tmf cooperations

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A.K.A. “Team B.O.S.S.”

Goal: understand tmf_*tmf

(The ring of tmf cooperations)

Goal: understand $tmf_*tmf_{(2)}$

(The ring of tmf cooperations)

2-local

[There is something interesting to say at every prime p]

Review: (co)homology (co)operations

- $E =$ generalized cohomology theory (spectrum)

- E^*E = graded endomorphisms of the functor
$$E^*(-): Spectra \to Graded\ Abelian\ Gps$$
= ring of stable cohomology operations

- $E_*E =$ "cooperations"

Review: (co)homology (co)operations

E_*E = "cooperations"

Assume E is a ring spectrum ⇒ E_*E is a ring

- E_*E and E^*E are dual if certain hypotheses are satisfied
  (most importantly E_*E is flat over E_*)

- In this case E_*E is a Hopf algebroid, and there is an Adams spectral sequence:

$$Ext_{E_*E}(E_*X, E_*) \Rightarrow \pi_*(X_E^\wedge)$$

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$tmf_* tmf$ NOT flat / tmf_*

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[flatness only needed for nice E_2-page]

Famous rings of cooperations

Computation

- $(H\mathbb{F}_2)_*H\mathbb{F}_2 = \mathbb{F}_2[\xi_1, \xi_2, \dots]$

[Milnor]

- $BP_*BP = BP_*[t_1, t_2, \dots]$

[Quillen]

- $(E_n)_*E_n = Map^c(\mathbb{G}_n, \pi_*E_n)$

[Morava]

Algebro-Geometric interpretation (apply "spec(-)")

- automorphisms of the additive formal group law over $\mathbb{F}_2$

- Isomorphisms

$$f: F_1 \to F_2$$

between p-typical formal grp laws

- $\mathbb{G}_n$ = nth Morava stabilizer grp

K-theory cooperations

- $K_*K \subset K_*K_{\mathbb{Q}} = \mathbb{Q}[u^{\pm}, v^{\pm}]$ (since K_*K is torsion-free)
- $K_*K = K_* \otimes K_0K$ and $K_0K \subset K_0K_{\mathbb{Q}} = \mathbb{Q}[w^{\pm}]$ $\left(w := \frac{v}{u}\right)$
- $K_0K_{(p)} = \{f(w) \mid f(x) \in \mathbb{Z}_p \text{ for all } x \in \mathbb{Z}_p^{\times}\}$ [Adams-Harris]
- $KO_*KO = KO_* \otimes KO_0KO$ and $KO_0KO \subset KO_0KO_{\mathbb{Q}} = \mathbb{Q}[w^{\pm 2}]$
- $KO_0KO_{(p)} = \{f(w^2) \mid f(x^2) \in \mathbb{Z}_p \text{ for all } x \in \mathbb{Z}_p^{\times}\}$ [Adams-Switzer]

Connective K-theory cooperations

Numerical polynomial perspective

From this point on everything in the talk is 2-local

$H_* =$ *homology with mod 2 coefficients*

$bo_*bo \ \rightarrow KO_*KO$ (NOT injective, "kernel = wedge of $H\mathbb{F}_2$'s")

Image of the above map consists of

$\{f(u^2, v^2) \in KO_*KO \mid f \in \mathbb{Q}[u^2, v^2], f$ has Adams filtration $\geq 0\}$

Connective K-theory cooperations

Brown-Gitler spectrum perspective

- $H\mathbb{Z} = \bigcup_i B_i$ (B_i = i^{th} integral Brown-Gitler spectrum)

