Normalization and Mixed Degrees of Integration in Cointegrated Time Series Systems

Robert J. Rossana*
Department of Economics, 2074 F/AB,
Wayne State University, Detroit MI 48202
E-Mail: r.j.rossana@wayne.edu

June 2004

Abstract

This paper provides one possible explanation of why cointegrated time series systems can have constituent time series displaying mixed degrees of integration. If exogenous forcing variables display differing degrees of integration, choice variables may do so as well if separability restrictions are present in an optimization framework. Pre-testing data for unit roots can thus reveal the presence of separability in an optimizing framework. A method for normalizing cointegrating vectors is also developed for systems with multiple cointegrating vectors. The method permits the normalized cointegrating vectors to contain the parameters of static factor demand functions and the resulting adjustment matrix contains all the adjustment parameters familiar from the neoclassical investment literature. VARs are derived for systems with I(2) forcing series and the I(2) case results in cointegration between the levels and differences of the exogenous and choice variables, a property known as multicointegration. Empirical examples are provided from the literature on money demand and inventories. The price level and nominal cash balances display differing degrees of integration, irrespective of how the money stock is measured. From the perspective of this paper, this finding is difficult to explain unless real balances, rather than nominal balances, are the appropriate choice variable for the household. Evidence is provided that inventories, production workers, and sales in selected two-digit industries do not have the same degree of integration, unlike the predictions of the production smoothing model of inventories. In the selected industries, labor input decisions appear to be separable from decisions on finished goods inventories.

*Thanks are due to David A. Dickey for helpful discussions and to Richard T. Baillie for useful comments on a previous draft of this paper. I also benefitted from the comments of seminar participants at the Federal Reserve Board and Wayne State University. The usual disclaimer applies regarding responsibility for the remaining limitations of this paper.

1 Introduction

Cointegration tests and estimation methods are routinely used in applied macroeconomic research. Almost all of the work using these methods has focused on the situation where economic time series in a system are I(1) since the application of tests for unit roots typically results in the finding that time series require one degree of differencing to achieve apparent stationarity. However there are cointegration methods that apply to time series systems where there are mixed degrees of integration displayed by the data but there appears to be no available guidance from economic theory as to the conditions under which such a time series system might arise. One purpose of this paper is to examine a series of economic models with the goal of explaining why the time series in a cointegrated error correction vector autoregression (VAR) can display mixed degrees of integration.

Any VAR can be written in the error correction form

$$\Delta X_t = \sum_{i=1}^{k-1} \Delta X_{t-i} + \Pi X_{t-1} + \epsilon_t$$

where Δ is the differencing operator, X refers to a vector of time series, and ϵ is a vector of iid disturbances.[1] For I(1) systems, cointegration between economic time series is a condition regarding the rank of the parameter matrix, Π, attached to the lagged levels, X_{t-1}, in the error correction VAR. If the vector of time series has dimension n, then a cointegrated time series system has the property that $0 < \text{rank}(\Pi) < \text{n}$. If this parameter matrix has reduced rank, then it may be written as $\Pi = \alpha\beta'$ where α is referred to as the adjustment matrix and β is the cointegrating matrix. The elements of the latter matrix contain the parameters of the long-run relationships maintained by the time series in this VAR. In the terminology of Engle and Granger (1987), if each of the time series in a system is I(d), then a linear combination of these series may be I(d-b), a fact denoted by $X_t \sim CI(d, b)$.

As stated above, the presumption in the definition of a cointegrated time series system by Engle and Granger (1987) is that all time series are I(d). But there are cointegration tests and estimators of the elements of cointegrating vectors that apply to systems where mixed degrees of integration are manifested by time series. For example, the cointegration tests and estimation procedures

[1] The vector X_t could contain one entry where every element is unity to accommodate an intercept term in the long-run equilbrium relationships maintained by the elements of X_t. See Johansen and Juselius (1990) or Johansen (1991) for further discussion on handling intercepts in cointegrated time series systems.

in Johansen (1988, 1991) are appropriate to systems where the data are either I(0) or I(1). The estimator of cointegrating vectors developed by Stock and Watson (1993) is suitable for systems with mixed degrees of integration and where the series might be integrated of order greater than unity. Further, economic time series do not always display the same degree of integration. Stock and Watson (1993) apply unit root tests to the time series relevant to the demand for money and find that, within their data sample, the price level and the money stock appear to be I(2) whereas all other series in their data set are at most I(1). The question then arises as to how a cointegrated time series system may arise where there are mixed degrees of integration manifested by the data.

In this paper, a series of economic models are examined with the goal of establishing the conditions under which the time series in an error correction VAR, arising from optimizing behavior by economic agents, can display mixed degrees of integration. These maximization models will employ quadratic forms, familiar from previous macroeconomic research, so that a standard VAR may obtained from the models. Exogenous forcing variables will be assumed to obey differing degrees of integration and they will sometimes be I(2) (degrees of integration above two seem not to be relevant to economic data). Euler equations and their implied VARs will be obtained and we will seek to establish the degree of integration that the choice variables appear to obey as a consequence of optimizing behavior by economic agents. Conditions will be developed in this paper permitting the choice variables in the model to display varying degrees of integration. However, in order to address this issue, two other matters must be examined.

The cointegrating vectors that arise from the models considered here will require normalization so that the adjustment and cointegrating vectors can be given plausible economic interpretations. In one state variable optimizing models, such a normalization is obvious; as observed elsewhere (see Rossana (1998)), normalizing the state variable to unity produces long-run relationships consistent with economic theory. But in cases where there are multiple cointegrating vectors, it is not clear how normalization should be done in order to produce cointegrating vectors containing parameters of economic interest.[2] In this paper, a method will be developed showing that, within the class of models considered, a particular normalization of cointegrating vectors should always be used

[2] In Rossana (1995, p. 15), there was a discussion of the normalized cointegrating vectors that could be obtained from a model with multiple state variables. However that analysis did not establish the precise normalization that would produce parameters that would be of economic interest in the normalized cointegrating vectors nor did it address the resulting adjustment matrix in this case.

because the resulting normalized cointegrating vectors will be consistent with the implications of static economic theory. Conditional on this normalization, the adjustment matrix will be shown to contain parameters familiar from the dynamic theory of the firm.

A final by-product of the analysis is that we will look at the implications of higher orders of integration (that is, a degree of integration above unity) on the properties of error correction VARs and the adjustment and cointegrating matrices contained in them, a subject largely overlooked in previous economic studies of cointegrated time series systems. As discussed in Johansen (1992, 1995), it is convenient to write a VAR in the form

$$\Delta^2 X_t = \sum_{i=1}^{k-2} \Pi_i \Delta^2 X_{t-i} + \Pi X_{t-2} + \Gamma \Delta X_{t-1} + \epsilon_t$$

for systems where the time series may possibly be I(2). Cointegration in this context now involves rank conditions on the parameter matrices associated with the lagged levels and differences of the vector X_t. It is possible in this case for cointegration to arise between the levels and differences of the time series in the economic framework. For the optimization models in this paper, these rank conditions will be used to deduce the presence of cointegration in the levels and differences as implied by the restrictions of economic theory. This case will serve as one benchmark for the study of mixed systems of integration that follows. The analysis will reveal what, if any, effects there are upon the error correction VAR of the presence of degrees of integration exceeding unity, as well as differing degrees of integration, in any exogenous series that are present in an optimization model.

It is shown in this paper that if exogenous forcing variables obey time series processes with mixed degrees of integration, all time series in the system set optimally by economic agents will appear to contain the number of unit roots equal to the largest number of unit roots in any forcing variable appearing in the maximization problem. The reason is that choice variables, through the Euler equations describing optimal behavior in an economic model, will be linearly related to all exogenous forcing variables present in the optimization framework so that the time series characteristics of these exogenous series will be propagated throughout the time series system arising from the maximization model. As a consequence of optimizing behavior then, all choice variables in an error correction VAR will appear to have the same number of unit roots. Thus the case discussed by Engle and Granger (1987), namely that a cointegrated time series system will have all series with the same degree of integration, is seen to be the case arising when exogenous forcing variables are integrated of the same order. But it will also be shown that if the exogenous forcing variables in an optimization model display mixed degrees of integration, it is possible to rationalize differing degrees of integration for the choice variables in these models by the use of separability restrictions that effectively isolate one or more of these magnitudes from a subset of the exogenous forcing variables. Thus it is possible to justify differing degrees of integration in an error correction VAR for the choice variables in an economic model but this characteristic must be explained by functional form restrictions generally requiring empirical justification. The analysis implies that in pre-testing economic data for unit roots, an applied econometrician can gain information about the structure of an underlying optimization problem if it is found that time series manifest different numbers of unit roots because these unit root test results can reveal the presence of separability restrictions in an economic model. Empirical examples are provided below that are taken from the literature on money demand and inventories. It will be seen that the price level seems to have more unit roots than the money stock or other series relevant to the demand for money. From the viewpoint of the results in this paper, this finding seems hard to rationalize if the traditional money demand relationship is thought to arise as the solution to some type of optimization problem solved by a representative household where the Euler equation, describing the optimal choice of nominal cash balances, contains the price level and the money stock among other time series. The results can be rationalized if real balances are the choice variable in the optimization model as sometimes argued. In selected two-digit industries, it is found that data on inventories, sales, and production workers do not contain the same number of unit roots as predicted by the production smoothing model of inventories. These results suggest that the decision on the choice of labor input in production is separable from decisions on inventories of finished goods.

Regarding the normalization of cointegrating vectors, it will be shown that, within the profit maximization models of an optimizing firm that are studied in the paper, a normalization is available that will cause the resulting cointegrating vectors to contain the parameters of the static factor demand functions that arise in the static theory of the firm. This normalization is easy to carry out and conveniently enables an applied researcher to obtain estimates of these parameters of economic interest by estimating the elements of the cointegrating vectors. Further, the adjustment matrix that results from this normalizing transformation will contain all of the neoclassical adjustment parameters familiar from the neoclassical adjustment cost literature on investment. These results are not specific to models of the firm as they should be available within any optimizing model of the firm or household. This normalizing transformation has the advantage that, given a suitable estimator of the elements of cointegrating vectors with desirable statistical properties, the transformation can always be used to obtain estimates of the parameters of static demands for inputs. Other normalizations may not produce estimates of the parameters of static factor demands because it may not be possible to uniquely identify all of the underlying structural parameters needed to construct the parameters of these long-run demands for inputs.

