A NEW PROOF OF THE CHEEGER-GROMOLL SOUL CONJECTURE AND THE TAKEUCHI THEOREM

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Abstract. Let $M^n$ be a complete, non-compact Riemannian manifold with nonnegative sectional curvature. We derive a new broken flat strip theorem associated with the Cheeger-Gromoll convex exhaustion, in the case when $M^n$ is not diffeomorphic to $\mathbb{R}^n$. This leads to a new proof of the Cheeger-Gromoll soul conjecture without using Perelman’s flat strip theorem.

Using the Cheeger-Gromoll inward equidistant evolution and the Calabi’s barrier surface technique, we also provide a new proof of the Takeuchi Theorem.

§1. A new broken flat strip theorem and applications to the soul conjecture

Suppose that $M^n$ is a $C^\infty$-smooth, complete and non-compact Riemannian manifold with nonnegative sectional curvature. Cheeger-Gromoll [ChG] established a fundamental theory for such a manifold. Among other things, they showed that $M^n$ admits a totally convex exhaustion $\{\Omega_u\}_{u \geq 0}$ of $M^n$, where $\Omega_0 = S$ is a totally geodesic and compact submanifold without boundary. Furthermore, $M^n$ is diffeomorphic to the normal vector bundle of the soul $S$. In particular, if the soul $S$ is a point, then $M^n$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$.

The Cheeger-Gromoll soul conjecture asserts that “if a complete and non-compact Riemannian manifold $M^n$ has nonnegative sectional curvature and if $M^n$ contains a point $p_0$ where all sectional curvatures are positive, then $M^n$ must be diffeomorphic to the Euclidean space $\mathbb{R}^n$.” This is true if $M^n$ has positive sectional curvature everywhere by the earlier work of Gromoll and Meyer [GrM]. This conjecture was solved by G. Perelman [Per] by his remarkable flat strip theorem. Earlier partial results on the Cheeger-Gromoll soul conjecture were obtained by Marenich, Walschap and Strake, see references in [Per].

If there is a totally geodesic isometric immersion

$$\phi: \mathbb{R} \times [0, \ell] \to M^n$$

$$(s, t) \to \phi(s, t), \quad (1.1)$$

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then \( \phi(\mathbb{R} \times [0, \ell]) \) is said to be an immersed flat strip in \( M^n \). If \( \{\Omega_u\} \) is a Cheeger-Gromoll convex exhaustion, it is known that, for each \( u > 0 \), \( \Omega_u \) is a totally convex (and hence totally geodesic) submanifold with boundary. We let \( \partial\Omega_u \) be the relative boundary of \( \Omega_u \). A flat strip \( \phi(\mathbb{R} \times [0, \ell]) \) is said to be compatible with the Cheeger-Gromoll convex exhaustion \( \{\Omega_u\} \) if each horizontal geodesic \( \phi(\mathbb{R} \times \{t\}) \) of the flat strip is contained in \( \partial\Omega_{u(t)} \) for some \( u(t) > 0 \) where \( t > 0 \).

In this paper, we obtain a different type of flat strip theorem using a different approach.

**Theorem 1.1.** Let \( M^n \) be a complete and non-compact Riemannian manifold with nonnegative sectional curvature. Suppose that \( M^n \) is not diffeomorphic to the Euclidean space \( \mathbb{R}^n \). Then for each \( x \in M^n \) there is a non-trivial broken flat strip \( \Phi = \{\phi_i\}_{1 \leq i \leq N} \) passing through \( x \). Moreover, each flat strip \( \phi_i \) is compatible with the Cheeger-Gromoll exhaustion.

Consequently, then the Cheeger-Gromoll soul conjecture is true.

A result similar to Theorem 1.1 was obtained in [CaS] via a different method. The broken flat strip stated in Theorem 1.1 above is indeed determined by the Cheeger-Gromoll exhaustion in a canonical way, which we now describe in the next subsection

**a. The Cheeger-Gromoll broken geodesics.**

Let us briefly recall the Cheeger-Gromoll convex exhaustion.

