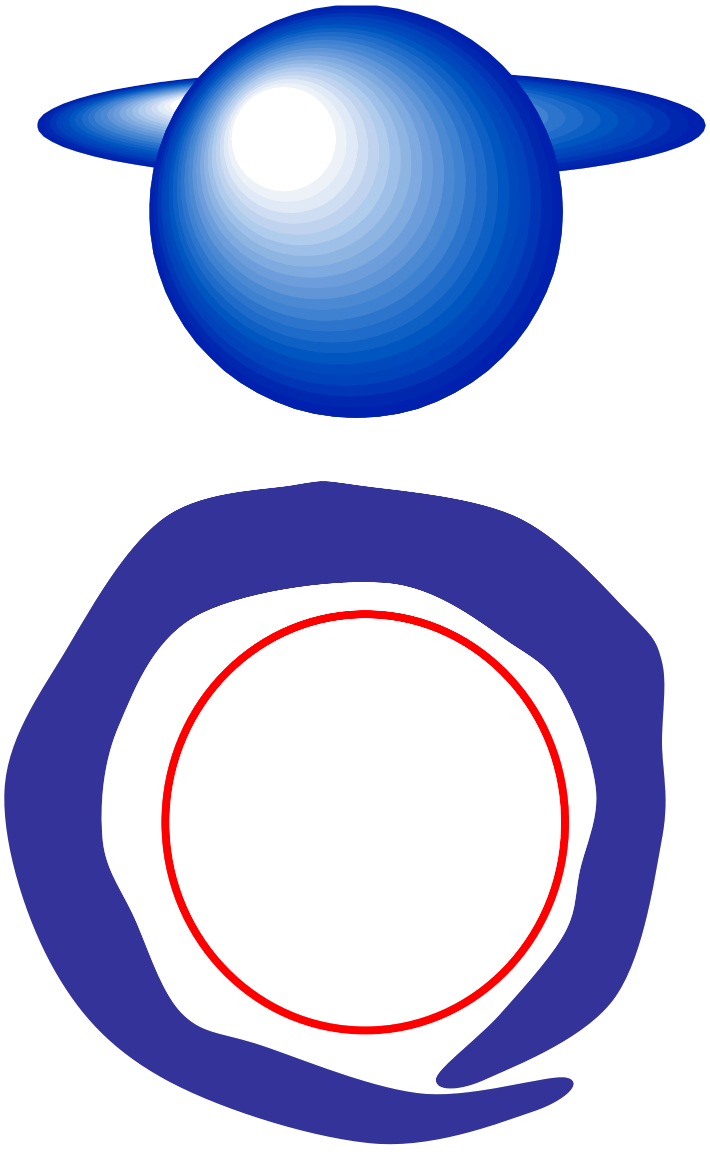
-Wrapping circle around sphere
-Wrapping rubber band around globe

…and another example.

Wrapping S^2 around S^1

- Wrapping sphere around circle

- Flatten balloon, stretch around circle



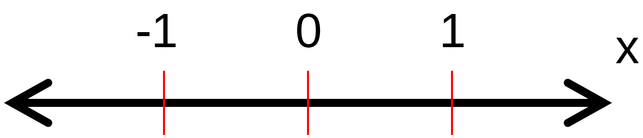
Goal: Understand all of the ways to wrap S^k around S^n !

- n and k are positive numbers
- Classifying the ways you can wrap is VERY HARD!
- Turns out that interesting patterns emerge as n and k vary.
- We'd like to do this for not just spheres, but for other geometric objects – spheres are just the simplest!

Plan of talk

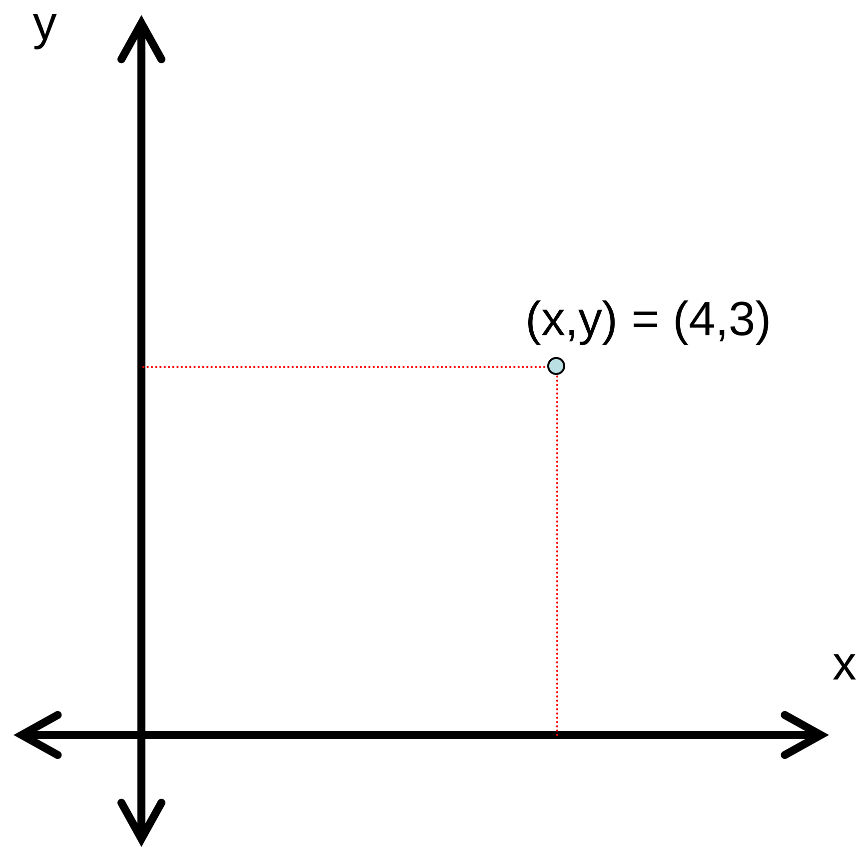
- Explain what I mean by “higher dimensional spheres”
- Work out specific low-dimensional examples
- Present data for what is known
- Investigate number patterns in this data

n-dimensional space



1-dimensional space:
The real line

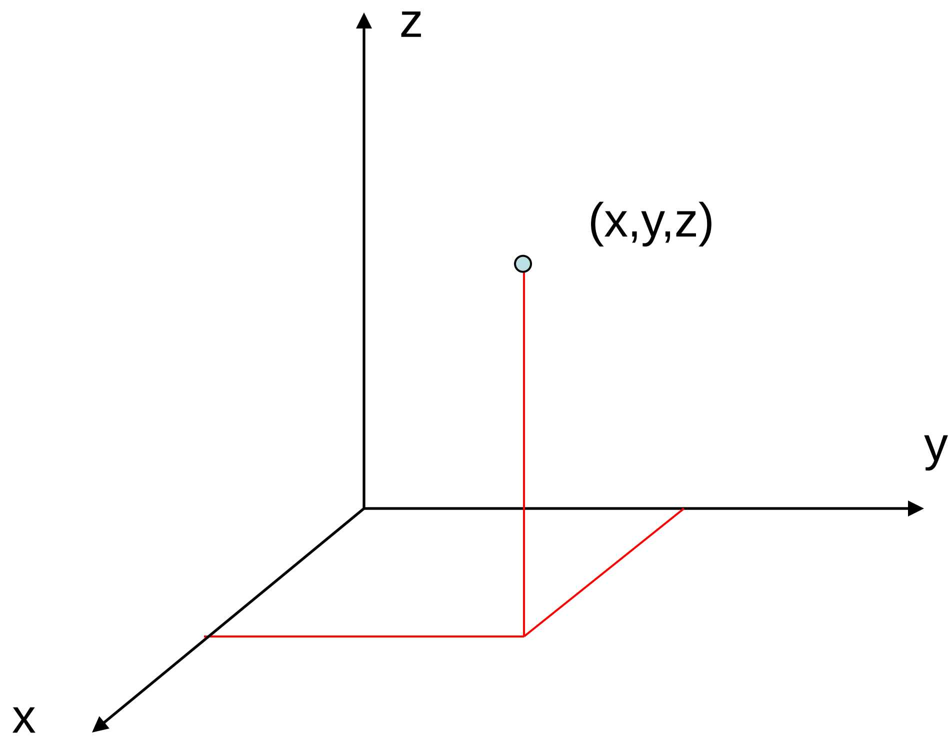
To specify a point,
give 1 number



2-dimensional space:
The Cartesian plane

To specify a point,
give 2 numbers

3-dimensional space



-The world we live in
- To specify a point, give 3 numbers (x,y,z).