Finally, higher degrees of integration (that is, a system with I(2) data) are shown to have no effect upon the adjustment or cointegrating matrices associated with lagged levels of time series that arise in error correction models; lagged difference terms are however added to the error correction model that is obtained from the optimizing framework. But long-run relationships between time series will not be identical to the case where time series data are I(1). In the I(2) case, stationary linear combinations of the time series will involve both the levels and first differences of economic time series. Since shocks can permanently change the growth rates of exogenous forcing variables, the growth rates of choice variables will also respond to these shocks because optimal behavior requires that choice variables be tied to these exogenous variables. The assumption of I(2) exogenous forcing variables thus has an important impact upon the nature of the equilibrium relationships implied by optimizing behavior, inducing considerably more complex equilibrium relationships in an economic system. This economic framework provides an optimization-based explanation for how it is that multicointegration (Granger and Lee (1989)) arises in time series systems.

This paper is organized as follows. The next section examines normalization of cointegrating vectors within single state variable and multiple state variable models of the firm, establishing the preferred manner of normalizing cointegrating vectors in these models. In Section 3, the role of the degree of integration on error correction VARs is explored, establishing the characteristics of the error correction VAR in the situation where forcing variables are I(2). The analysis also illustrates the effects on the error correction VAR of differing degrees of integration in the exogenous forcing variables, showing that separability assumptions can exist that allow choice variables to display differing degrees of integration. Section 4 provides the empirical examples on the demand for money and inventories where unit root tests imply differing degrees of integration between choice variables and exogenous forcing series. A final section summarizes results. A brief appendix contains some results for the I(2) models establishing that cointegration in the I(2) case involves both the levels and differences of the time series in the economic model.

2 Normalization

This section provides a discussion of the time series properties of VARs arising from the solution of models of a price-taking representative firm producing a nonstorable output. The first model is one where the firm holds one state variable, a quasi-fixed input in production, and a variable factor input (one that is not subject to adjustment costs). The description of the model will be brief because a more complete discussion of it can be found elsewhere (see Rossana (1995)). It will thus be possible to draw from the results of this earlier study. The model has two choice variables and the cointegrating matrix incorporating both inputs has rank above unity. The first task will be to determine the appropriate way to normalize the cointegrating vectors in the error correction VAR arising from the model. Once this has been established, it will be possible to show that the results generalize into an optimization model where there are an arbitrary number of state variables held by the firm. Having determined the proper way to normalize cointegrating vectors, we can use this particular normalization in the subsequent parts of the analysis. In this section, forcing variables are assumed to be I(1), the typical case in economic applications. For convenience, functional forms are chosen in part to eliminate constants from all decision rules.

2.1 A Model with One State Variable

Using E_0 to denote the conditional expectation operator, the optimization framework for the firm is to maximize

$$E_0 \sum_{t=0}^{\infty} \gamma^t \left\{ y_t + (\psi_{1t} - v_{1t})x_{1t} + (\psi_{2t} - v_{2t})x_{2t} - (\rho/2)(x_{1t} - x_{1t-1})^2 \right\} \tag{1}$$

$$y_t = -(\delta_0/2)x_{1t}^2 - (\delta_1/2)x_{2t}^2 - \delta_2 x_{1t}x_{2t} \tag{2}$$

where we have the parameter restrictions $\delta_i > 0$ i = 0, 1 and $\rho > 0$. In these expressions, y_t refers to output, inputs in production are denoted by x_{it}, v_{it} are the real factor prices (the firm's output price is normalized to unity) associated with each input in production, stochastic shocks to the firm's technology are given by ψ_{it}, and γ is the discount rate $(0 < \gamma < 1)$. There are diminishing returns in production and the technology is assumed to be strictly concave in its arguments $(\delta_0\delta_1 - \delta_2^2 > 0)$. The cross-derivative of the production function is unrestricted as is traditionally the case in the ordinary theory of the firm. Finally there are installation costs associated with the input x_{1t} and these costs rise at the margin. Thus this input is quasi-fixed but the other input in production has no such adjustment costs associated with it and is thus a variable factor input in production. If the firm has contemporaneous information at its disposal, the following optimality criteria describe the optimal choices of the two inputs in production.

$$\gamma\rho E_t x_{1t+1} - [\delta_0 + \rho(1+\gamma)]\, x_{1t} + \rho x_{1t-1} - \delta_2 x_{2t} = v_{1t} - \psi_{1t} \tag{3}$$

$$-(\delta_1 x_{2t} + \delta_2 x_{1t}) = v_{2t} - \psi_{2t} \tag{4}$$

The characteristics of the VAR that can be derived from these conditions will depend critically upon the assumptions that are made about the stochastic processes obeyed by the exogenous factor prices, v_{it}, and the shocks to the technology, ψ_{it}. In what follows, $\psi_{it} \backsim iid(0, \sigma_{\psi i}^2)$ because there cannot be any cointegration between the factor inputs and the factor input prices if there are unit roots in the stochastic processes associated with these magnitudes.[3] But alternative assumptions will be made about the stochastic processes obeyed by the factor input prices, v_{it}, in order to study the effects of various assumptions upon the VAR that can be derived from this model.

2.2 Error Correction Models with I(1) Factor Prices

We begin the analysis by looking at the standard case in the literature, namely, the situation where the factor prices obey independent univariate time series processes that are each I(1). This case will be one benchmark against which we will compare the VARs obtained under alternative

[3] Linear combinations of these inputs and their associated factor prices will not be stationary in the case where the shocks to the technology are not I(0). Rather, to be stationary, these linear combinations will require differencing of degree equal to the number of unit roots in these stochastic processes. See Rossana (1995, p. 10) for a demonstration of these properties.

assumptions about these stochastic processes. Thus it will now be assumed that

$$\Delta v_{it} = \varepsilon_{it}^{v}, \varepsilon_{it}^{v} \sim iid(0, \sigma_{vi}^{2}).$$

Therefore shocks, ε_{it}^{v}, are iid stochastic processes and $\Delta = (1-L)$ with L denoting the lag operator. Thus factor input prices obey independent univariate stochastic processes and so there are no cointegrating relationships between the exogenous forcing variables confronting the firm nor do other variables Granger-cause any of these series. More will be said below about permitting cointegration between these factor prices. Further, it will not add anything of substance to the analysis if we were to add a polynomial in the lag operator preceding the stationary processes Δv_{it}. This element would add lagged difference terms to the VAR that we obtain but it will not have any effect on the cointegrating and adjustment matrices that will be derived.[4] Since these matrices are the principal interest that we have in the VARs we obtain, this possible complication is inessential and is therefore not pursued here.

The Euler equations imply that the level of the quasi-fixed input, x_{1t}, will depend upon the discounted sequences of factor prices and technology shocks. An assumption must be made regarding expectation formation and it will be assumed that linear least squares projections are used in forming expectations of these magnitudes. These sequences require the use of the Weiner-Kolmogorov prediction formula (Sargent (1987, p. 304))

$$\sum_{j=0}^{\infty} (\phi_1 \gamma)^j E_t v_{it+j} = \frac{\theta^i(\phi_1 \gamma) - \phi_1 \gamma L^{-1} \theta^i(L)}{\theta^i(\phi_1 \gamma)(1 - \phi_1 \gamma L^{-1})} v_{it} \tag{5}$$

where $\theta^i(L)$ is a polynomial in the lag operator appearing in the stochastic process for v_{it} and ϕ_1 is a stable characteristic root $(0 < \phi_1 < 1)$. Evaluating this prediction formula using the assumptions about the time series processes obeyed by the factor prices leads to the following VAR (Rossana (1995, p. 9)).

$$\begin{bmatrix} \Delta x_{1t} \\ \Delta v_{1t} \\ \Delta v_{2t} \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ v_{1t-1} \\ v_{2t-1} \end{bmatrix} + \begin{bmatrix} \Phi_t \\ \varepsilon_{1t}^{v} \\ \varepsilon_{2t}^{v} \end{bmatrix} \tag{6}$$

[4] It is, however, important to the interpretation of the adjustment matrix if there are unobservable shocks in this model that obey stationary stochastic processes with autoregressive elements. See footnote 7 below.

where

$$
\begin{aligned}
\pi_{11} &= \phi_1 - 1 < 0, \pi_{12} = \frac{\delta_1(\phi_1 - 1)}{\delta_0\delta_1 - \delta_2^2} < 0, \pi_{13} = \frac{\delta_2(\phi_1 - 1)}{\delta_0\delta_1 - \delta_2^2} && \text{(7a)} \\
\Xi &= (1 - \phi_1)(1 - \phi_1\gamma)(\delta_0\delta_1 - \delta_2^2) > 0 && \text{(7b)} \\
\Phi_t &= \Xi\left[\delta_1\psi_{1t} - \delta_2\psi_{2t} + (1 - \phi_1\gamma)^{-1}(\delta_1\varepsilon_{1t}^v - \delta_2\varepsilon_{2t}^v)\right]. && \text{(7c)}
\end{aligned}
$$

The matrix of coefficients attached to the lagged levels of time series has rank one, given the size of the adjustment parameter ϕ_1. As a consequence, this matrix can be written as $\Pi = \alpha\beta'$ where α is the adjustment matrix and β is the cointegrating matrix. Normalizing on the state variable x_{1t}, these matrices are easily found to be

$$
\alpha = \begin{bmatrix} \phi_1 - 1 \\ 0 \\ 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 \\ \delta_1(\delta_0\delta_1 - \delta_2^2)^{-1} \\ -\delta_2(\delta_0\delta_1 - \delta_2^2)^{-1} \end{bmatrix}.
$$

The adjustment matrix contains the speed of adjustment for the quasi-fixed input, x_{1t}. The cointegrating vector contains the parameters of the static factor demand function for x_{1t}. The own-factor price effect is negative but the cross-factor price effect is unrestricted, given the assumptions made above about the technology. Since adjustment costs disappear in the equilibrium of this model, the equilibrium corresponds to the solution of a static profit maximization problem so that the findings about the adjustment and cointegrating matrices are plausible.

The error correction model in (6) correctly summarizes the dynamic behavior of the series that it contains. However this system is not adequate for the purposes of this paper because, in order to determine the proper method of normalization when there are multiple choice variables in an economic model, we need to incorporate the variable factor input, x_{2t}, into this error correction system. This can be accomplished in the following way.