- $H_*(B_i) \subset H_*(H\mathbb{Z}) = \mathbb{F}_2[\zeta_1^2, \zeta_2, \ldots]$

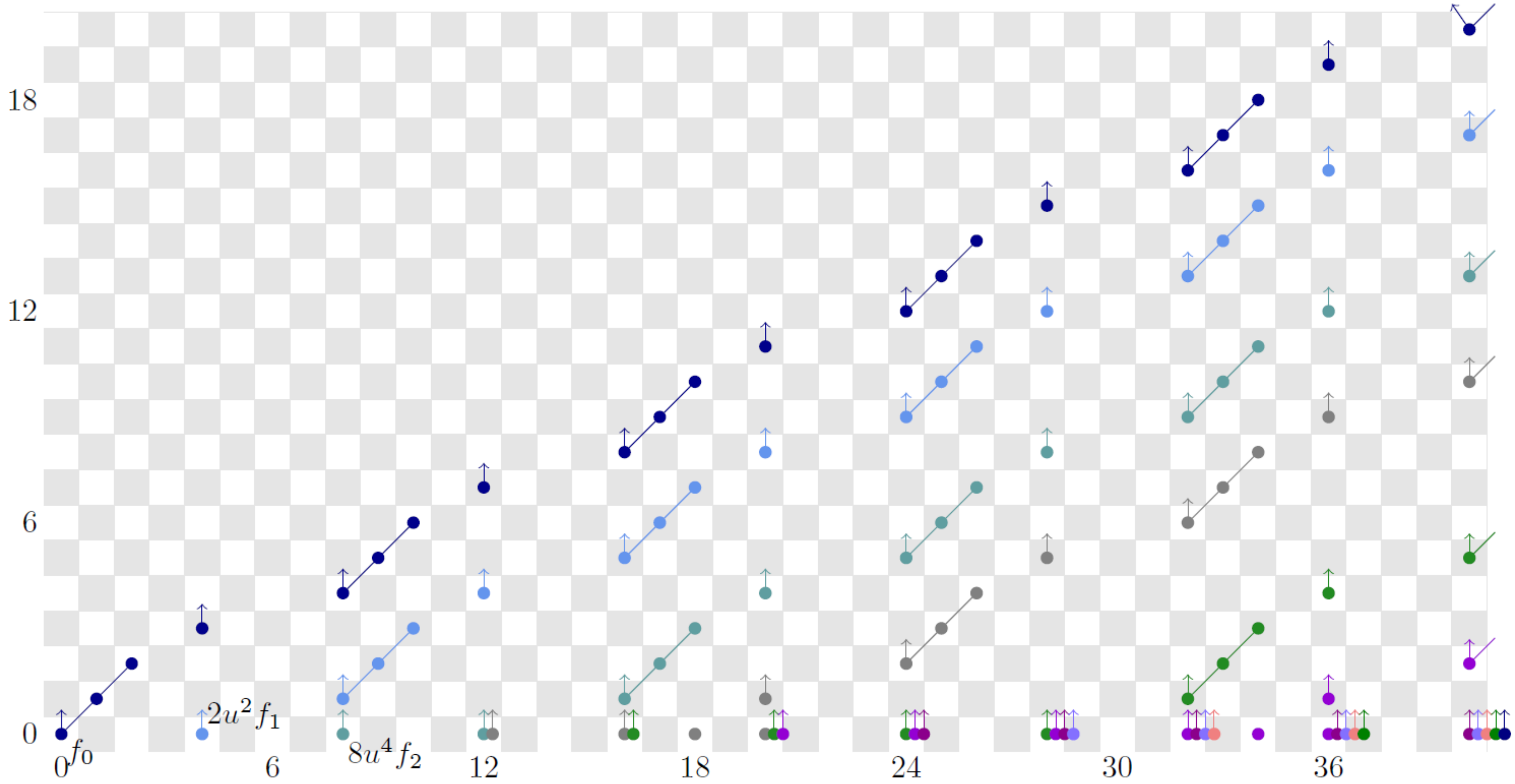
subspace spanned by monomials of weight $\leq 2i$ (weight(ζ_i) = 2^{i-1})