Higher dimensional space

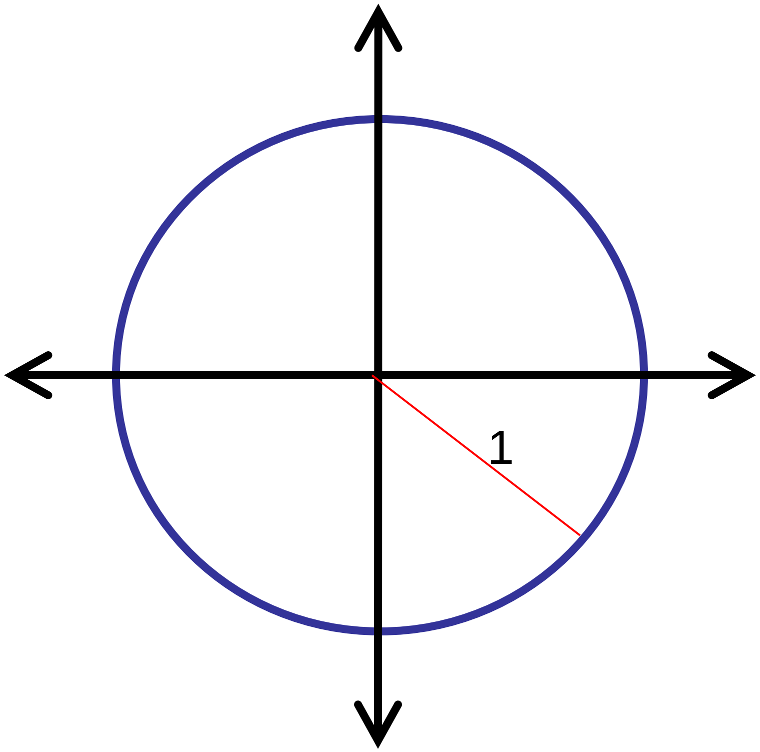
- Points in 4-dimensional space are specified with 4 numbers (x,y,z,w)

- Points in n-dimensional space are specified with n numbers:

$$(x_1, x_2, x_3, \ldots\ldots, x_n)$$

Higher dimensional spheres:

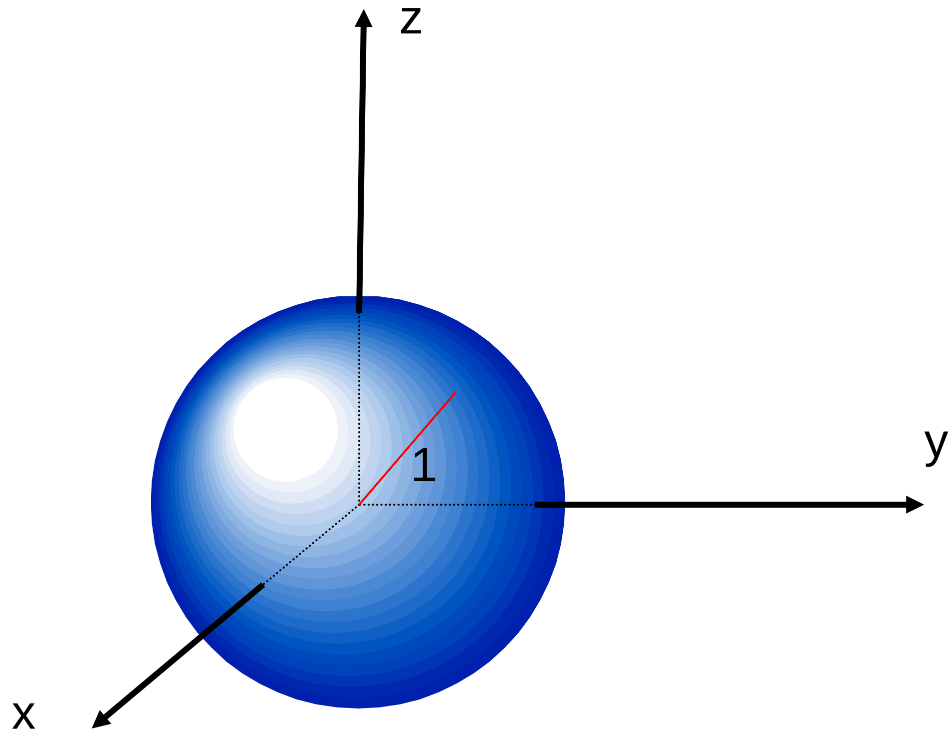
The circle S^1 is the collection of all points (x,y) in 2-dimensional space of distance 1 from the origin (0,0).



$$\sqrt{x^2 + y^2} = 1$$

Higher dimensional spheres:

The sphere S^2 is the collection of all points (x,y,z) in 3-dimensional space of distance 1 from the origin (0,0,0).



$$\sqrt{x^2 + y^2 + z^2} = 1$$

Higher dimensional spheres:

S^3 is the collection of all points (x,y,z,w) in 4-dimensional space of distance 1 from the origin (0,0,0,0).

$$\sqrt{x^2 + y^2 + z^2 + w^2} = 1$$

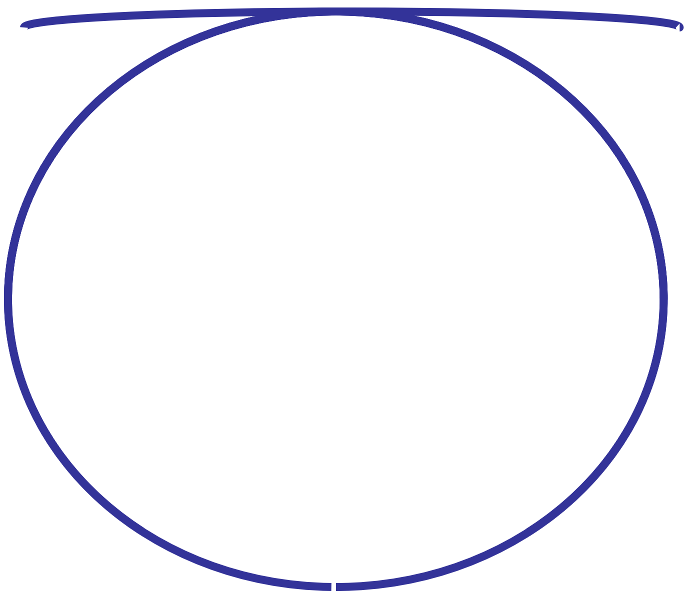
S^{n-1} is the collection of all points $(x_1,\ldots,x_n)$ in n-dimensional space of distance 1 from the origin.

$$\sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2 + \cdots + x_n{}^2} = 1$$