The Euler equation describing the optimal choice of x_{2t} is given by (4). Invert this expression, solving for x_{2t} as a function of the quasi-fixed input x_{1t}, the factor price v_{2t}, and the technology shock ψ_{2t}. Eliminate the state variable from this expression using the first row of (6) and stack the resulting expression into (6). The end result of these operations is the extended error correction model

$$\begin{bmatrix} \Delta x_{1t} \\ \Delta x_{2t} \\ \Delta v_{1t} \\ \Delta v_{2t} \end{bmatrix} = \begin{bmatrix} \pi_{11} & 0 & \pi_{13} & \pi_{14} \\ \pi_{21} & -1 & \pi_{23} & \pi_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \\ v_{1t-1} \\ v_{2t-1} \end{bmatrix} + \begin{bmatrix} \Phi_t \\ \Omega_t \\ \varepsilon^v_{1t} \\ \varepsilon^v_{2t} \end{bmatrix} \tag{8}$$

where coefficient definitions are given below.

$$\pi_{11} = \phi_1 - 1 < 0, \pi_{13} = \frac{\delta_1(\phi_1 - 1)}{\delta_0\delta_1 - \delta_2^2} < 0, \pi_{14} = \frac{\delta_2(1-\phi_1)}{\delta_0\delta_1 - \delta_2^2} \tag{9a}$$

$$\pi_{21} = -\frac{\delta_2\phi_1}{\delta_1}, \pi_{23} = \frac{\delta_2(1-\phi_1)}{\delta_0\delta_1 - \delta_2^2}, \pi_{24} = \frac{\phi_1\delta_2^2 - \delta_0\delta_1}{\delta_1(\delta_0\delta_1 - \delta_2^2)} \tag{9b}$$

$$\Omega_t = \delta_1^{-1}(\psi_{2t} - \varepsilon^v_{2t} - \delta_2\Phi_t). \tag{9c}$$

Note that there is no speed of adjustment in this system for the variable factor, x_{2t}, as there is for the quasi-fixed input, x_{1t}. The reason is that partial adjustment is not present here for the variable factor since there are no installation or planning costs associated with changing the level of the variable factor input. Thus there is no adjustment parameter appearing in the row two, column two entry of the parameter matrix above. Equivalently, the variable factor is always at its optimal level (determined by the level of the quasi-fixed state variable, the factor input prices, and the exogenous shocks) since adjustment is complete and instantaneous for this factor input. Inspection of this coefficient matrix shows that it has reduced rank. It's rank is two and there are now two cointegrating vectors in the system corresponding to the two choice variables that are present in the optimization problem solved by the firm. The question then arises as to how to normalize the cointegrating matrix in this system.

Normalization is statistically arbitrary and therefore economic theory must be used to choose a suitable normalization of cointegrating vectors. Indeed the adjustment and cointegrating matrices only have interesting economic content once this normalization is imposed. If there were one choice variable in this cointegrated system, it would be easy to choose a normalizing transformation since it would only make sense to attach a unit coefficient to the choice (state) variable. This would be consistent with the implications of economic theory since choice variables, in equilibrium, are determined by the exogenous parameters in an optimization problem. The issue here is somewhat more complicated because unit coefficients can be attached to coefficients in both cointegrating vectors attached to either or both inputs in production. On the basis of previous analysis, we would expect that the adjustment matrix should contain the speed of adjustment as it did above, suggesting that we should choose a normalizing transformation so as to achieve this result. This suggests normalizing β_{11} to unity, the coefficient in the first row and column of the cointegrating matrix associated with the quasi-fixed state variable x_{1t}. It will be instructive to consider two normalizations, each of which has $\beta_{11} = 1$.

In the first normalization, set $\beta_{11} = 1 = \beta_{12}$, the normalization used in Rossana (1995, p. 15) so that coefficients attached to x_{1t} are set to unity in each cointegrating vector. The resulting adjustment and cointegrating matrices are given below.

$$\alpha = \begin{bmatrix} \phi_1 - 1 & 0 \\ 0 & -\delta_2\phi_1\delta_1^{-1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 1 \\ 0 & \delta_1(\delta_2\phi_1)^{-1} \\ \delta_1(\delta_0\delta_1 - \delta_2^2)^{-1} & \delta_1(\phi_1 - 1)[\phi_1(\delta_0\delta_1 - \delta_2^2)]^{-1} \\ -\delta_2(\delta_0\delta_1 - \delta_2^2)^{-1} & (\delta_0\delta_1 - \phi_1\delta_2^2)[\delta_2\phi_1(\delta_0\delta_1 - \delta_2^2)]^{-1} \end{bmatrix}$$

The results from this choice of normalization are unappealing for the following reasons.

The parameters of the cointegrating matrix are intended to capture the long-run equilibrium behavior of the firm. That equilibrium is the solution of a static profit maximization problem involving the optimal choices of the two inputs in production. This is due to the fact that adjustment costs disappear in the equilibrium of this model because installation costs depend upon net investment in stocks and the latter are zero in equilibrium. So we would expect to find the parameters of the long-run factor input demand functions appearing in these cointegrating vectors. Under this particular normalization, those parameters do not directly appear in both cointegrating vectors, appearing in only the first column of the cointegrating matrix. Three elements of the second cointegrating vector contain the adjustment parameter, ϕ_1, which should not appear in the long-run demands that are sought. Therefore to obtain the long-run factor demand coefficients in applied work requires using these parameters to solve for the coefficients of interest.[5] Thus

[5] Given empirical estimates of this normalized cointegrating matrix, it might be possible to take the parameters from this set of cointegrating vectors and solve them to derive estimates of the long-run factor demand coefficients. This approach would require that it be possible to uniquely identify all relevant parameters needed to construct the coefficients of interest. As a general rule, it seems unlikely that identification could be achieved for all parameters that would be required to derive the coefficients of long-run demands in this manner.

it would be desirable to seek an alternative normalization that would cause these long-run factor demand parameters to appear directly in the cointegrating matrix, thus avoiding the need for an applied econometrician to manipulate estimated elements of the cointegrating vectors. It will now be shown that a second normalization can be found that will generate cointegrating vectors with elements consistent with static profit maximization by the firm.

To show this, set $\beta_{11} = 1 = \beta_{22}$ so that coefficients attached to x_{1t} and x_{2t} are set to unity in each cointegrating vector respectively. The resulting adjustment and cointegrating matrices are as follows.

$$\alpha = \begin{bmatrix} \phi_1 - 1 & 0 \\ -\delta_2\phi_1\delta_1^{-1} & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \delta_1(\delta_0\delta_1 - \delta_2^2)^{-1} & -\delta_2(\delta_0\delta_1 - \delta_2^2)^{-1} \\ -\delta_2(\delta_0\delta_1 - \delta_2^2)^{-1} & \delta_0(\delta_0\delta_1 - \delta_2^2)^{-1} \end{bmatrix}$$

Now if we form $\beta' X_t = 0$, we obtain the static factor demands for each input, displaying negative own-factor price effects and symmetric (though unrestricted) cross-factor price effects. In addition, the adjustment matrix continues to have the speed of adjustment for the quasi-fixed factor in the element occupying its first row and column. Further, there is a zero coefficient in the row one, column two element of the adjustment matrix reflecting the fact that x_{2t} is a variable factor input and, as a result, its level has no impact on quasi-fixed factors (see Lucas (1967, p. 83)). The row two, column one element of the adjustment matrix captures the impact of the quasi-fixed factor input (again unrestricted in sign due to the specification of the technology) upon the variable factor input. This normalization is clearly to be preferred to the previous alternative because of its consistency with the implications of static profit maximization. This illustrates the fact that multiple cointegrating vectors arise in systems with more than one choice variable incorporated into the error correction system, whether or not these choice variables are quasi-fixed. There will be one cointegrating vector for every choice variable present in the optimization problem.

Next we take up the question of how this normalization scheme carries over into a problem with multiple state variables, that is, where there are multiple inputs in production that are subject to installation costs.

2.3 Multiple State Variables

The normalization that was shown to produce cointegrating vectors corresponding to the static theory of the firm can be generalized to the case where the firm holds an arbitrary number of quasi-fixed inputs. Before turning to this case, it is useful to look somewhat further at the case just considered for it provides a precise guide to the general case described below.

Towards this end, rewrite the coefficient matrix in (8) as

$$\begin{aligned} \Pi &= \begin{bmatrix} \Lambda_{11} - I_2 & \Lambda_{12} \\ 0_2 & 0_2 \end{bmatrix}, \Lambda_{11} = \begin{bmatrix} \phi_1 & 0 \\ -\delta_2\phi_1\delta_1^{-1} & 0 \end{bmatrix}, \\ \Lambda_{12} &= (\delta_0\delta_1 - \delta_2^2)^{-1} \begin{bmatrix} \delta_1(\phi_1 - 1) & -\delta_2(\phi_1 - 1) \\ -\delta_2(\phi_1 - 1) & -(\delta_0\delta_1 - \phi_1\delta_2^2) \end{bmatrix}. \end{aligned}$$

where I_2 and 0_2 are identity and zero matrices, respectively, of order two. Observe that the preferred normalization, illustrated above, leads to the following adjustment and cointegrating matrices.

$$\alpha\beta' = \begin{bmatrix} \Lambda_{11} - I_2 \\ 0_2 \end{bmatrix} \begin{bmatrix} I_2 & (\Lambda_{11} - I_2)^{-1}\Lambda_{12} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} - I_2 & \Lambda_{12} \\ 0_2 & 0_2 \end{bmatrix} = \Pi$$

Thus the preferred normalization can be illustrated rather simply when written in this way. It will be seen shortly that this is the preferred manner of normalizing the cointegrating matrix when there are multiple state variables present in an optimization model.

Now consider the model of the firm discussed above but with one additional component. Here it will be assumed that the input x_{2t} is quasi-fixed because it now is the case that there are installation costs attached to this input, just as there are for the input x_{1t}. With this additional component, this problem has two state variables (generalization to an arbitrary number of state variables is straightforward and will be considered below). The optimization problem to be solved is to maximize

$$E_0 \sum_{t=0}^{\infty} \gamma^t \left\{ y_t + (\psi_{1t} - v_{1t})x_{1t} + (\psi_{2t} - v_{2t})x_{2t} - (1/2)(x_{1t} - x_{1t-1})^2 - (1/2)(x_{2t} - x_{2t-1})^2 \right\} \quad (10)$$

where the production function is defined as it was previously. As should be clear from the analysis above, it is inessential to simplify this problem by eliminating the parameter preceding the installation cost terms in the objective function (that is, setting $\rho = 1$ in the first problem studied above). This simplification has the further advantage that the Euler equations for this problem can be cast in a manner consistent with Hansen and Sargent (1981). The Euler conditions describing the optimal choices of the two productive inputs are given below.

$$E_t \left\{ \left[\begin{bmatrix} \delta_0 & \delta_2 \\ \delta_2 & \delta_1 \end{bmatrix} + \begin{bmatrix} 1-\gamma L^{-1} & 0 \\ 0 & 1-\gamma L^{-1} \end{bmatrix} \begin{bmatrix} 1-L & 0 \\ 0 & 1-L \end{bmatrix} \right] \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \right\}$$
$$= - \begin{bmatrix} v_{1t} - \psi_{1t} \\ v_{2t} - \psi_{2t} \end{bmatrix} \tag{11}$$

Each of these Euler equations may be interpreted with the usual marginal cost equals discounted marginal benefit description of the firm's behavior in choosing its inputs in production.