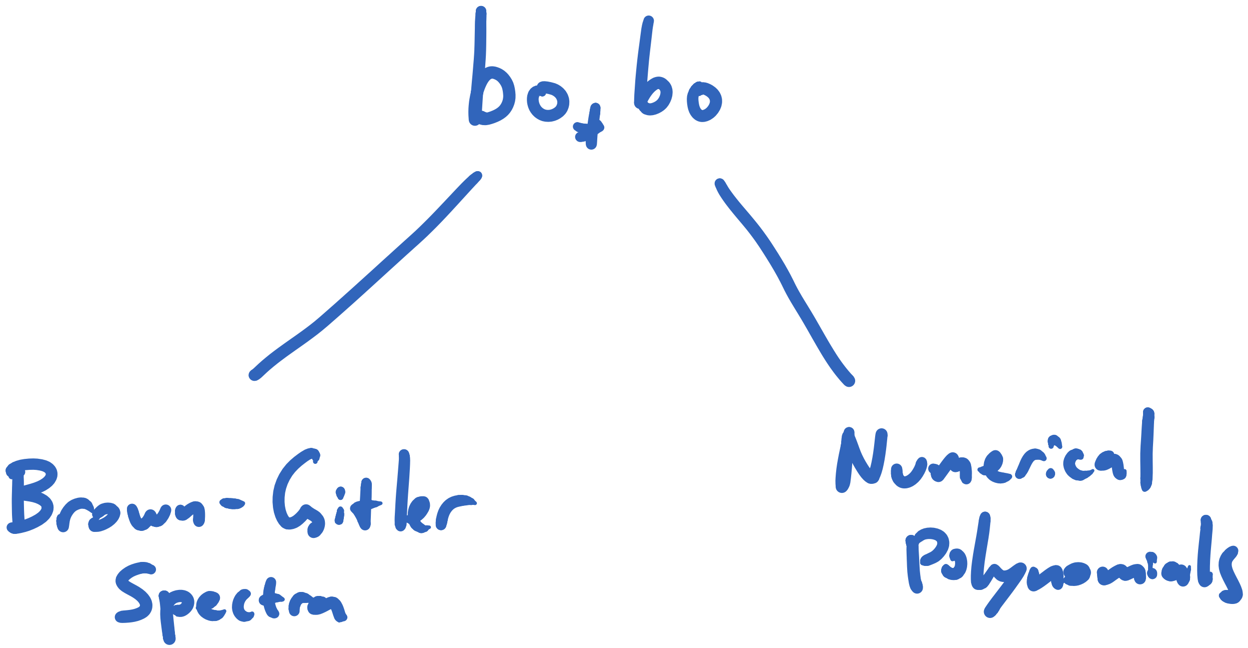
- $bo \wedge bo \simeq \bigvee_i \Sigma^{4i}\, bo \wedge B_i$ [Mahowald-Milgram]

Bottom cell of $\Sigma^{4i} bo \wedge B_i$ corresponds to the function $2^{2i-\alpha(i)} u^{2i} f_i(w^2)$

$$f_i(w^2) := \frac{(w^2-9)(w^2-9^2)\cdots(w^2-9^{i-1})}{(9^i-1)(9^i-9)\cdots(9^i-9^{i-1})}$$

Adams spectral sequence for $bo \wedge bo$





Mahowald's analysis of the bo-based Adams spectral sequence hinged on this understanding of $bo \wedge bo$.

It led to his proof of the telescope conjecture at chromatic level 1 at the prime 2. "bo resolutions"

$tmf_* tmf$

- bo-Brown-Gitler Spectra
- 2-variable modular forms
- Isogenies of elliptic curves

tmf cooperations: bo-Brown-Gitler spectra

- $bo = \bigcup_i bo_i$ (bo_i = i^{th} bo-Brown-Gitler spectrum)

- $H_*(bo_i) \subset H_*(bo) = \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots]$