Spheres: another approach

(This will help us visualize S^3)

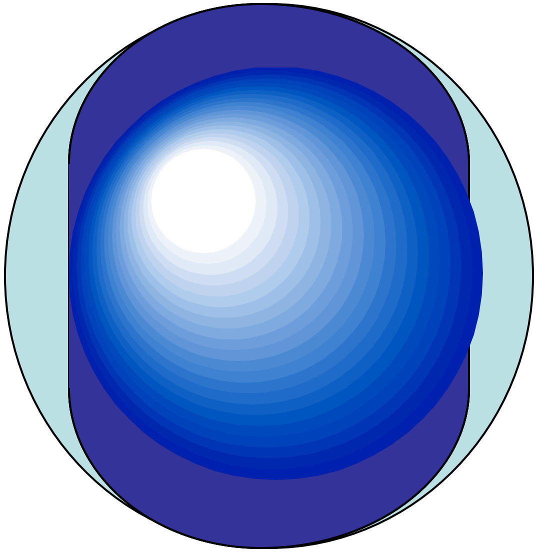
S^1 is obtained by taking a line segment and gluing the ends together:



Spheres: another approach

(This will help us visualize S^3)

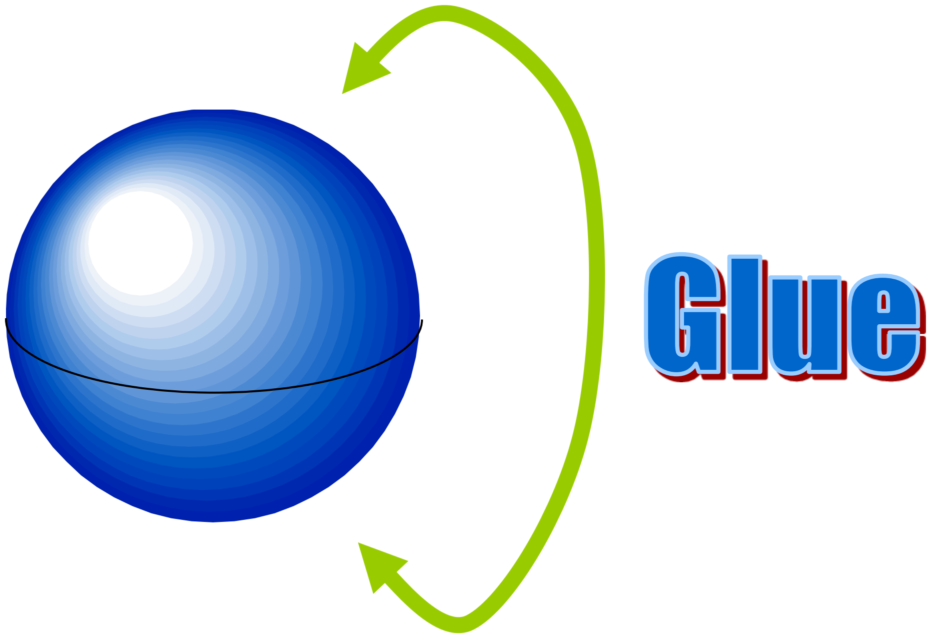
S^2 is obtained by taking a disk and gluing the opposite sides together:



Spheres: another approach

(This will help us visualize S^3)

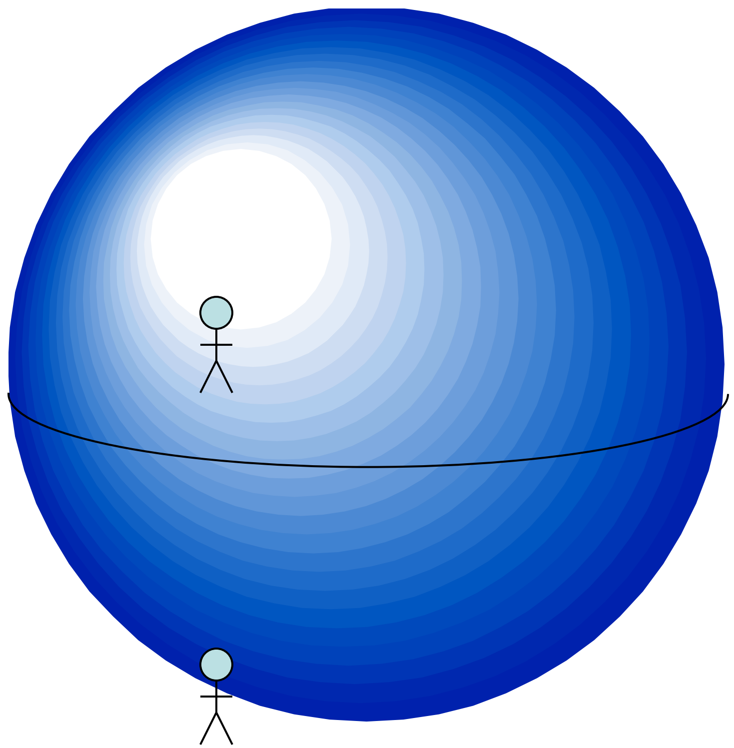
S^3 is obtained by taking a solid ball and gluing the opposite hemispheres together:



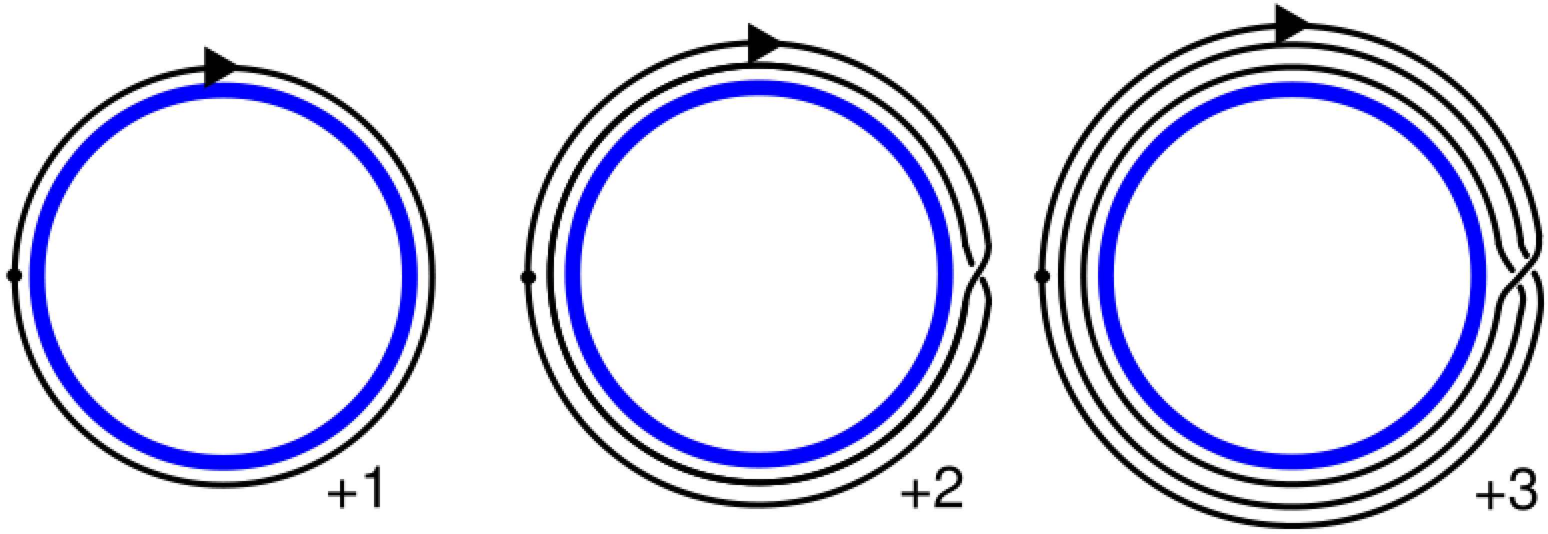
Spheres: another approach

(This will help us visualize S^3)

You can think of S^3 this way: If you are flying around in S^3, and fly through the surface in the northern hemisphere, you reemerge in the southern hemisphere.