The investment demand equations can be derived with the same methods used in the previous model although the addition of a second state variable makes the solution of this problem a much more formidable task. The investment demand equations will each display partial adjustment because each input in production is quasi-fixed. Optimal investment in each input in production will again depend upon the entire future sequences of discounted factor prices and technology shocks. The Weiner-Kolmogorov prediction formula may be applied to these future sequences, assuming that it is the optimal forecasting formula used by the firm.

From the work of Lucas (1967) and Hansen and Sargent (1981), it is well known that the optimal investment demand schedules arising from this optimizing model will form a multivariate flexible accelerator model where investment in each quasi-fixed input depends upon all factor input prices, its own lagged level, and the lagged levels of all other quasi-fixed inputs in production. If the factor prices obey driftless random walks as they did above, then this investment demand system may be written as

$$\begin{bmatrix} x_t \\ v_t \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0_2 & I_2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ v_{t-1} \end{bmatrix} + \begin{bmatrix} \kappa_{1t} \\ \kappa_{2t} \end{bmatrix}$$

where κ_{it} are vectors comprised of linear combinations of iid technology shocks and thus the elements of these vectors are I(0). The matrices Λ_{ij} are non-diagonal and capture the impacts of the state variables and factor input prices on the investment demands of the firm. The error correction form of this expression is

$$\begin{bmatrix} \Delta x_t \\ \Delta v_t \end{bmatrix} = \begin{bmatrix} \Lambda_{11} - I_2 & \Lambda_{12} \\ 0_2 & 0_2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ v_{t-1} \end{bmatrix} + \begin{bmatrix} \kappa_{1t} \\ \kappa_{2t} \end{bmatrix}. \tag{12}$$

The error correction system in (12) is quite similar to (8). The essential difference is that it is no longer true that x_{2t} is a variable factor input so that, in the earlier model, there was no adjustment speed associated with this factor input; now there is an own-adjustment parameter associated with this input. Beyond this difference, the structure of the two problems is identical. Given these similarities, it should be no surprise then that we can proceed with normalization in this problem just as we did above. We can define the adjustment and cointegrating matrices to be

$$\alpha = \begin{bmatrix} \Lambda_{11} - I_2 \\ 0_2 \end{bmatrix}, \beta = \begin{bmatrix} I_2 \\ ((\Lambda_{11} - I_2)^{-1}\Lambda_{12})' \end{bmatrix}.$$

The cointegrating matrix contains the parameters of the reduced form arising from this optimization model.[6]

Generalization to the case of an arbitrary number of quasi-fixed inputs is clear; with n quasi-fixed factor inputs, the cointegrating matrix will have rank n and the parameter matrix attached to the lagged levels of time series in the system will be of order 2n x 2n. There will be n stochastic trends in the model, one corresponding to each I(1) factor price confronting the firm. Using the preferred normalization scheme, the cointegrating matrix contains the parameters of the static factor demand equations and the adjustment matrix will contain all of the neoclassical adjustment parameters that arise in the problem.[7] The reader is referred to Rossana (1995, pp. 12-15) for further discussion. This illustrates how crucial it is for normalization to be done in the proper

[6] Unlike the textbook case of the reduced form of an econometric model, the cointegrating matrix does not contain parameters for predetermined (lagged) values of the choices variables because the cointegrating matrix pertains to an equilibrium situation where choice variables are constant over time.

[7] There is one important qualification to this statement about the adjustment matrix. This matrix will also contain the parameters from the stochastic processes obeyed by any unobservable shocks in the model. See Rossana (1998, pp. 430-434) for a proof of this property. Thus estimating the elements of the adjustment matrix will not necessarily provide estimates of structural parameters of economic interest unless the analyst observes that the error correction model that is estimated contains no more lags than are implied by the economic model in use.

way so that the relevant parameter matrix, attached to the lagged levels of time series in an error correction VAR, decomposes into two parameter matrices that have useful economic information.

These results extend the findings in Rossana (1998) where it was shown that the speed of adjustment appears in the adjustment matrix as suggested by the work of Johansen and Juselius (1990). Here we find that all neoclassical adjustment parameters will appear in the adjustment matrix of an error correction model under a suitable normalization of the cointegrating matrix. The cointegrating matrix, once normalized, will have elements capturing the responses of equilibrium stocks to factor input prices (equivalently, it contains the parameters of the reduced form emerging from the economic model). The product of these two matrices will reveal the effects of factor input prices on the dynamic demand schedules for these quasi-fixed inputs in production. Using the appropriate normalization of the cointegrating matrix, the parameter matrix attached to the lagged levels of economic time series neatly decomposes into two matrices summarizing all relevant economic information contained in a neoclassical investment demand system. Thus an applied econometrician has a ready way to recover estimates of all of the parameters of economic interest associated with the steady-state equilibrium and disequilibrium behavior in a neoclassical investment demand model by estimating the elements of the adjustment and cointegrating matrices using statistical methods with desirable statistical properties. The crucial step is that this estimation strategy will be feasible as long as the appropriate normalizing transformation of the cointegrating matrix is carried out. This transformation is easy to do and will always be available in any optimizing model of a firm or household. Finally, this normalization is appropriate in other contexts as well.[8]

With these results in place, we may now turn to the question of the circumstances under which an optimization framework can produce a cointegrated time series system manifesting mixed degrees of integration in its component time series.

[8] For example, suppose that the VAR containing cointegrating relationships applies to an ordinary market setting where two magnitudes are set in the market and where there are exogenous variables positioning the demand and supply curves in the market. Normalizing on the endogenous variables results in a cointegrating matrix containing the parameters of the reduced form for the market, thereby providing a set of parameters with interesting economic content. The normalized adjustment matrix would continue to contain parameters associated with the dynamics in the market (adjustment speeds and the like). Thus this normalization appears to be the right choice in any standard market setting.

3 Mixed Degrees of Integration

It is helpful to begin with the case where exogenous forcing variables are I(2). This will be useful for subsequent results when forcing variables are of differing degrees of integration. It will also permit us to study cointegration in the context of higher order degrees of integration.

3.1 A One State Variable Model with I(2) Forcing Variables

Consider the model described above where the firm holds one quasi-fixed input as well as a variable factor input. Factor input prices will now be assumed to obey the independent I(2) processes

$$\Delta^2 v_{it} = \varepsilon_{it}^v, \varepsilon_{it}^v \sim iid(0, \sigma_{vi}^2).$$

Euler equations need not be repeated here because they will be identical to those given previously for this problem and the same methods can be used to derive the time series implications of this model as summarized by the VAR that arises. This VAR is given below.

$$\begin{aligned}
\begin{bmatrix} x_t \\ v_t \end{bmatrix} &= \Pi_1 \begin{bmatrix} x_{t-1} \\ v_{t-1} \end{bmatrix} + \Pi_2 \begin{bmatrix} x_{t-2} \\ v_{t-2} \end{bmatrix} + \begin{bmatrix} \omega_{1t} \\ \omega_{2t} \end{bmatrix} && \text{(13a)} \\
\Pi_1 &= \begin{bmatrix} \Pi_1^{11} & \Pi_1^{12} \\ 0_2 & 2I_2 \end{bmatrix}, \Pi_1^{11} = \begin{bmatrix} \phi_1 & 0 \\ -\delta_2\phi_1\delta_1^{-1} & 0 \end{bmatrix} && \text{(13b)} \\
\Pi_1^{12} &= [(1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)]^{-1} \bullet && \text{(13c)} \\
& \begin{bmatrix} \delta_1(1-\phi_1)(\phi_1\gamma - 2) & \delta_2(1-\phi_1)(2-\phi_1\gamma) \\ \delta_2(1-\phi_1)(2-\phi_1\gamma) & \delta_1^{-1}[\delta_2^2\phi_1[2-\gamma(1+\phi_1)]] - 2(1-\phi_1\gamma)\delta_0 \end{bmatrix} \\
\Pi_2 &= \begin{bmatrix} 0_2 & \Pi_2^{12} \\ 0_2 & -I_2 \end{bmatrix}, && \text{(13d)} \\
\Pi_2^{12} &= [(1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)]^{-1} \bullet \\
& \begin{bmatrix} \delta_1(1-\phi_1) & \delta_2(\phi_1 - 1) \\ \delta_2(\phi_1 - 1) & \delta_1^{-1}[\delta_2^2(1-\phi_1) + (1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)] \end{bmatrix} && \text{(13e)}
\end{aligned}$$

The elements of the disturbance vector, ω_{it}, are I(0) because they are weighted sums of the iid disturbances that are present in the model. As is evident from this system, the VAR is considerably more complex when compared to that arising from the case where forcing variables were I(1). We now have higher order lags in this VAR because the factor input prices are I(2) and the parameters in these coefficient matrices continue to be highly nonlinear.

For the purpose of comparison to the previous model, it is useful to rewrite this VAR as

$$\begin{bmatrix} \Delta x_t \\ \Delta v_t \end{bmatrix} = -\Pi_2 \begin{bmatrix} \Delta x_{t-1} \\ \Delta v_{t-1} \end{bmatrix} + (\Pi_1 + \Pi_2 - I_4) \begin{bmatrix} x_{t-1} \\ v_{t-1} \end{bmatrix} + \begin{bmatrix} \omega_{1t} \\ \omega_{2t} \end{bmatrix} \tag{14a}$$

$$-\Pi_2 = \begin{bmatrix} 0_2 & -\Pi_2^{12} \\ 0_2 & I_2 \end{bmatrix}, \Pi = \Pi_1 + \Pi_2 - I_4 = \begin{bmatrix} \Pi_1^{11} - I_2 & \Pi_1^{12} + \Pi_2^{12} \\ 0_2 & 0_2 \end{bmatrix} \tag{14b}$$

$$\Pi_1^{11} = \begin{bmatrix} \phi_1 & 0 \\ -\delta_2 \phi_1 \delta_1^{-1} & 0 \end{bmatrix}, \Pi_1^{12} + \Pi_2^{12} = (\delta_0 \delta_1 - \delta_2^2)^{-1} \begin{bmatrix} \delta_1(\phi_1 - 1) & -\delta_2(\phi_1 - 1) \\ -\delta_2(\phi_1 - 1) & \phi_1 \delta_2^2 - \delta_0 \delta_1 \end{bmatrix} \tag{14c}$$

$$\Pi_2^{12} = [(1 - \phi_1 \gamma)(\delta_0 \delta_1 - \delta_2^2)]^{-1} \bullet \begin{bmatrix} \delta_1(1 - \phi_1) & \delta_2(\phi_1 - 1) \\ \delta_2(\phi_1 - 1) & \delta_1^{-1}[\delta_2^2(1 - \phi_1) + (1 - \phi_1 \gamma)(\delta_0 \delta_1 - \delta_2^2)] \end{bmatrix}. \tag{14d}$$

There are a number of observations to be made about this VAR.