subspace spanned by monomials of weight $\leq 4i$

Note: $H^*(tmf) = A\,//\,A(2)$

- Theorem[Mahowald] There is a splitting of A(2)-modules

$$H^*(tmf) = \bigoplus_i H^*(\Sigma^{8i} bo_i)$$

tmf cooperations: bo-Brown-Gitler spectra

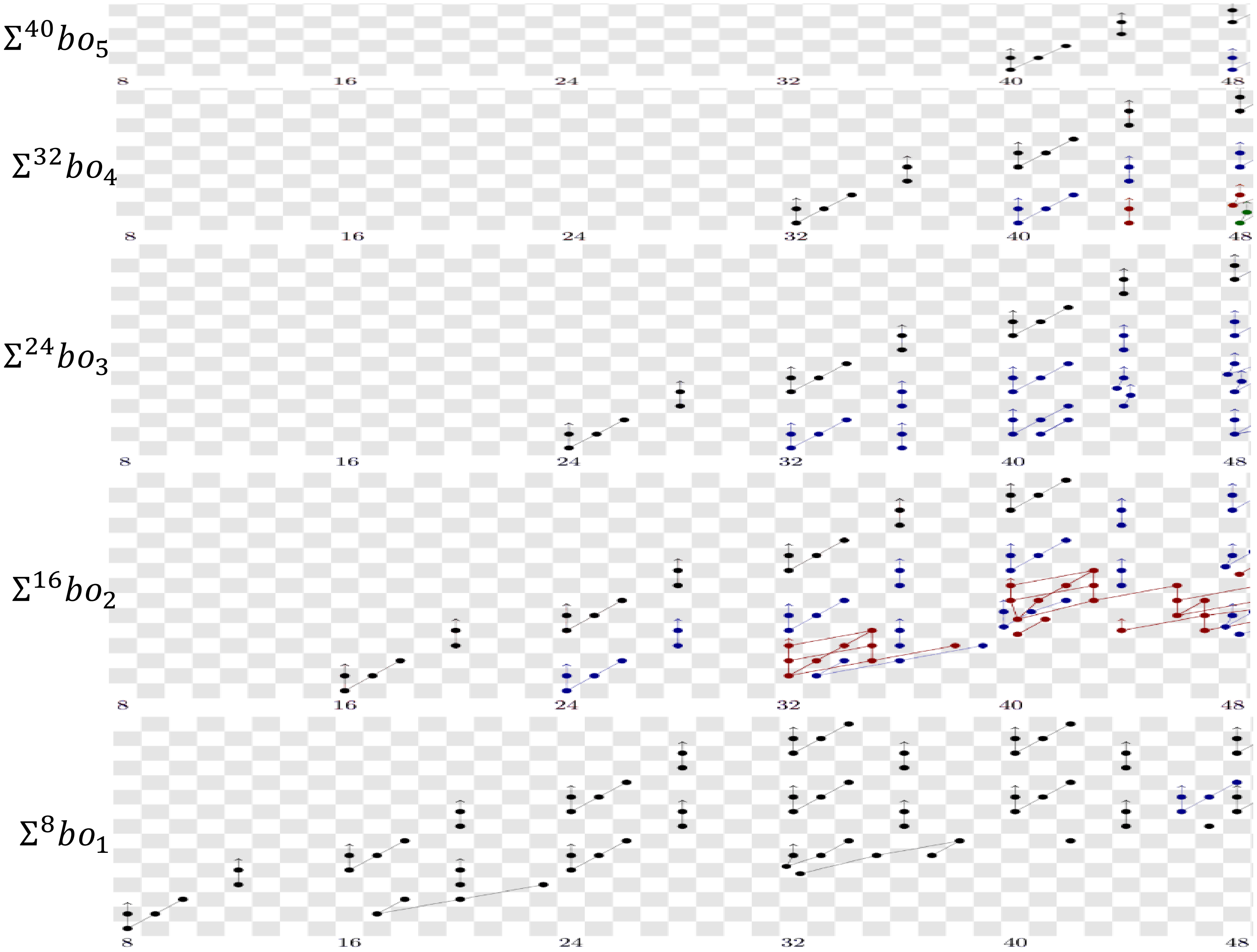
Theorem[Mahowald] There is a splitting of A(2)-modules

$$H^*(tmf) \cong \bigoplus_i H^*(\Sigma^{8i} bo_i)$$

Corollary There is a splitting of Adams spectral sequence E_2-terms:

$$Ext_A(H^*(tmf \wedge tmf), \mathbb{F}_2) \cong \bigoplus_i Ext_{A(2)}(H^*(\Sigma^{8i} bo_i), \mathbb{F}_2)$$

The Adams spectral sequence for $tmf \wedge tmf$



So, is there a splitting

$$tmf \wedge tmf \simeq \bigvee_i \Sigma^{8i} tmf \wedge bo_i \quad ???$$

NO! [Davis-Mahowald-Rezk]

(we'll revisit this point at the end of the talk)

tmf cooperations: 2-variable modular forms

- $TMF_* TMF_{\mathbb{Q}} = \mathbb{Q}[c_4, c_6, \bar{c}_4, \bar{c}_6][\Delta^{-1}, \bar{\Delta}^{-1}]$