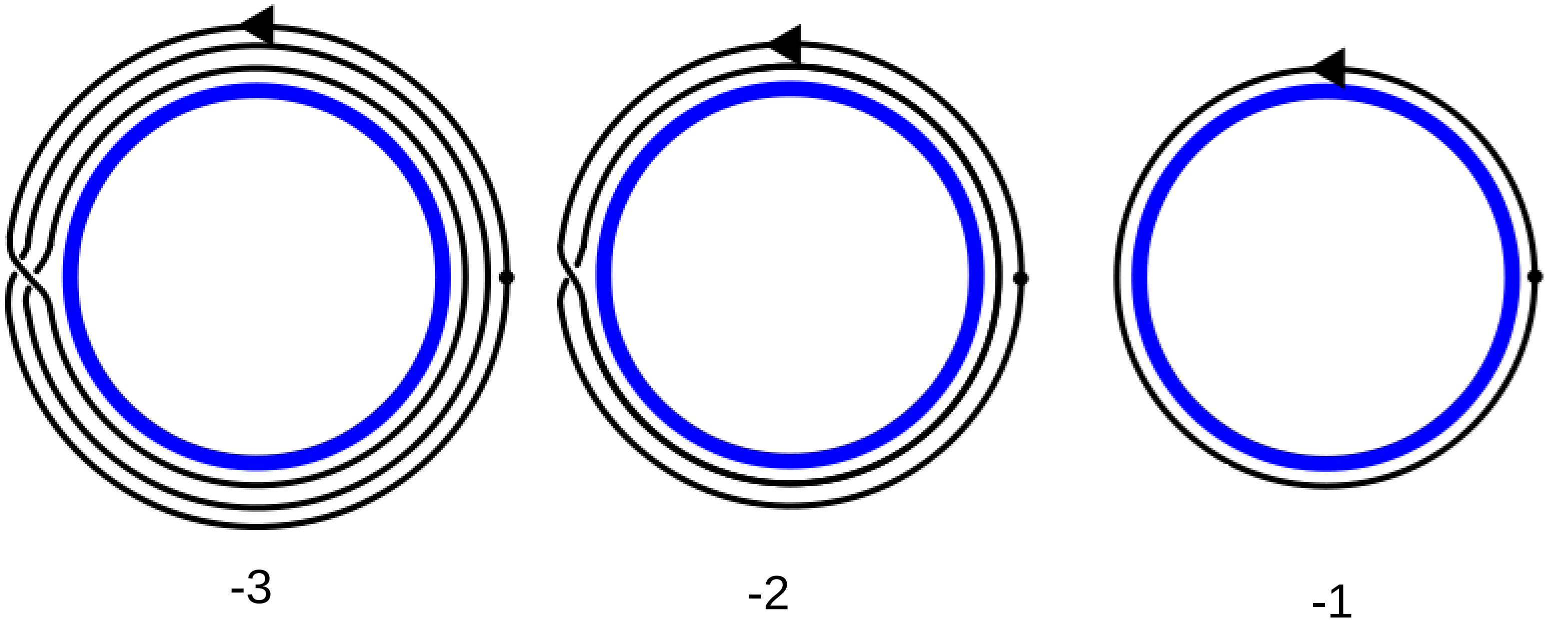


Wrapping S^1 around S^1



For each positive integer n, we can wrap the circle around the circle n times

Wrapping S^1 around S^1

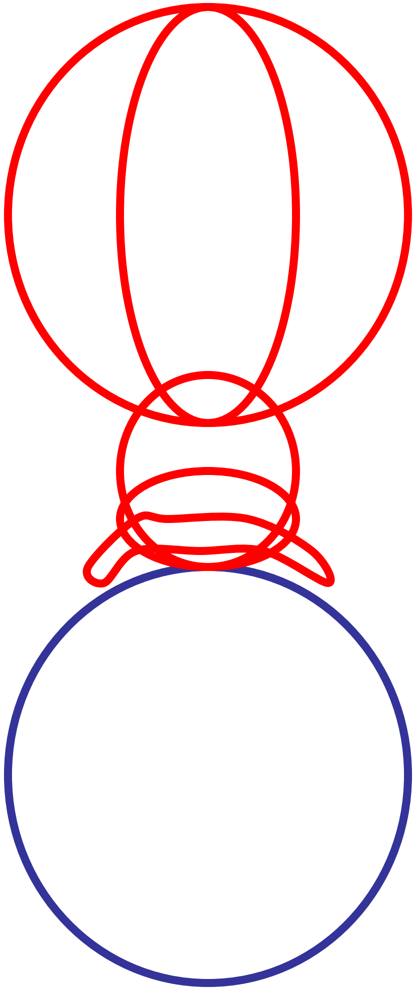


We can wrap counterclockwise to get the negative numbers

The unwrap

A trivial example: just drop the circle onto the circle.

The unwrap wraps 0 times around



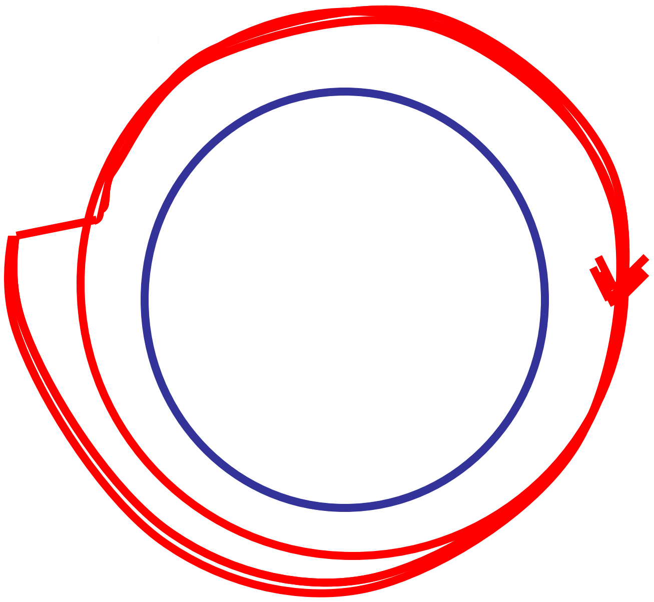
Equivalent wrappings

We say that two wrappings are <u>equivalent</u> if one can be adjusted to give the other

For example:

This wrapping
is equivalent to…

…this wrapping.
(the "wrap 1")