The dynamics displayed by the I(2) VAR will generally differ from the VAR for the I(1) case because longer lags appear in the present model and there are lagged differences appearing in the I(2) VAR. But some of the characteristics of the I(1) case discussed earlier carry over into the current I(2) model. The parameter matrix attached to the lagged levels of time series, related to cointegration in the levels of the time series, has two eigenvalues that are zero implying that it has rank two, just as in the previous model, because there are two rows of zeros in the matrix. Because of its reduced rank, it may again be factored into the product of two matrices. It is easy to see that the order of differencing has no impact on the rank of this matrix by looking at the "error correction" form of the univariate models

$$\Delta v_{it} = \Delta v_{it-1} + \varepsilon_{it}^v.$$

Under either degree of differencing considered, there are zero coefficients attached to the lagged levels of the time series in these univariate stochastic processes. For this reason, and because the factor prices are assumed to obey univariate stochastic processes that are independent of all other series in the economic environment, it is clear that the degree of integration of these forcing variables is irrelevant to the rank of the parameter matrix Π. By implication, the rank of the adjustment and cointegrating matrices for the levels of the data are independent of the degree of integration of these series.

Second, note that the parameter matrix, $\Pi = \Pi_1 + \Pi_2 - I_4$, preceding the lagged levels of the time series, is in fact identical to the I(1) case above. As a result, the adjustment and cointegrating matrices, associated with cointegrating relationships in the levels of the time series in the system, are identical to the I(1) case under any selected normalization scheme and so may be interpreted as they were above. The cointegrating matrix continues to have rank two. But now, the choice variables are I(2). To see this, follow Dolado, Galbraith, and Banerjee (1991, p. 925), and use the dynamic demand for the quasi-fixed factor, given by

$$\begin{aligned} x_{1t} &= \phi_1 x_{1t-1} + \frac{\delta_1(1-\phi_1)(1-\phi_1\gamma)}{\delta_0\delta_1 - \delta_2^2} \bullet \\ &\quad \left[\sum_{i=0}^{\infty}(\phi_1\gamma)^i E_t(\psi_{1t+i} - v_{1t+i}) + \frac{\delta_2}{\delta_1}\sum_{i=0}^{\infty}(\phi_1\gamma)^i E_t(v_{2t+i} - \psi_{2t+i})\right]. \end{aligned}$$

If we use the Weiner-Kolmogorov prediction formula in (5) to eliminate the discounted expectations in this demand schedule, this will give

$$x_{1t} = \phi_1 x_{1t-1} + \frac{\delta_1(1-\phi_1)(1-\phi_1\gamma)}{\delta_0\delta_1 - \delta_2^2}\left[\psi_{1t} - \frac{1-\phi_1\gamma L}{(1-\phi_1\gamma)^2}v_{1t} + \frac{\delta_2}{\delta_1}\left[\frac{1-\phi_1\gamma L}{(1-\phi_1\gamma)^2}v_{2t} - \psi_{2t}\right]\right]. \tag{15}$$

Since there are two unit roots in the factor input prices, the above expression shows that there will also be two unit roots in the state variable x_{1t} as long as the factor input prices are not themselves cointegrated.[9] Differencing this equation twice results in

$$(1-\phi_1 L)\Delta^2 x_{1t} = \frac{\delta_1(1-\phi_1)(1-\phi_1\gamma)}{\delta_0\delta_1 - \delta_2^2}\left[\Delta^2\psi_{1t} - \frac{1-\phi_1\gamma L}{(1-\phi_1\gamma)^2}\varepsilon_{1t}^v + \frac{\delta_2}{\delta_1}\left[\frac{1-\phi_1\gamma L}{(1-\phi_1\gamma)^2}\varepsilon_{2t}^v - \Delta^2\psi_{2t}\right]\right],$$

an expression showing that the choice variable is linearly related to I(0) shocks and obeys a mixed

[9] If it happens that an expression such as this one emerges from an economic model where a linear combination of exogenous variables is CI(d,b), then the choice variable will be I(d-b), the order of integration of the linear combination of the exogenous series.

ARIMA stochastic process ($x_{1t} \sim$ ARIMA(1,2,2)). The necessary condition for the optimal choice of the variable factor, x_{2t}, will similarly show that this input is also I(2) because there is a linear relationship between the choice variables and the factor price, v_{2t}. It is clear that this same result will arise for a model with an arbitrary number of quasi-fixed inputs.

As noted by Johansen (1995), it is useful to write the VAR for the I(2) case in a somewhat different form for the purposes of cointegration testing. To study the properties of these cointegrating relationships, write the VAR in (13) the equivalent form

$$\begin{bmatrix} \Delta^2 x_t \\ \Delta^2 v_t \end{bmatrix} = \Gamma \begin{bmatrix} \Delta x_{t-1} \\ \Delta v_{t-1} \end{bmatrix} + \Pi \begin{bmatrix} x_{t-2} \\ v_{t-2} \end{bmatrix} + \begin{bmatrix} \omega_{1t} \\ \omega_{2t} \end{bmatrix}, \Gamma = \Pi_1 - 2I_4, \Pi = \Pi_1 + \Pi_2 - I_4.$$

As observed in Johansen (1995), cointegration in I(2) models may involve both the levels and the differences of the time series, a property of time series systems referred to as multicointegration by Granger and Lee (1989).[10] Thus cointegration testing involves a joint null hypothesis about reduced-rank parameter matrices associated with both the levels and the differences of time series. As in the I(1) case, $\Pi = \alpha\beta'$ is one possible reduced rank restriction regarding the levels of time series and, to account for cointegration in the differences, Johansen (1995) shows that it may also be the case that the matrix $\alpha'_\perp \Gamma \beta_\perp$ has reduced rank. In this expression, $\alpha_\perp$ and $\beta_\perp$ are orthogonal complements to the matrices α and β respectively.[11]

The parameter matrix attached to the lagged levels of the time series is identical to the matrix discussed above for the I(1) case and so there will be cointegration involving the levels of the time series in the system with the same interpretation of the cointegrating vectors under the preferred normalization scheme discussed earlier. It is shown in the appendix that $\alpha'_\perp \Gamma \beta_\perp$ does have reduced rank; in fact, $\alpha'_\perp \Gamma \beta_\perp = 0$. In addition, Johansen (1995, p. 31) shows that stationary combinations of the levels and differences of the time series in I(2) VARs are given by

$$\beta' \begin{bmatrix} x_t \\ v_t \end{bmatrix} + \left(\alpha \left(\alpha'\alpha\right)^{-1}\right)' \Gamma \begin{bmatrix} \Delta x_t \\ \Delta v_t \end{bmatrix}$$

[10] The analyses of multicointegration by Granger and Lee (1989) and Johansen (1992, 1995) do not provide an economic motivation for multicointegration, confining their studies to statistical and econometric issues.

[11] For any p x r matrix α that is of full rank, the orthogonal complement is defined as a p x (p-r) matrix $\alpha_\perp$, also of full rank, such that $\alpha'\alpha_\perp = 0$.

and, in the appendix, it is shown that the matrix preceding the differences of the series in the above expression is nonzero. Unlike the I(1) case, it is no longer true that $\beta' X_t$ is I(0). Thus cointegration is a more complex process in the I(2) case as compared to the I(1) models above since cointegration now occurs between the levels and differences of the time series. But it is plausible, for the following reasons, that cointegration should involve the differences of the data.

In the I(1) models, exogenous factor prices were assumed to be driftless random walks. Whether or not the drift term is zero, shocks have no effect on the growth of these exogenous magnitudes, changing only their levels. Thus in setting the optimal relationship between choice variables and these factor prices, the firm will maintain optimal relationships between the levels of inputs and factor prices but shocks have no effect on the growth rates of the firm's choice variables because the growth rates of factor prices are independent of these shocks. In the I(2) case however, shocks change both the level and growth rates of exogenous factor prices and so the firm will find it optimal to change both the growth rates and the levels of its inputs because of the impact of shocks to factor input prices.

The matrix preceding the differences of the time series apparently does not have the same intuitive appeal as the parameter matrix associated with the levels of the time series (see the appendix). But there is one interesting aspect of this matrix; it contains the speed of adjustment for the quasi-fixed input x_{1t}. Adjustment speeds are normally associated with disequilibrium behavior since, in previous research, these parameters traditionally appear in dynamic demand schedules describing the adjustment of quasi-fixed inputs to their static demand functions. In the I(2) case, these parameters are now a part of equilibrium behavior. Since the equilibrium growth rate of the choice variables is affected by realizations of exogenous shocks, the magnitude of this response depends in part on the speed of adjustment of inputs subject to installation costs.

These results show that as long as the exogenous forcing variables are I(2), so too will all of the choice variables in the optimization problem. This is the standard case considered by Engle and Granger (1987). Thus to rationalize differences in the number of unit roots evident in the time series in this system, we will need to impose some changes in the economic structure of this problem and it will now be shown that, in part, separability assumptions must be used to permit these differences in the time series behavior of the choice variables in this VAR.

3.2 Systems with I(1) and I(2) Series

There is one (trivial) sense in which we can get both I(1) and I(2) behavior in the context of this optimization model and that would be if both factor prices are strictly exogenous and one is I(2) and one I(1). But the more interesting issue to be examined is under what circumstances the choice variables can display differing degrees of integration and, to address this issue, begin by assuming that $v_{1t} \backsim I(2)$ and $v_{2t} \backsim I(1)$.