- Theorem[Laures]

$$TMF_* TMF[1/6] = \left\{ f \in TMF_* TMF_{\mathbb{Q}} \,\middle|\, f(q, \bar{q}) \in \mathbb{Z}[1/6][[q^{\pm 1}, \bar{q}^{\pm 1}]] \right\}$$

- Theorem[B.O.S.S] There is an injection

$$\frac{tmf_* tmf_{(2)}}{tors} \hookrightarrow \{ f \in \mathbb{Q}[c_4, c_6, \bar{c}_4, \bar{c}_6] \mid f(q, \bar{q}) \in \mathbb{Z}_{(2)}[[q, \bar{q}]] \}$$

The Adams spectral sequence for $tmf \wedge tmf$

$$f_1 := (-\bar{c}_4 + c_4)/16$$

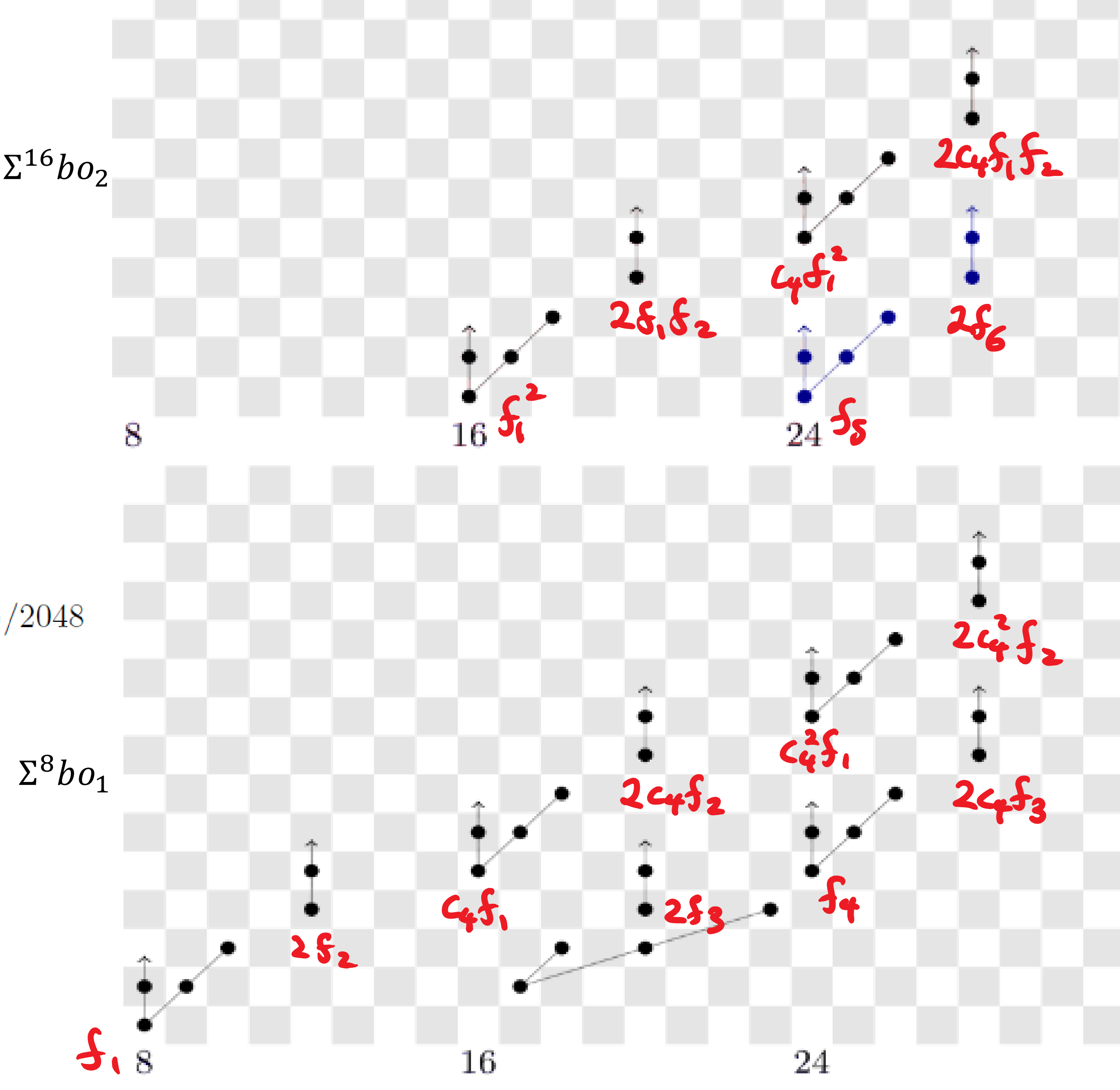
$$f_2 := (-\bar{c}_6 + c_6)/8$$

$$f_3 := (5 f_1 c_6 + 21 f_2 c_4)/8$$

$$f_4 := (5 f_2 c_6 + 21 f_1 c_4^2)/8$$

$$f_5 := (-f_1^2 c_4 + f_2^2)/16$$

$$f_6 := (-c_4^2 c_6 + c_4^2 c_6 + 544 f_2 c_4^2 + 768 f_3 c_4 + 1792 f_1 f_2 c_4)/2048$$



tmf cooperations: isogenies of elliptic curves

$TMF \leftrightarrow$ moduli space of elliptic curves C

$TMF \wedge TMF \leftrightarrow$ moduli space of tuples (C_1, C_2, α)

$C_i =$ elliptic curves

$\alpha: \hat{C}_1 \to \hat{C}_2$ [iso of FGL's]

Unfortunately, the above perspective seems rather impractical for computations.