Winding number

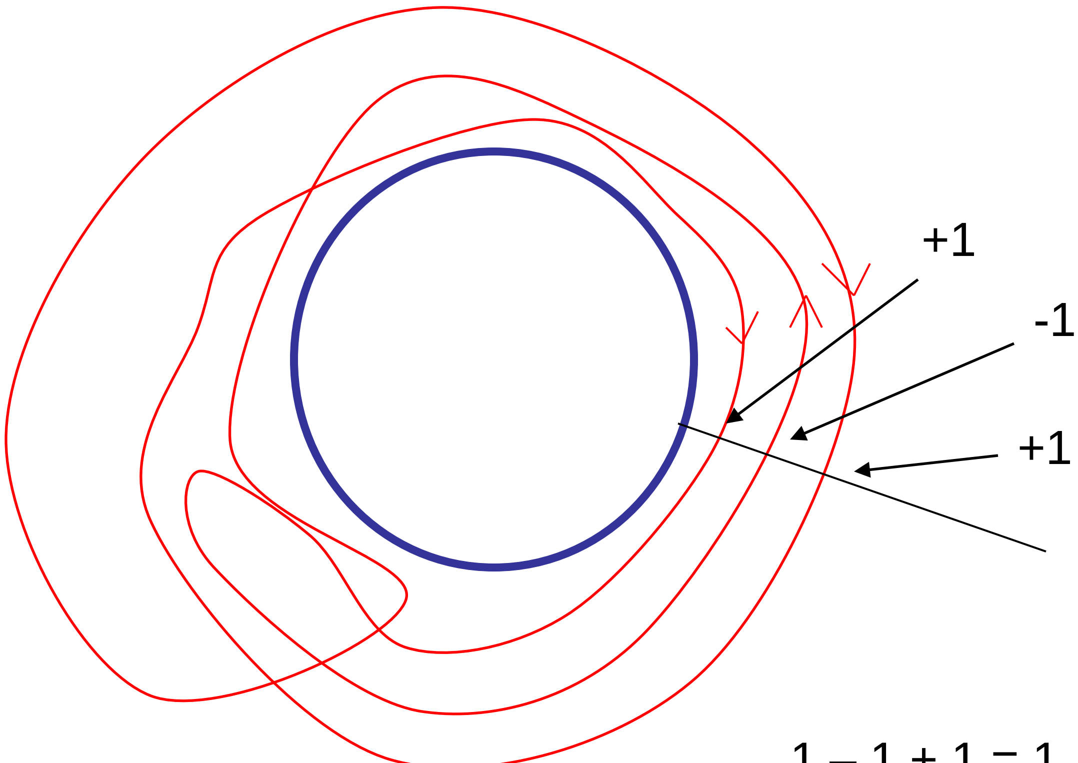
Every wrapping of S^1 by S^1 is equivalent to "wrap n" for some integer n. Which wrap is this equivalent to?

Handy trick:

1) Draw a line perpendicular to S^1

2) Mark each intersection point with + or – depending on direction of crossing

3) Add up the numbers – this is the "winding number"



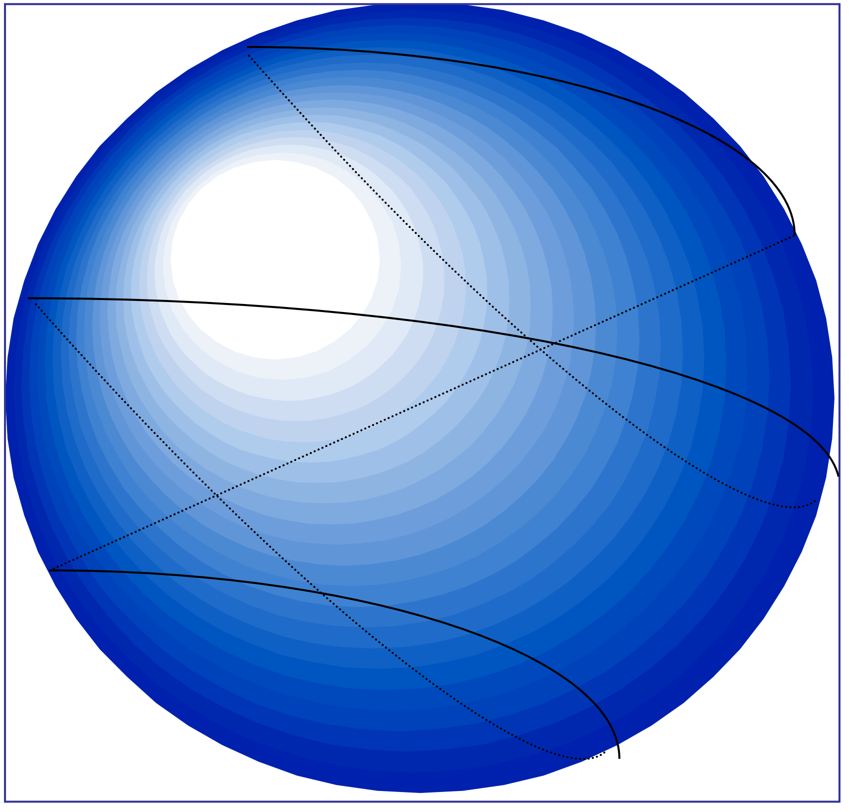
1 – 1 + 1 = 1

What have we learned:

The winding number gives a correspondence:

(Ways to wrap S^1 around S^1) ⟷ (The integers: …-2, -1, 0, 1, 2, …)

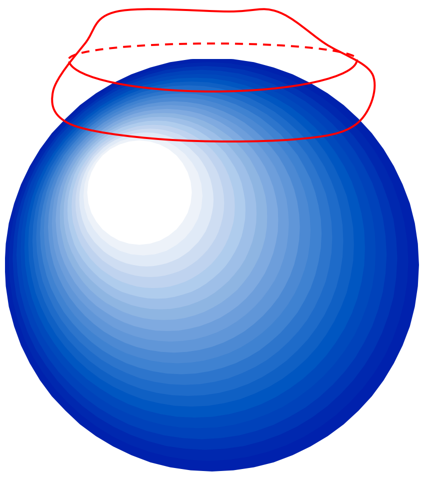
Wrapping S^1 around S^2



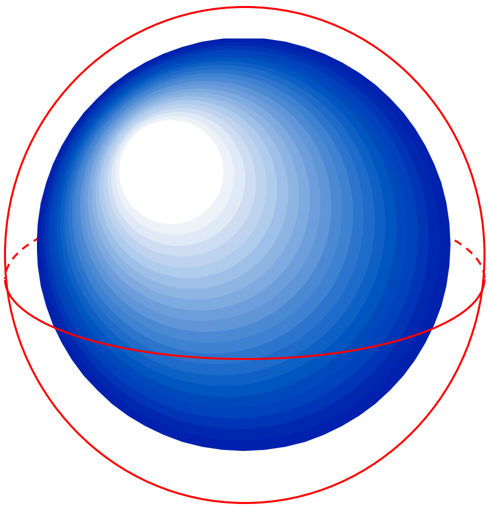
What have we learned:

- Every way of wrapping S^1 around S^2 is equivalent to the “unwrap”
- FACT: the same is true for wrapping any sphere around a *larger* dimensional sphere.
- REASON: there will always be some place of the larger sphere which is uncovered, from which you can “push the wrapping off”.