As is evident from (15), there will continue to be two unit roots in the time series process for x_{1t} because this series is linearly related to an I(2) forcing variable. But if we were to difference (15) twice, we would now observe that there were additional moving average elements in the time series process for x_{1t} because $v_{2t} \backsim I(1)$. Similarly the variable factor input will continue to be I(2) because of its linear relationship to the quasi-fixed factor input. Further, notice that, in a model with an arbitrary number of exogenous forcing variables, if there is just one forcing series that is both I(2) and exogenous, its effects will be propagated throughout the model, causing all choice variables to be I(2). To see this, consider the Euler equations for the one state variable problem rewritten here for convenience.

$$
\begin{aligned}
\gamma\rho E_t x_{1t+1} - [\delta_0 + \rho(1+\gamma)]x_{1t} + \rho x_{1t-1} - \delta_2 x_{2t} &= v_{1t} - \psi_{1t} \\
\delta_1 x_{2t} + \delta_2 x_{1t} &= \psi_{2t} - v_{2t}
\end{aligned}
$$

Optimality criteria similar to these would arise for all of these additional quasi-fixed and variable factor inputs. Each condition would contain all exogenous, variable, and quasi-fixed choice variables. Therefore as long as separability assumptions are not imposed in the economic model, all choice variables would be linearly related to the I(2) forcing variable, thus picking up the unit root characteristics of this I(2) series. Put differently, a consequence of the fact that optimizing behavior causes Euler equations to hold, all choice variables will display the highest degree of integration observed in the exogenous forcing variables. To prevent this property from arising, it is necessary that we break the relationships that tie together the choice variables in this optimization problem. That is, an additional element must be added to the model to allow choice variables to obey time series processes with differing numbers of unit roots.

To see what restrictions must be imposed to achieve mixed degrees of integration among the choice variables, we can use the Euler equations, given above, that are necessary for the solution of the intertemporal problem solved by the firm. A separability restriction must be imposed that would permit each choice variable to display differing degrees of integration. For illustration, continue to assume that the factor price associated with the quasi-fixed factor, v_{1t}, continues to be I(2) and that the factor price for the variable factor input is I(1). The required separability assumption that is needed is simply to set $\delta_2 = 0$. Euler equations for the optimization problem now reduce to

$$\begin{aligned} \gamma\rho E_t x_{1t+1} - [\delta_0 + \rho(1+\gamma)]x_{1t} + \rho x_{1t-1} &= v_{1t} - \psi_{1t} \\ \delta_1 x_{2t} &= \psi_{2t} - v_{2t}. \end{aligned}$$

There is now a very simple demand schedule for the variable factor input x_{2t}. It continues to be static as it was previously but now there is no connection between the variable and quasi-fixed factor inputs and so the time series properties of v_{1t} will have no impact on x_{2t}. Thus if $v_{2t} \sim I(1)$, so too will x_{2t}. The quasi-fixed factor, x_{1t}, continues to obey a dynamic demand schedule. It will now be

$$x_{1t} = \widetilde{\phi}_1 x_{1t-1} + \frac{\delta_1(1-\widetilde{\phi}_1)(1-\widetilde{\phi}_1\gamma)}{\delta_0\delta_1}\sum_{i=0}^{\infty}(\widetilde{\phi}_1\gamma)^i E_t(\psi_{1t+i} - v_{1t+i}).$$

Applying the Weiner-Kolmogorov prediction formula to this expression, we obtain

$$x_{1t} = \widetilde{\phi}_1 x_{1t-1} + \frac{(1-\widetilde{\phi}_1)(1-\widetilde{\phi}_1\gamma)}{\delta_0}\left[\psi_{1t} - \frac{1-\widetilde{\phi}_1\gamma L}{(1-\widetilde{\phi}_1\gamma)^2}v_{1t}\right]$$

a relationship between the I(0) shock, ψ_{1t}, the quasi-fixed factor and its associated factor price, v_{1t}. The univariate characteristics of x_{1t} may be determined by differencing this expression twice to obtain

$$\left(1-\widetilde{\phi}_1 L\right)\Delta^2 x_{1t} = \frac{(1-\widetilde{\phi}_1)(1-\widetilde{\phi}_1\gamma)}{\delta_0}\left[\Delta^2\psi_{1t} - \frac{1-\widetilde{\phi}_1\gamma L}{(1-\widetilde{\phi}_1\gamma)^2}\varepsilon_{1t}^{v}\right],$$

an exercise revealing that $x_{1t} \backsim I(2)$. Note that it is still true that $x_{1t} \sim$ ARIMA(1,2,2) so that the orders of the autoregressive and moving average processes evident in the state variable are unaffected by the imposition of the separability restriction on the parameter δ_2. The variable factor, however, is now IMA(1,1); this factor input now has only one unit root in its stochastic process and has fewer moving average components than it did in the I(2) case above.[12]

Concerning the VAR that arises, inspection of (14c) shows that the VAR continues to have lagged differences appearing in the model but it has been simplified by the elimination of a unit root in the series v_{2t} (for example, two coefficients are eliminated from the parameter matrix Π_2^{12}). It has already been established that the adjustment and cointegrating matrices, associated with the levels of the time series, are independent of the degree of integration in the constituent time series. Setting $\delta_2 = 0$, the reduced rank coefficient matrix, associated with the levels of the time series (recall that cointegration may still involve the differences of these series as it did in the I(2) case above), the adjustment and cointegrating matrices reduce to

$$\Pi = \begin{bmatrix} \widetilde{\phi}_1 - 1 & 0 & \left(\widetilde{\phi}_1 - 1\right)\delta_0^{-1} & 0 \\ 0 & -1 & 0 & -\delta_1^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \alpha = \begin{bmatrix} \widetilde{\phi}_1 - 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \delta_0^{-1} & 0 \\ 0 & \delta_1^{-1} \end{bmatrix}.$$

Thus we find that the adjustment matrix, conditional on using the preferred normalization advocated above, continues to contain the neoclassical speed of adjustment (although its magnitude differs from the previous model because of the separability restriction that is now in effect), and the resulting cointegrating matrix is considerably simplified by the elimination of cross-factor price effects in the long-run static factor input demands.[13] The parameters of the long-run factor demands now are comprised of the coefficients obtained from simple static conditions for each individual input in production where each input is chosen only in relation to its own factor price.

The implication of this analysis is that elements of the optimization structure, namely separability assumptions, can be used to justify differing numbers of unit roots apparent in the choice variables of an optimization framework, conditional on evidence that exogenous forcing variables also display mixed degrees of integration. That is, exogenous forcing variables must display dif-

[12] There is no autoregressive component to the stochastic process for the variable factor input because, by assumption, there are no adjustment costs associated with this input in producton.

[13] It is well-known that this stable characteristic root, as well as the unstable one in this problem, are functions of the exogenous parameters contained in this economic problem. Thus by changing the magnitude of δ_2, we change the magnitude of the resulting characteristic roots.

fering degrees of integration (this is a necessary, but not sufficient, condition for differing degrees of integration in the choice variables) and, if this is the case, choice variables may do so as well in the presence of the appropriate separability assumptions. These structural assumptions must ultimately be justified on empirical grounds, just as it must be empirically justified in assuming the separability between adjustment costs and the marginal products of factor inputs in production. One method of providing this justification would be to estimate Euler equations to try to verify the unit root findings that $\delta_2 = 0$. But the analysis shows that there must be a connection between the forcing variables and the choice variables. That is, the finding of differing degrees of integration of the exogenous time series, and the resulting differences in the degree of integration of the choice variables, could enable an applied econometrician to determine the precise separability restrictions needed to be consistent with unit root test results. In the example just considered, the observation that x_{1t} and v_{1t} are each I(2) and that x_{2t} and v_{2t} are each I(1) requires the restriction that $\delta_2 = 0$.[14] Absent separability assumptions, the standard case to be found is that assumed by Engle and Granger (1987) where all time series in the system manifest the same degree of integration, a result requiring that all of the forcing variables have the same degree of integration.

Therefore, the analysis indicates that pre-testing economic data for unit roots (and perhaps the use of other diagnostic procedures such as inspection of sample autocorrelations), a standard preliminary step in using cointegration methods, can reveal information about the economic structure under examination. If an applied econometrician were to observe differing degrees of integration in data that is claimed to be consistent with an optimizing framework, then this evidence should lead to a search for the structural parameter restrictions that can justify the findings of preliminary unit root tests and other diagnostic procedures.

In the next section, empirical examples are given that illustrate how tests for unit roots can be used to draw out information on the structure of an economic problem using time series that might be contained in a study of the demand for money and inventory investment. In previous studies on money demand, there has been some evidence of mixed degrees of integration among the time series used in applied work. Thus an example from this literature provides a good illustration of some of the results in this section where it was shown that Euler equations can be used to infer

[14] The problem illustrated here is sufficiently simple that there is only one parameter restriction that permits choice variables to differ in their degrees of integration. In more complex problems, these separability restrictions are unlikely to be unique, thus requiring that Euler equations be estimated to sort out the various separability possibilities.

the unit root properties of choice variables conditional on the degree of integration of exogenous variables in an optimizing model. The inventory examples are drawn from the literature on the production smoothing model of inventories which is the standard framework for the analysis of inventory behavior. These test results provide some evidence on the possible separability of the employment decision in production from inventory decisions.

4 Empirical Illustrations

This section contains two examples where economic time series display differing degrees of integration. The first set of results are drawn from the literature on the demand for money. Cointegration techniques have been used in the past to establish the characteristics of the long-run demand for money. Examples of this line of research are Johansen and Juselius (1990), Hoffman and Rasche (1991), Hafer and Jansen (1991), and Stock and Watson (1993). The second set of results come from the literature on inventory investment. The production smoothing model of the firm has been the traditional framework for the analysis of the inventory decisions of firms (see Blinder and Maccini (1991) for a survey of research on inventory investment). Time series methods have been used to test the implications of the production smoothing model and there has been some controversy about the suitability of this model of the firm. Unit root tests will be used in both examples to see if the data are consistent with optimizing behavior by economic agents.

To carry out unit root testing, a test for multiple numbers of unit roots will be required because we will be interested in seeing if any of the time series examined have two unit roots. Dickey and Pantula (1987) provide a test procedure, suitable for present purposes, that is an application of the standard Dickey-Fuller ADF test and is thus quite simple to carry out. This test is done in the following way.