tmf cooperations: isogenies of elliptic curves

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C_i= elliptic curves

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$\prod_{\substack{i \in \mathbb{Z} \\ j \geq 0}} TMF_0(\ell^j) \leftrightarrow$ moduli space of tuples (C_1, C_2, ϕ)

C_i= elliptic curves

$\phi : C_1 \to C_2$ [quasi-isogeny of deg ℓ^k]

tmf cooperations: isogenies of elliptic curves

$$\{\text{moduli space of tuples } (C_1, C_2, \phi)\} \to \{\text{moduli space of tuples } (C_1, C_2, \alpha)\}$$
$$\phi \mapsto \hat{\phi}$$

Theorem [B.O.S.S.] For primes $\ell \neq 2$ the maps of moduli spaces above induce maps of ring spectra:

$$TMF \wedge TMF_{(2)} \to \prod_{\substack{i \in \mathbb{Z} \\ j \geq 0}} TMF_0(\ell^j)$$

The product map

$$TMF \wedge TMF_{(2)} \to \prod_{\substack{i \in \mathbb{Z} \\ j \geq 0}} TMF_0(3^j) \times TMF_0(5^j)$$

induces a monomorphism on homotopy groups.

Application: connective covers

Theorem [Davis-Mahowald-Rezk]

- There is a subcomplex

$$\Sigma^8 tmf \wedge bo_1 \cup \Sigma^{16} tmf \wedge bo_2 \subset tmf \wedge tmf$$