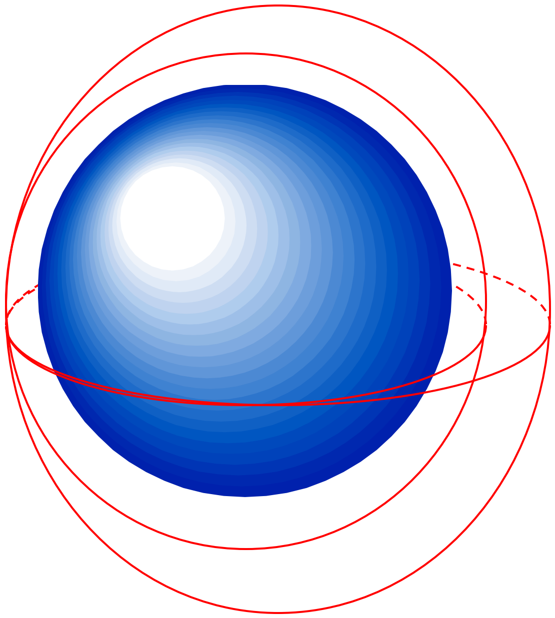
Wrapping S^2 around S^2:



Wrap 0



Wrap 1



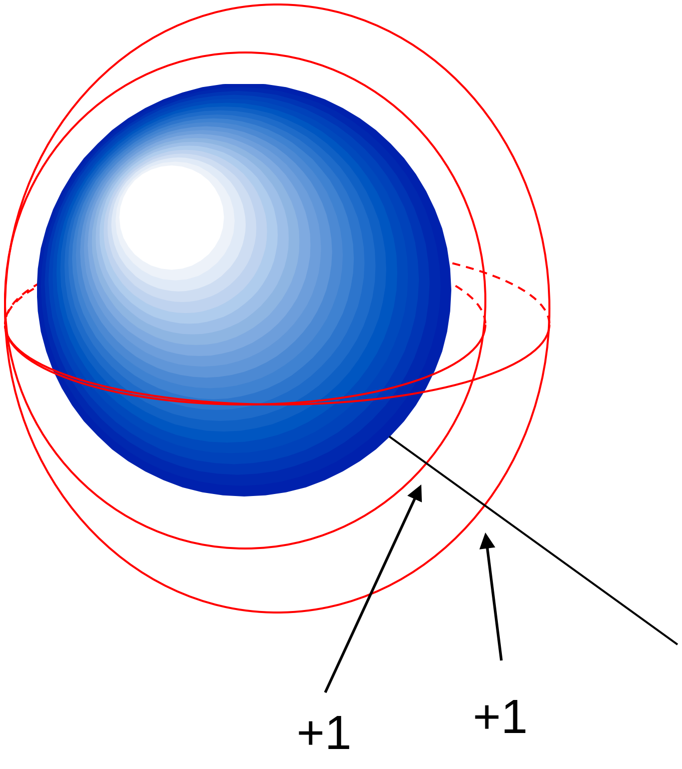
Wrap 2

(Get negative wraps by turning sphere inside out)

“Winding number”

Same trick for S^1 works for S^2
for computing the “winding number”

Winding number = 1 + 1 = 2



Fact:

The winding number gives a correspondence:

(Ways to wrap S^2 around S^2) ⟷ (The integers: …-2, -1, 0, 1, 2, …)

General Fact!

The winding number gives a correspondence:

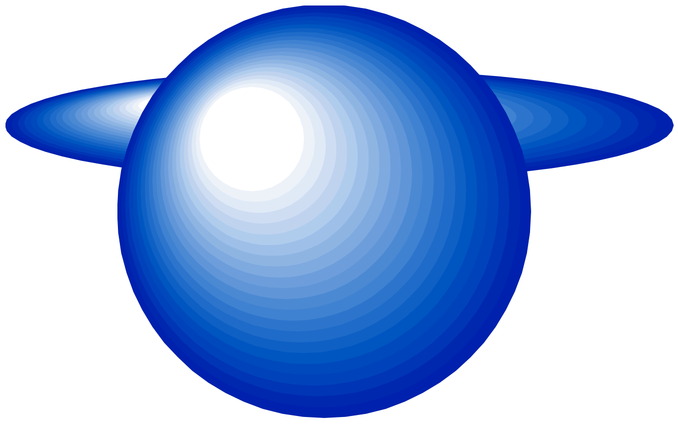
Ways to wrap S^n around S^n ⟷ The integers: …-2, -1, 0, 1, 2, …

Summary:

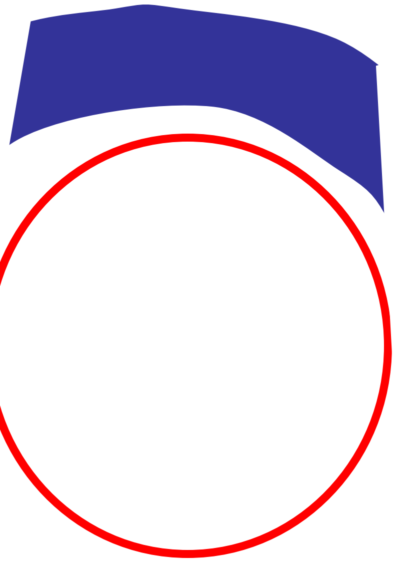
Ways to wrap S^n around S^n	⟷	The integers: …-2, -1, 0, 1, 2, …
Ways to wrap S^k around S^n $k < n$	⟷	Only the unwrap
Ways to wrap S^k around S^n $k > n$	⟷	**???**

Wrapping S^2 around S^1:

Consider the example given earlier:



In fact, this wrap is equivalent to
The unwrap, because you can
“shrink the balloon”



What have we learned:

This sort of thing always happens, and we have:

$\left(\text{Ways to wrap } S^2 \text{ around } S^1 \right) \longleftrightarrow \left(\text{Only the unwrap} \right)$

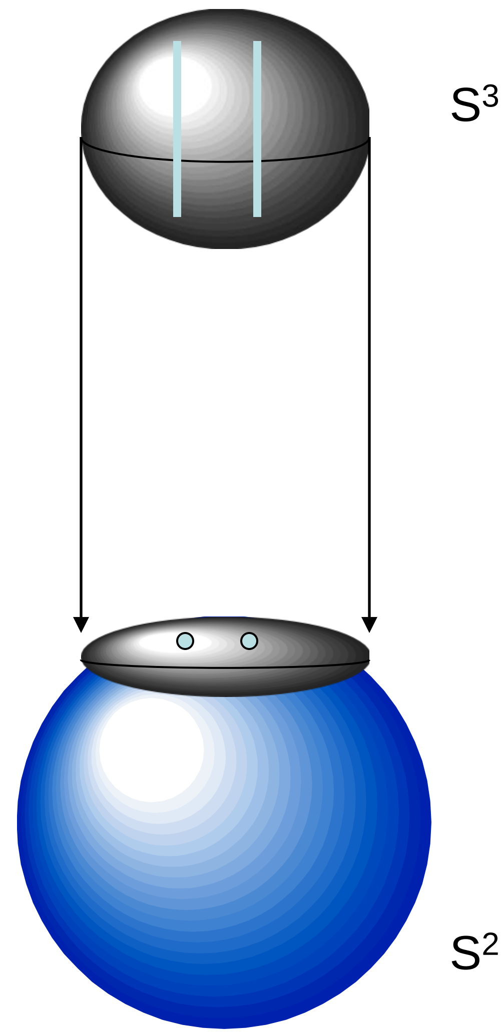
Turns out that this is just a fluke!

There are many interesting ways to
wrap S^{n+k} around S^n for $n > 1$, and $k > 0$.

Wrapping S^3 around S^2:

Recall: we are thinking of S^3 as a solid ball with the northern hemisphere glued to the southern hemisphere.