It is necessary to choose the maximum number of unit roots that will be permitted in the analysis and it will be assumed here that this maximum number is two. The test requires using OLS to estimate

$$\Delta y_t = \eta_0 + \eta_1 y_{t-1} + \sum_{i=1}^{k} \lambda_i \Delta y_{t-i} + \chi_t.$$

The lag length, k, must be chosen in conducting the test and the statistical criterion of Hannan and Quinn (1979) will be used to select lag lengths since this method consistently estimates the order of an autoregression.[15] The maximum value of k is set to three years of lagged observations. The test is carried out by first testing for the maximum number of unit roots and, if this null hypothesis is rejected, the test is then carried out for a reduced number of unit roots. So here we begin by defining y_t as the first difference of the relevant data series. Critical values for the test can be found in Fuller (1976, p. 373).

4.1 Money Demand

This section provides evidence on the number of unit roots in the data frequently used in applied work on the demand for money. The following data series will be used. The Federal Reserve Board provides data on M1 and M2, as well as data for the interest rate on three month Treasury Bills. The Department of Commerce provides data on constant-dollar Gross Domestic Product and its associated implicit price deflator. The data are measured at quarterly frequency in natural logarithms (with the exception of the nominal interest rate) over the period 1959:1 to 2001:2 and they are adjusted for seasonality (although the nominal interest rate has no apparent seasonal fluctuations and is thus unadjusted for seasonality). Two measures of the money stock will be tested because there has been considerable professional debate in the past about the appropriate measure of the money stock that should be used for the purposes of economic analysis. So it is useful to see if there are any observed differences in the time series behavior of alternative money stock measures from the point of view of this paper. It will also be useful to test real balances for their unit root properties because it has been argued in previous research that real balances, rather than nominal balances, is the relevant choice variable for households. This will permit us to see if changing the choice variable affects any conclusions that might be drawn from unit root tests.

Table 1 provides unit root test results for these series. For the null hypothesis of double unit roots in the time series, the results indicate that, with the exception of the price level, all series reject a double unit root null hypothesis at the one percent level. The test statistic for the price level, however, does not reject this double unit root null hypothesis.[16] These results confirm the findings of Stock and Watson (1993, p. 818) for the price level but do not agree with their results

[15] Unit root tests generally retain their validity when data-based methods are used in testing for unit roots. See Hall (1994) for further discussion.

[16] It should be pointed out that there is a competing view regarding the degree of integration of the price level. The analysis here restricts the degree of differencing to integer values but Baillie, Chung, and Tieslau (1996) suggest that the price level may in fact obey a fractionally integrated stochastic process.

for M1; they suggest that M1 may have two unit roots. As for test results for the presence of single unit roots in all series but the price level, the unit root null hypothesis cannot be rejected for any series at the one percent level.

If it is believed that the time series in this system are generated by the solution of some sort of optimization problem regarding the choice of nominal cash balances by a representative household, then there will be an Euler equation emerging from this economic model describing the optimal choice of this magnitude; in this condition, there should appear the level of cash balances, the price level, and the other variables appearing in the optimizing model. Based upon the analysis above, the variable chosen by an economic agent, nominal cash balances, should manifest the highest degree of differencing displayed by its determinants. Since the price level appears to have two unit roots, so too should the money stock but, under either definition of money, the unit root test statistics are inconsistent with the theoretical results above. The results of Stock and Watson (1993) are consistent with the results in this paper since they find that both the price level and nominal cash balances are I(2). But these empirical results could also arise if there is a cointegrating relationship between real GDP, the price level, and the nominal interest rate such that a linear combination of these series is I(1), causing nominal cash balances to be I(1) as well. Providing empirical evidence on this possibility, thus reconciling these findings about the number of unit roots in the time series, and providing an economic interpretation of a possible cointegrating restriction in the data, is beyond the scope of this paper but, for an applied econometrician examining these data series, there is certainly some reason to be skeptical about the hypothesis that this data arises from an optimizing framework of a representative household without a convincing structural explanation for the number of unit roots evident in the data.

If it is believed that real balances are the appropriate choice variable for this optimization problem as has been sometimes argued, then the unit root tests are consistent with the results in this paper because all series are I(1) and thus there need not be any cointegrating relationship between the exogenous variables that is needed to rationalize unit root tests applied to real balances. Thus how the choice variable is defined in this context is crucial to the conclusions that might be drawn about the consistency between optimizing behavior by economic agents and these economic time series.

4.2 Inventories

In the production smoothing model of the firm, it is assumed that the firm buffers output demand fluctuations by holding a stock of finished goods. With rising marginal costs of production, the firm will wish to avoid costly changes in production in the face of demand fluctuations but it will wish to hold a planned stock of inventories positively related to its expected sales. In choosing its stock of inventories, an Euler equation will arise involving the stock of inventories and sales (see, e.g., Rossana (1998, p. 431)) and that condition clearly shows that the stock of inventories should contain the same number of units roots as found in the stochastic process for the firm's sales. Thus unit root testing can reveal information regarding the consistency of inventories and sales data with the Euler equation arising from the production smoothing model of the firm.

But if we want to use unit root tests to find evidence of separability in the optimization structure of an economic problem, additional choice variables must be tested for unit roots. The production smoothing model has been extended in a number of studies to incorporate factor input decisions into the production smoothing model (see Humphreys, Maccini, and Schuh (2001) for a recent example). Input decisions, such as choosing the optimal labor force, will be conditional on the stock of inventories (Bils and Kahn (2000, p. 463)) and so the stock of labor should reflect the unit root properties of inventories which, in turn, are determined by the stochastic process for sales.[17] Thus we can apply unit root tests to a measure of the labor force to see if they suggest the presence of any separability characteristics in the optimizing framework.

The data used for these tests are the following. Labor input is measured by the stock of production workers and is provided by the Bureau of Labor Statistics. The Bureau of Economic Analysis, U. S. Department of Commerce, provides constant-dollar finished goods inventories and sales. The data are at monthly frequency, seasonally adjusted, and they are in natural logarithms, covering 1967:1 through 1997:12. Data are used for four industries from the durable goods manufacturing sector: Lumber and Wood Products (SIC 24), Furniture and Fixtures (SIC 25), Paper and Allied Products (SIC 26), and Industrial Machinery (SIC 35). More will be said below about the industries chosen for testing.

[17] Equivalently, the firm will evaluate the marginal products of factor inputs using the shadow value of the stock of finished goods because productive inputs are used to produce output into the stock of inventories. Since this shadow value is tied to the stock of inventories, it should have the unit root characteristics of output inventories and sales.

Table 2 contains the unit root results. Sales are I(1) in all four industries as double unit roots are decisively rejected in all cases but a single unit root null cannot be rejected even at the ten percent significance level. As a result, inventories should also be I(1). The test results imply that two unit roots in the stock of inventories are decisively rejected in all industries. Three industries cannot reject one unit root in inventories but, in one industry (SIC 24), inventories appear stationary, a finding that is inconsistent with the implications of the production smoothing model of the firm. Turning to production workers, double unit roots are rejected in all industries as are single unit roots. Thus the stock of production workers is stationary in these industries and so the employment decision appears to be separable from the inventory-sales relationships built into the production smoothing model. To confirm these findings, unit root tests should be applied to measures of labor costs, such as real wages. If real wages are I(0), these findings ought to lead to a search for the precise set of parameter restrictions allowing the employment decision to be separable from decisions on inventories and sales.[18]

A final point is worth noting about these tests. All of these industries are part of the durable manufacturing sector where unfilled orders data exist. As a result, there is the possibility of misspecification in these tests because there is no allowance made for production to order in these industries despite the fact that this type of production may occur. In the industries from the nondurable sector where unfilled orders do not exist, unit roots tests for the same time series (not shown here) reveal that all time series are I(1), thus indicating no separability in production from the inventory-sales decisions of firms. Thus it would appear prudent to be somewhat cautious about interpreting the findings in Table 2 in view of the potential for misspecification in durable goods industries because of the possible presence of production to order in these industries.

5 Concluding Remarks

Engle and Granger (1987) provided a definition of cointegrated time series where all series in a time series system have the same degree of integration. Test procedures and estimation methods have been developed for time series systems that display mixed degrees of integration. This paper

[18] It should also be noted that it will not matter if labor services are measured by manhours, rather than production workers, since hours appear to be I(0) in these industries. Based upon inspection of the hours and employment data and unit root tests that are not provided in this paper, manhours appear to be I(0).

provides conditions under which cointegrated time series systems can arise with differing degrees of integration in its component series.

In order to provide conditions under which a VAR can contain time series with varying degrees of integration, two other issues must be addressed. The time series systems that are studied here contain multiple cointegrating vectors and a method must be found showing how to normalize cointegrating vectors in a manner that provides coefficients of economic interest in the resulting cointegrating and adjustment matrices. In addition, the VARs that are examined involve I(2) series and so it is necessary that we observe how the number of unit roots in economic time series affects the error correction VAR that arises from the optimizing behavior of economic agents.

Regarding the normalization of a cointegrating matrix with rank above unity, it is shown in this paper that there is a simple transformation of the cointegrating matrix that produces normalized cointegrating vectors containing the parameters arising from the solution of a static profit maximization problem by the firm. This transformation attaches a unit coefficient associated with each state or choice variable in each cointegrating vector. This implies that it will always be possible to estimate the parameters of the static demands for these choice variables with any of the available estimators of cointegrating vectors having desirable statistical properties, without the need to recover estimates of any underlying individual parameters in the optimization problem, parameters that it may not be possible to identify in applied work. The adjustment matrix resulting from this normalization is found to contain all of the neoclassical adjustment parameters familiar from the adjustment cost literature of investment. These results should apply to any optimizing framework describing the behavior of a representative firm or household.

As for the effect of the degree of differencing of the data in an error correction VAR, it is shown that the rank of the parameter matrix attached to the lagged levels of the time series is unaffected by the degree of integration of the data. By implication, the rank and the elements of the adjustment and cointegrating matrices are also unaffected by the degree of integration. Thus the long-run level relationships maintained by the data, reflecting the restrictions implied by economic theory, are intact under any degree of integration. But in the I(2) case, cointegration generally arises between the levels and the differences of the time series, defined in previous research as a multicointegrated time series system, essentially because shocks affect the growth rates of exogenous variables and so the firm adjusts the growth rates of its choice variables as a result.

Finally, it is shown in this paper that choice variables in an optimization model can display differing degrees of integration if the forcing or exogenous variables in the problem also display mixed degrees of integration. For this to be true, it is necessary that separability assumptions be invoked, permitting the choice variables to have differing degrees of integration. These structural assumptions must be justified on empirical grounds but, absent any such restrictions, all choice variables will display a degree of integration equal to the highest degree displayed by any exogenous forcing variable. If the forcing variables are all I(1), so too will be all of the choice variables. Thus the case considered by Engle and Granger (1987) is that consistent with no separability restrictions in the optimizing framework and where all exogenous forcing variables in the problem are of the same degree of integration.