- There is a cofiber sequence

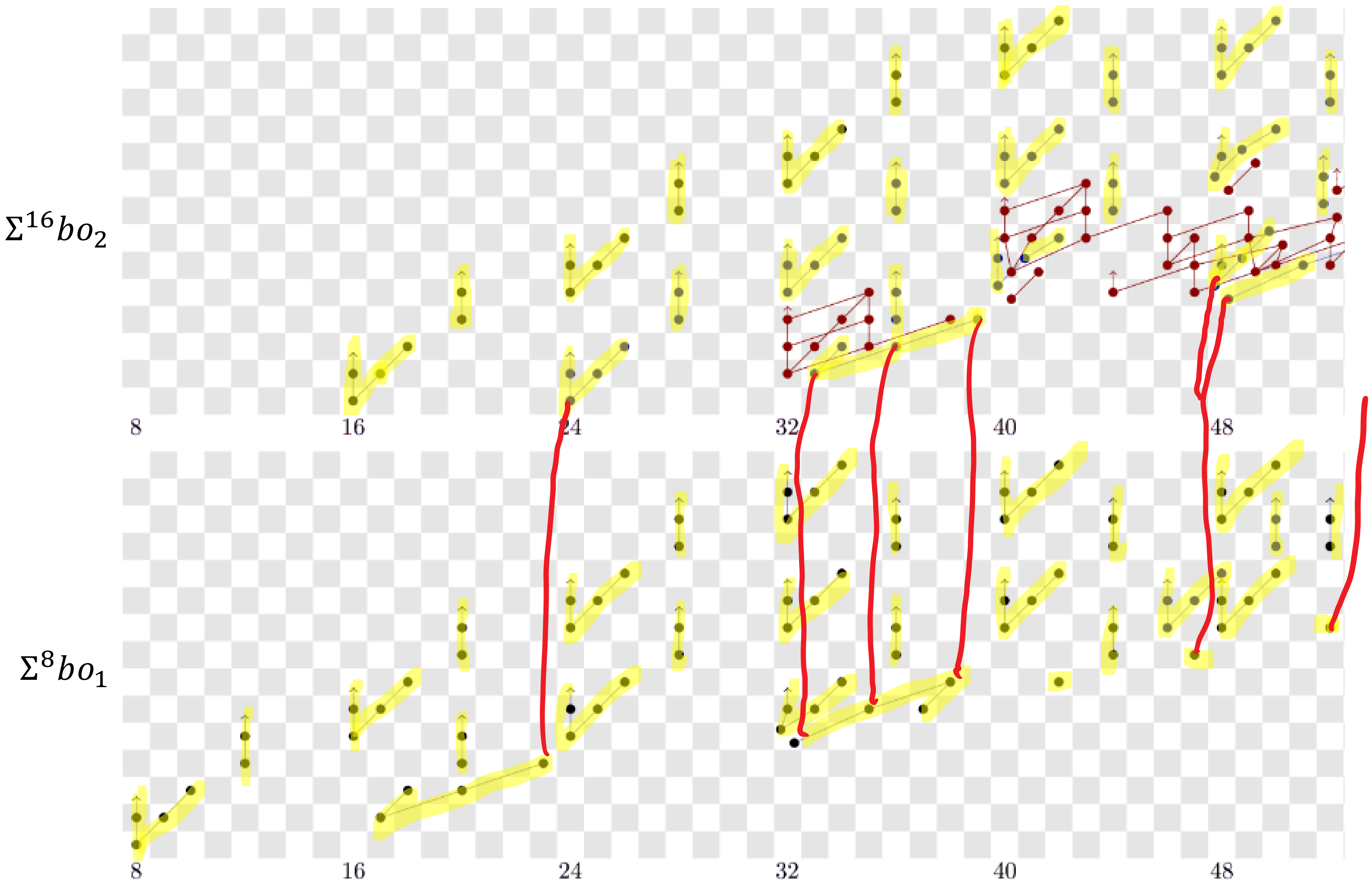
$$\Sigma^{32} tmf \to \Sigma^{16} tmf \wedge bo_2 \to \Sigma^{16} \overline{tmf \wedge bo_2}$$

- There is a factorization

$$\begin{array}{ccccc} \Sigma^8 tmf \wedge bo_1 \cup \Sigma^{16} tmf \wedge bo_2 & \to & tmf \wedge tmf & \to & TMF \wedge TMF \\ \downarrow & & & & \downarrow \\ \Sigma^8 tmf \wedge bo_1 \cup \Sigma^{16} \overline{tmf \wedge bo_2} & & \longrightarrow & & TMF_0(3) \end{array}$$

such that the bottom arrow is a connective cover of $TMF_0(3)$

The Adams Spectral Sequence for $\Sigma^8 tmf \wedge bo_1 \cup \Sigma^{16}\overline{tmf \wedge bo_2}$



Theorem [B.O.S.S.]