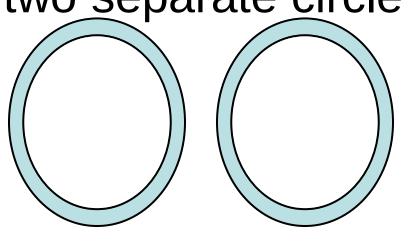
Consider the unwrap:



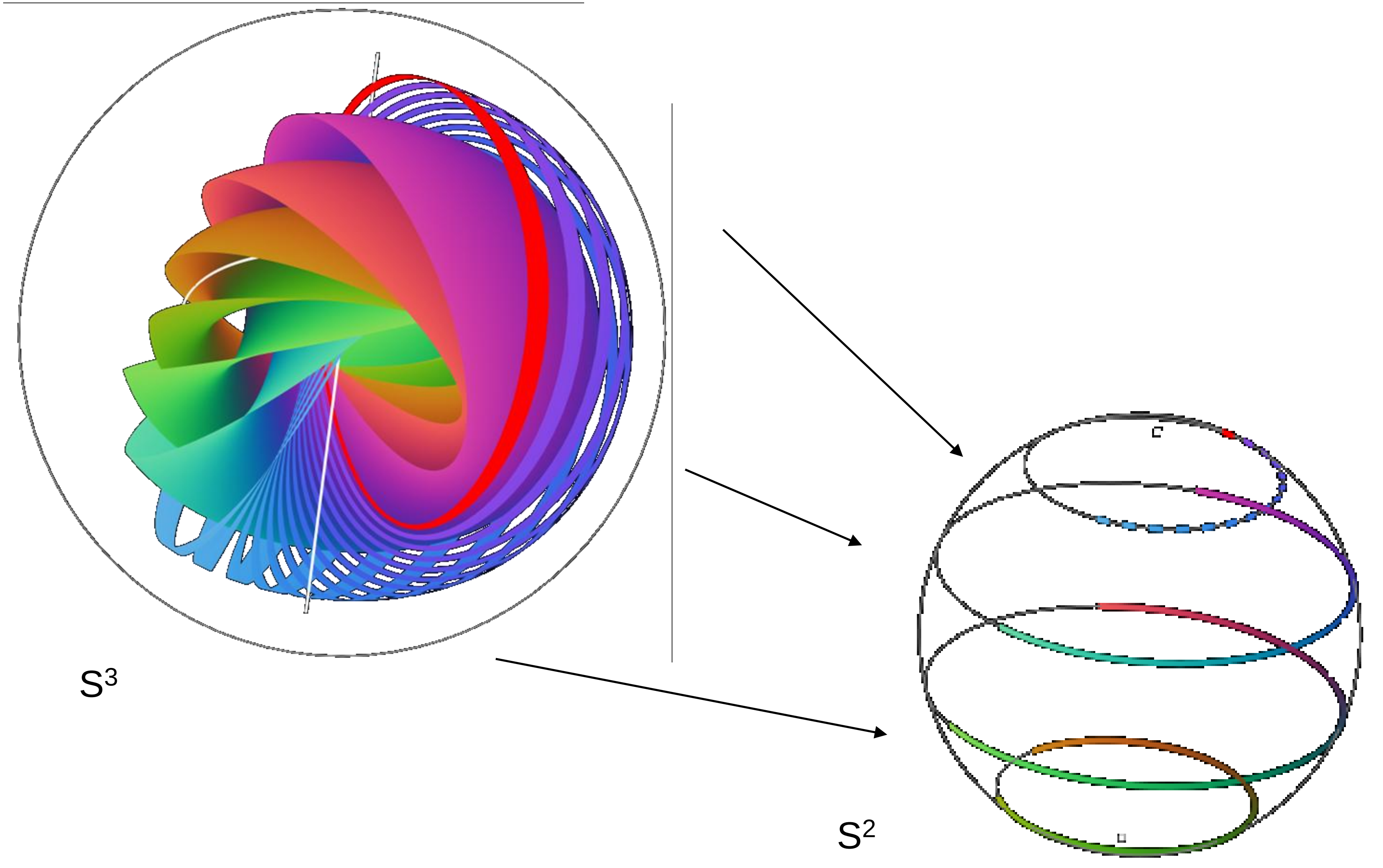
1) Take two points in S^2

2) Examine all points in S^3 that get sent to these two points.

3) Because the top and bottom are identified, these give two separate circles in S^3.

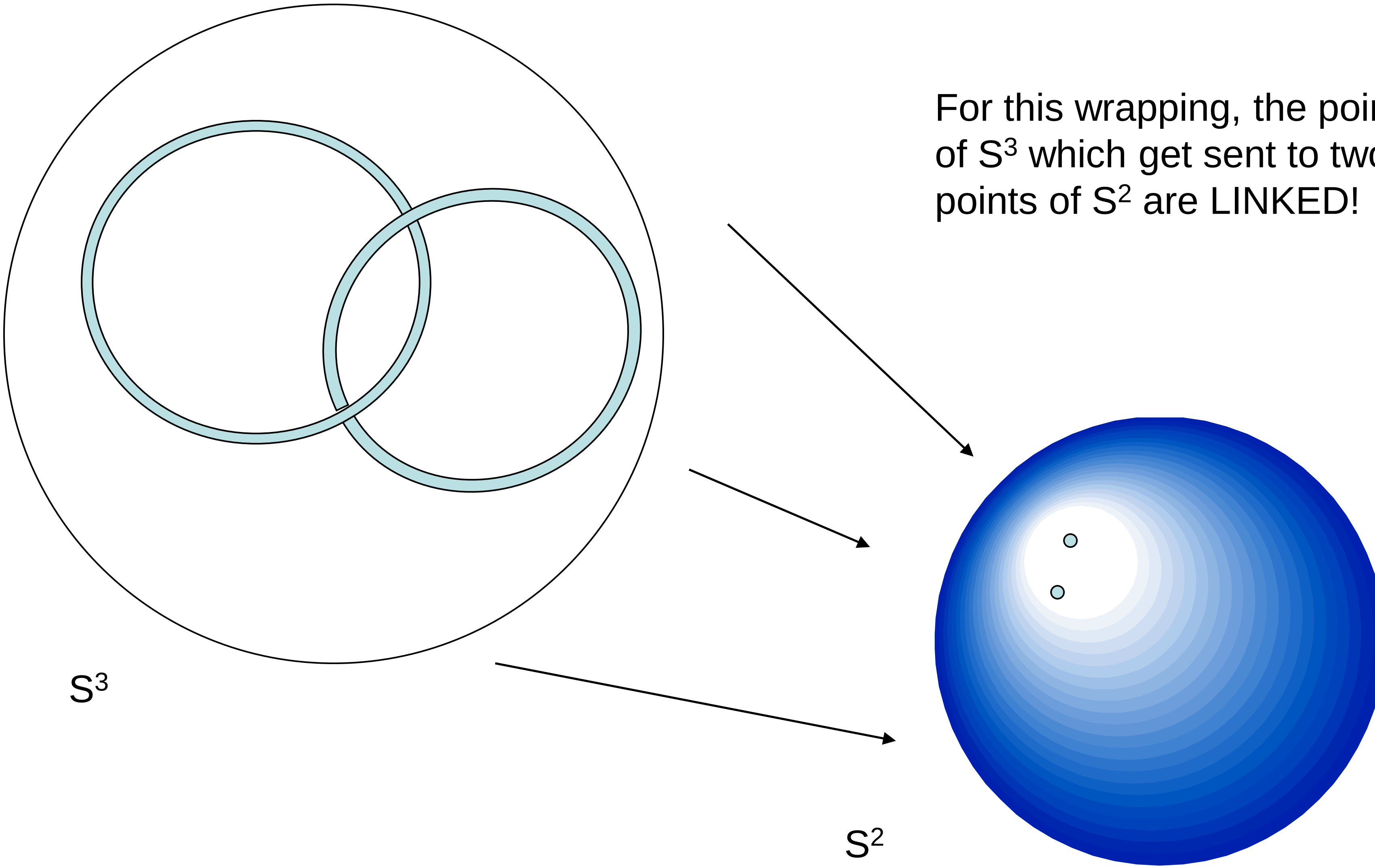


Hopf fibration: a way to wrap S^3 around S^2 different from the unwrap



Hopf fibration: a way to wrap S^3 around S^2 different from the unwrap

For this wrapping, the points of S^3 which get sent to two points of S^2 are LINKED!



