Boundary Hölder and $L^p$ Estimates for local solutions of the tangential Cauchy-Riemann equation

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1 Introduction

In this paper we study the local solvability of the tangential Cauchy-Riemann equation $\partial_b$ on an open neighborhood $\omega$ of a point $z_0 \in M$ when $M$ is a generic CR manifold of real codimension $k$ in $\mathbb{C}^n$, where $1 \leq k \leq n - 1$. We assume that $M$ is $q$-concave near $z_0$ (see Definition 2.2.1). Our method is to first derive an homotopy formula for $\partial_b$ in $\omega$ when $\omega$ is the intersection of $M$ with a strongly pseudoconvex domain. The homotopy formula gives a local solution operator for any $\partial_b$-closed form on $\omega$ without shrinking. We obtain Hölder and $L^p$ estimates up to the boundary for the solution operator.

Let $C^\alpha(\omega)$, $0 < \alpha < 1$, be the space of Hölder continuous functions of order $\alpha$ in $\omega$. We use $C_{n,s}^\alpha(\omega)$ to denote the space of $(n, s)$-forms with $C^\alpha(\omega)$ coefficients. The norm in $C_{n,s}^\alpha(\omega)$ is defined to be the sum of $C^\alpha(\omega)$ norm of each coefficient. We also denote by $L^p_{(n,s)}(\omega)$ the space of $(n, s)$-forms with $L^p(\omega)$ coefficients, $1 \leq p \leq \infty$. The norm in $L^p_{(n,s)}(\omega)$ is denoted by $\| \|$ for $(n, s)$-forms. Our main results are the following:

**Theorem 1.0.1.** (Homotopy formula for $\partial_b$.) Let $M$ be a strictly $q$-concave generic CR manifold in $\mathbb{C}^n$ and $z_0 \in M$. Let $\Omega$ be a strictly pseudoconvex domain containing $z_0$ in $\mathbb{C}^n$ with $C^3$ boundary and $\omega = M \cap \Omega$. For any $s$, $n - k - q + 1 \leq s \leq n - k$, there exists a continuous operator $T_{s-1}$ from $C_{n,s}(\omega)$ into $C_{n,s-1}^3(\omega)$ such that for any $f \in C_{n,s}(\omega)$ with $\partial_b f \in C_{n,s+1}(\omega)$,

$$f = \partial_b T_{s-1} f + T_s \partial_b f.$$

**Theorem 1.0.2.** (Hölder and $L^p$ estimates for $\partial_b$.) Let $M$ be a strictly $q$-concave generic CR manifold in $\mathbb{C}^n$ and $z_0 \in M$. Let $\Omega$ be a strictly pseudoconvex domain containing $z_0$ in $\mathbb{C}^n$ with $C^3$ boundary and $\omega = M \cap \Omega$. For any $f \in L^p_{(n,s)}(\omega)$ with $\partial_b f = 0$ in $\omega$, $1 \leq p \leq \infty$ and $n - k - q + 1 \leq s \leq n - k$, there exists an operator $\tilde{T}_{s-1}$ satisfying $\partial_b \tilde{T}_{s-1} f = f$ in $\omega$ and the following estimates hold:

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