The results indicate that pre-testing the data for unit roots, a preliminary step in applying cointegration methods, can reveal information to the applied econometrician about the structure of the optimization problem that has presumably generated the data under study. If all exogenous series are I(1) and I(0) say, and if the choice variables are not all I(1) as well, this finding is a signal about the presence of structural restrictions that must be present in the optimizing model to permit such empirical findings. Empirical evidence is provided in this paper from the inventory investment literature suggesting that, in selected two-digit industries, labor input decisions are separable from the inventory-sales choices by firms, a finding typically not implied by the standard production smoothing model of the firm. But these unit root test results may be difficult to rationalize. Another example is given in the paper that is drawn from the money demand literature where it is shown that all series in a standard money demand exercise, whether money is defined as M1 or M2, seem to be I(1) but the price level appears to be I(2). Unless it is argued that real balances are the appropriate choice variable by economic agents, it is hard to see why money should obey a stochastic process that differs from the price level regarding its degree of integration since money is held ostensibly for the goods it will buy, suggesting that the time series processes obeyed by the money stock and prices should be the same. Nonetheless the results in this paper indicate that unit root testing can provide evidence about the economic structure of optimization models, a fact that may be useful in future applied studies using tests for unit roots in economic time series.

References

Baillie, R. T., C.-F. Chung, and M. A. Tieslau (1996). Analysing inflation by the fractionally integrated ARFIMA-GARCH model. *Journal of Applied Econometrics 11*(1), 23–40.

Bils, M. and J. A. Kahn (2000). What inventory behavior tells us about business cycles. *American Economic Review 90*(3), 458–481.

Blinder, A. S. and L. J. Maccini (1991). The resurgence of inventory research: What have we learned? *Journal of Economic Surveys 5*(4), 291–328.

Dickey, D. A. and S. G. Pantula (1987). Determining the ordering of differencing in autoregressive processes. *Journal of Business & Economic Statistics 5*(4), 455–461.

Dolado, J., J. W. Galbraith, and A. Banerjee (1991). Estimating intertemporal quadratic adjustment cost models with integrated series. *International Economic Review 32*(4), 919–936.

Engle, R. F. and C. W. J. Granger (1987). Co-integration and error correction: Representation, estimation, and testing. *Econometrica 55*(2), 251–276.

Fuller, W. A. (1976). *Introduction to Statistical Time Series*. John Wiley & Sons.

Granger, C. W. J. and T. H. Lee (1989). Investigation of production, sales and inventory relationships using multicointegration and non-symmetric error correction models. *Journal of Applied Econometrics 4*(Supplement), S145–S159.

Hafer, R. W. and D. W. Jansen (1991). The demand for money in the united states: Evidence from cointegration tests. *Journal of Money, Credit, and Banking 23*(2), 155–168.

Hall, A. R. (1994). Testing for a unit root in time series with pretest data-based model selection. *Journal of Business & Economic Statistics 12*(4), 461–470.

Hannan, E. J. and B. G. Quinn (1979). The determination of the order of an autoregression. *Journal of the Royal Statistical Society, Series B 41*(2), 190–195.

Hansen, L. P. and T. J. Sargent (1981). Linear rational expectations models for dynamically interrelated variables. In R. E. Lucas and T. J. Sargent (Eds.), *Rational Expectations and Econometric Practice*, Number 1, pp. 127–156. Minneapolis: The University of Minnesota Press.

Hoffman, D. L. and R. H. Rasche (1991). Long-run income and interest elasticities of money demand in the united states. *Review of Economics and Statistics 73*(4), 665–674.

Humphreys, B. R., L. J. Maccini, and S. Schuh (2001). Input and output inventories. *Journal of Monetary Economics 47*(2), 347–375.

Johansen, S. (1988). Statistical analysis of cointegration vectors. *Journal of Economic Dynamics and Control 12*(2-3), 231–254.

Johansen, S. (1991). Estimation and hypothesis testing of cointegration vectors in gaussian vector autoregressive models. *Econometrica 59*(6), 1551–1580.

Johansen, S. (1992). A representation of vector autoregressive processes integrated of order 2. *Econometric Theory 8*(2), 188–202.

Johansen, S. (1995). A statistical analysis of cointegration for i(2) variables. *Econometric Theory 11*(1), 25–59.

Johansen, S. and K. Juselius (1990). Maximum likelihood estimation and inference on cointegration-with applications to the demand for money. *Oxford Bulletin of Economics and Statistics 52*(2), 169–210.

Lucas, R. E. (1967). Optimal investment policy and the flexible accelerator. *International Economic Review 8*(1), 78–85.

Rossana, R. J. (1995). Technology shocks and cointegration in quadratic models of the firm. *International Economic Review 36*(1), 5–17.

Rossana, R. J. (1998). On the adjustment matrix in error correction models. *Journal of Monetary Economics 42*(2), 427–444.

Sargent, T. J. (1987). *Macroeconomic Theory* (Second ed.). Academic Press, Inc.

Stock, J. H. and M. W. Watson (1993). A simple estimator of cointegrating vectors in higher order integrated systems. *Econometrica 61*(4), 783–820.

A Cointegration in the I(2) Case

This appendix provides information establishing that, in the I(2) case, there will be cointegration involving the levels and the differences of the series contained in the optimization model solved by the firm. When factor input prices are I(2), the VAR is

$$
\begin{aligned}
\begin{bmatrix} \Delta^2 x_t \\ \Delta^2 v_t \end{bmatrix} &= \Gamma \begin{bmatrix} \Delta x_{t-1} \\ \Delta v_{t-1} \end{bmatrix} + \Pi \begin{bmatrix} x_{t-2} \\ v_{t-2} \end{bmatrix} + \begin{bmatrix} \omega_{1t} \\ \omega_{2t} \end{bmatrix} \\
\Gamma &= \Pi_1 - 2I_4 = \begin{bmatrix} \Pi_1^{11} - 2I_2 & \Pi_2^{12} \\ 0_2 & 0_2 \end{bmatrix} \\
\Pi_1^{11} &= \begin{bmatrix} \phi_1 & 0 \\ -\delta_2 \phi_1 \delta_1^{-1} & 0 \end{bmatrix} \\
\Pi_2^{12} &= [(1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)]^{-1} \bullet \\
& \begin{bmatrix} \delta_1(1-\phi_1) & \delta_2(\phi_1 - 1) \\ \delta_2(\phi_1 - 1) & \delta_1^{-1}[\delta_2^2(1-\phi_1) + (1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)] \end{bmatrix}.
\end{aligned}
$$

It must first be shown that the expression $\alpha_\perp' \Gamma \beta_\perp$ has reduced rank (cointegration in the levels of the series was discussed previously). If α is the adjustment matrix, its orthogonal complement is denoted by $\alpha_\perp$ and, in the model at hand, these are defined as

$$
\alpha = \begin{bmatrix} I_2 \\ 0_2 \end{bmatrix}, \alpha_\perp = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}
$$

and the coefficient matrix, Γ, is defined above. It is easy to show that $\alpha_\perp' \Gamma = 0$ so that $\alpha_\perp' \Gamma \beta_\perp = 0$. There is cointegration in this model involving both the differences and the levels of the time series.

To find the cointegrating vectors associated with the differences of the series, it is necessary (Johansen (1995, p. 31)) that we evaluate the expression

$$
\left(\alpha \left(\alpha'\alpha\right)^{-1}\right)' \Gamma.
$$

Using the adjustment matrix conditional upon the preferred normalization of the cointegrating matrix for the levels, the cointegrating matrix for the differences is given below.

$$\begin{aligned}
\left(\alpha\left(\alpha'\alpha\right)^{-1}\right)'\Gamma &= \begin{bmatrix} \Psi_{11} & 0 & \Psi_{13} & \Psi_{14} \\ \Psi_{21} & 2 & \Psi_{23} & \Psi_{24} \end{bmatrix} \\
\Psi_{11} &= \left(\phi_1 - 2\right)\left(\phi_1 - 1\right)^{-1} \\
\Psi_{13} &= \left(2 - \phi_1\gamma\right)\delta_1[(1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)]^{-1} \\
\Psi_{14} &= \left(2 - \phi_1\gamma\right)\delta_2[(1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)]^{-1} \\
\Psi_{21} &= \delta_2\phi_1\delta_1^{-1}\left(\phi_1 - 1\right)^{-1} \\
\Psi_{23} &= \left(\phi_1\gamma - 2\right)\delta_2[(1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)]^{-1} \\
\Psi_{24} &= \left[\delta_2^2\delta_1^{-1}\left[2\phi_1\left(\phi_1\gamma - 2\right) + \phi_1\gamma\right] + 2\delta_0(1-\phi_1\gamma)\right][(1-\phi_1\gamma)(\delta_0\delta_1 - \delta_2^2)]^{-1}
\end{aligned}$$

Table 1[19]

Tests for Unit Roots in Money Demand

Series	d = 2	d = 1
Money Stock - M2	-5.127*	-1.394
k	0	1
Money Stock - M1	-4.318*	-0.580
k	1	2
Real GDP	-6.648*	-1.015
k	1	2
GDP Deflator	-2.016	-
k	4	-
Three Month T-Bill Rate	-5.704*	-2.157
k	6	7
Real Money Stock - M1/P	-5.843*	-0.747
k	0	1
Real Money Stock - M2/P	-5.465*	-1.109
k	0	1

[19] In the table, d refers to the degree of differencing, k is lag length, and (*) denotes rejection of the null hypothesis at the one percent level.

Table 2[20]

Tests for Unit Roots in the Production Smoothing Model

Industry	Series	d = 2	d = 1
SIC 24	Sales	-23.240*	-1.948
	k	0	1
	Inventories	-17.099*	-3.769*
	k	0	1
	Production Workers	-9.409*	-3.056*
	k	0	2
SIC 25	Sales	-11.327*	-0.748
	k	2	3
	Inventories	-20.353*	-1.880
	k	0	0
	Production Workers	-7.078*	-3.579*
	k	1	2
SIC 26	Sales	-10.419*	-1.591
	k	2	3
	Inventories	-21.323	-0.898
	k	0	1
	Production Workers	-6.382*	-4.256*
	k	3	3
SIC 35	Sales	-7.163*	1.203
	k	3	4
	Inventories	-8.80*	-0.613
	k	2	3
	Production Workers	-5.904*	-3.236
	k	1	4

[20] See Table 1 for definitions of table entries.