Keyring model of Hopf fibration

Fact:

Counting the number of times these circles are linked gives a correspondence:

Ways to wrap S^3 around S^2	⟷	The integers: …-2, -1, 0, 1, 2, …

Number of ways to wrap S^{n+k} around S^n

	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10	n=11
k=1	Z	2	2	2	2	2	2	2	2	2
k=2	2	2	2	2	2	2	2	2	2	2
k=3	2	12	Z*12	24	24	24	24	24	24	24
k=4	12	2	2^2	2	0	0	0	0	0	0
k=5	2	2	2^2	2	Z	0	0	0	0	0
k=6	2	3	24*3	2	2	2	2	2	2	2
k=7	3	15	15	30	60	120	Z*120	240	240	240
k=8	15	2	2	2	8*6	2^3	2^4	2^3	2^2	2^2
k=9	2	2^2	2^3	2^3	2^3	2^4	2^5	2^4	$Z*2^3$	2^3
k=10	2^2	12*2	$40*4*2*3^2$	18*8	18*8	24*2	8^2*2*3^2	24*2	12*2	2^2*3
k=11	12*2	$84*2^2$	$84*2^5$	$504*2^2$	504*4	504*2	504*2	504*2	504	504

Note: “Z” means the integers

Some of the numbers are factored to indicate that there are distinct ways of wrapping

Number of ways to wrap S^{n+k} around S^n

	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10	n=11
k=1	Z	2	2	2	2	2	2	2	2	2
k=2	2	2	2	2	2	2	2	2	2	2
k=3	2	12	Z*12	24	24	24	24	24	24	24
k=4	12	2	2^2	2	0	0	0	0	0	0
k=5	2	2	2^2	2	Z	0	0	0	0	0
k=6	2	3	24*3	2	2	2	2	2	2	2
k=7	3	15	15	30	60	120	Z*120	240	240	240
k=8	15	2	2	2	8*6	2^3	2^4	2^3	2^2	2^2
k=9	2	2^2	2^3	2^3	2^3	2^4	2^5	2^4	Z*2^3	2^3
k=10	2^2	12*2	40*4*2*3^2	18*8	18*8	24*2	8^2*2*3^2	24*2	12*2	2^2*3
k=11	12*2	84*2^2	84*2^5	504*2^2	504*4	504*2	504*2	504*2	504	504

The integers form an infinite set – the only copies of the integers are shown in red. This pattern continues. All of the other numbers are finite!

Number of ways to wrap S^{n+k} around S^n

	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10	n=11
k=1	Z	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**
k=2	2	2	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**
k=3	2	12	Z*12	**24**	**24**	**24**	**24**	**24**	**24**	**24**
k=4	12	2	2^2	2	**0**	**0**	**0**	**0**	**0**	**0**
k=5	2	2	2^2	2	Z	**0**	**0**	**0**	**0**	**0**
k=6	2	3	24*3	2	2	2	**2**	**2**	**2**	**2**
k=7	3	15	15	30	60	120	Z*120	**240**	**240**	**240**
k=8	15	2	2	2	8*6	2^3	2^4	2^3	**2^2**	**2^2**
k=9	2	2^2	2^3	2^3	2^3	2^4	2^5	2^4	Z*2^3	**2^3**
k=10	2^2	12*2	40*4*2*3^2	18*8	18*8	24*2	8^2*2*3^2	24*2	12*2	2^2*3
k=11	12*2	84*2^2	84*2^5	504*2^2	504*4	504*2	504*2	504*2	504	504

STABLE RANGE: After a certain point, these values become independent of n

Stable values

Below is a table of the stable values for various k.

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9
2	2	24	0	0	2	240	2^2	2^3

k = 10	k = 11	k = 12	k = 13	k = 14	k = 15	k = 16	k = 17	k = 18
2*3	504	0	3	2^2	480*2	2^2	2^4	8*2

Stable values

Below is a table of the stable values for various k.

Here are their prime factorizations.

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9
2	2	2^3*3	0	0	2	2^4*3*5	(2)(2)	(2)(2)(2)

k = 10	k = 11	k = 12	k = 13	k = 14	k = 15	k = 16	k = 17	k = 18
(2)(3)	2^3*3^2*7	0	3	(2)(2)	(2^5*3*5) (2)	(2)(2)	(2)(2) (2)(2)	$(2^3)(2)$

Stable values

Below is a table of the stable values for various k.

Note that there is a factor of 2^i whenever k+1 has a factor of 2^{i-1} and is a multiple of 4

k = 1	k = 2	k = 3	k = 4 $= 2^2$	k = 5	k = 6	k = 7	k = 8 $= 2^3$	k = 9
2	2	2^3*3	0	0	2	2^4*3*5	(2)(2)	(2)(2)(2)

k = 10	k = 11	k = 12 $= 2^2*3$	k = 13	k = 14	k = 15	k = 16 $= 2^4$	k = 17	k = 18
(2)(3)	2^3*3^2*7	0	3	(2)(2)	(2^5*3*5) (2)	(2)(2)	(2)(2) (2)(2)	$(2^3)(2)$

Stable values

Below is a table of the stable values for various k.

There is a factor of 3^i whenever k+1 has a factor of 3^{i-1} and is divisible by 4

k = 1	k = 2	k = 3	k = 4 = 4	k = 5	k = 6	k = 7	k = 8 = 4*2	k = 9
2	2	2^3*3	0	0	2	2^4*3*5	(2)(2)	(2)(2)(2)

k = 10	k = 11	k = 12 = 4*3	k = 13	k = 14	k = 15	k = 16 = 4*4	k = 17	k = 18
(2)(3)	2^3*3^2*7	0	3	(2)(2)	(2^5*3*5) (2)	(2)(2)	(2)(2) (2)(2)	$(2^3)(2)$

Stable values

Below is a table of the stable values for various k.

There is a factor of 5^i whenever k+1 has a factor of 5^{i-1} and is divisible by 8

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8 = 8	k = 9
2	2	2^3*3	0	0	2	2^4*3*5	(2)(2)	(2)(2)(2)

k = 10	k = 11	k = 12	k = 13	k = 14	k = 15	k = 16 = 8*2	k = 17	k = 18
(2)(3)	2^3*3^2*7	0	3	(2)(2)	(2^5*3*5)(2)	(2)(2)	(2)(2)(2)(2)	(2^3)(2)

Stable values

Below is a table of the stable values for various k.

There is a factor of 7^i whenever k+1 has a factor of 7^{i-1} and is divisible by 12

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9
2	2	2^3*3	0	0	2	2^4*3*5	(2)(2)	(2)(2)(2)

k = 10	k = 11	k = 12 = 12	k = 13	k = 14	k = 15	k = 16	k = 17	k = 18
(2)(3)	2^3*3^2*7	0	3	(2)(2)	(2^5*3*5)(2)	(2)(2)	(2)(2)(2)(2)	(2^3)(2)

What’s the pattern?

Note that:

$4 = 2(3-1)$

$8 = 2(5-1)$

$12 = 2(7-1)$

In general, for p a prime number, there is a factor of p^i if k+1 has a factor of p^{i-1} and is divisible by 2(p-1).

The prime 2 is a little different…

…..2(2-1) does not equal 4!

Beyond…

- It turns out that all of the stable values fit into patterns like the one I described.
- The next pattern is so complicated, it takes several pages to even describe.
- We don't even know the full patterns after this – we just know they exist!
- The hope is to relate all of these patterns to patterns in number theory.

Some patterns for the prime 5