- There is a map

$$\Sigma^{64} tmf \to \Sigma^{32} tmf \wedge bo_4$$

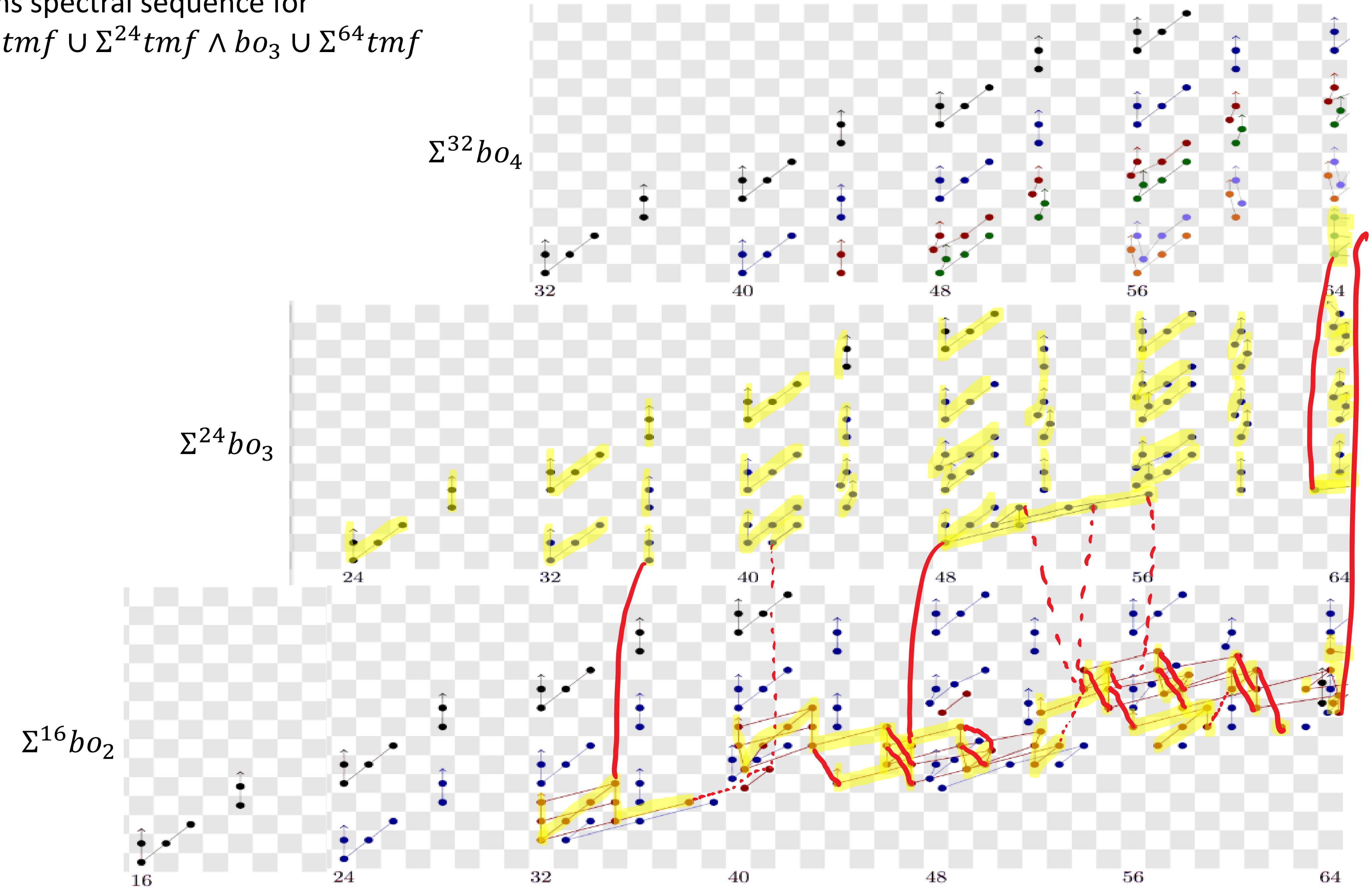
- There are subcomplexes

$$\begin{gathered} \Sigma^{32} tmf \cup \Sigma^{24} tmf \wedge bo_3 \cup \Sigma^{64} tmf \\ \cap \\ \hookrightarrow \Sigma^{8} tmf \wedge bo_1 \cup \Sigma^{16} tmf \wedge bo_2 \cup \Sigma^{24} tmf \wedge bo_3 \cup \Sigma^{64} tmf \\ \cap \\ \hookrightarrow tmf \wedge tmf \end{gathered}$$

- The following composite is a connective cover

$$\Sigma^{32} tmf \cup \Sigma^{24} tmf \wedge bo_3 \cup \Sigma^{64} tmf \to tmf \wedge tmf \to TMF \wedge \mathrm{TMF} \to TMF_0(5)$$

The Adams spectral sequence for
$\Sigma^{32}tmf \cup \Sigma^{24}tmf \wedge bo_3 \cup \Sigma^{64}tmf$



Thank You!