**Definition 1.2.** (A Cheeger-Gromoll exhaustion) For a complete and non-compact Riemannian manifold \( M^n \). A Cheeger-Gromoll exhaustion \( \{\Omega_u\}_{u \geq 0} \) has the following properties. There is a partition \( a_0 = 0 < a_1 < ... a_m < a_{m+1} = \infty \) of \( [0, \infty) \) and an exhaustion \( \{\Omega_u\}_{u \geq 0} \) of \( M^n \) such that the following holds:

(1.2.1) \( M^n = \bigcup_{u \geq 0} \Omega_u \). If \( u > a_m \) then \( \dim[\Omega_u] = n \). If \( u \leq a_m \), then \( \dim[\Omega_u] < n \).

(1.2.2) \( \Omega_0 = S \) is the soul of \( M^n \), which is a totally geodesic \( C^\infty \)-smooth compact submanifold without boundary.

(1.2.3) If \( u > 0 \), \( \Omega_u \) is a totally convex, compact subset of \( M^n \) and hence \( \Omega_u \) is a compact submanifold with a \( C^\infty \)-smooth relative interior. Furthermore, \( \dim(\Omega_u) = k_u > 0 \) and \( \Omega_u \) has a non-empty \( (k_u - 1) \)-dimensional relative boundary \( \partial\Omega_u \).

(1.2.4) For any \( u_0 \in [a_j, a_{j+1}] \) and \( 0 \leq u \leq u_0 - a_j \), the family \( \{\Omega_{u_0-u}\}_{u \in [0, u_0-a_j]} \) is given by the inward equidistant evolution:

\[
\Omega_{u_0-u} = \{x \in \Omega_{u_0} | d(x, \partial\Omega_{u_0}) \geq u\}. \tag{1.2}
\]

(1.2.5) If \( u > a_m \) then \( u - a_m = \max\{d(x, \partial\Omega_u) | x \in \Omega_u\} \). If \( 0 \leq j \leq m - 1 \) then \( a_{j+1} - a_j = \max\{d(x, \partial\Omega_{a_{j+1}}) | x \in \Omega_{a_{j+1}}\} \) and hence \( \dim[\Omega_{a_j}] < \dim[\Omega_{a_{j+1}}] \) for \( j \geq 0 \).
For each compact convex subset $\Omega \subset M^n$, we let $U_\epsilon(\Omega) = \{x \in M^n | d(x, \Omega) < \epsilon \}$. Its cut-radius is given by $\delta_\Omega = \sup \{\epsilon |$ there is a unique nearest point projection $P_\Omega : U_\epsilon(\Omega) \to \Omega\}$. 

For each given $T > a_m$, there is an estimate for the cut-radius of convex subsets $A$ in $\Omega_T$. For each $x \in M^n$, we let $\text{Inj}_{M^n}(x)$ be the injectivity radius of $M^n$ at $x$. Similarly, let $\text{Inj}_{M^n}(A) = \sup \{\text{Inj}_{M^n}(x) | x \in A\}$.

**Lemma 1.3.** ([ChG, p425]) Let $\Omega \subset \Omega_T$ be a connected, convex and compact subset in a Riemannian manifold $M^n$ with nonnegative sectional curvature and let $K_0 = \max \{K(x) | x \in \Omega_{T+1}\}$, $\text{Inj}_{M^n}(\Omega_T)$ be the upper bound of sectional curvature on $\Omega_{T+1}$ and $S$ be as above. Suppose that $\dim(\Omega_T) = n$. Then the subset $A$ has cut-radius bounded below by

$$\delta_A \geq \delta_0(T) = \frac{1}{4} \min \{\text{Inj}_{M^n}(\Omega_T), \frac{\pi}{\sqrt{K_0}}, 1\},$$

where $\delta_0(T)$ is independent of choices of $A$ with $A \subset \Omega_T$.

**Proof.** Suppose there are two points $q_1 \neq q_2$ in $A$ and $p \in U_{\delta_0(T)}(A)$ such that $d(p, q_1) = d(p, q_2) = d(p, A) = \ell$. Let $\sigma_i$ the geodesic segment from $q_i$ to $p$. By the first variation formula, the angle between $\sigma_i'(0)$ and the vector tangent to $A$ is at least $\frac{\pi}{2}$. Comparing the inner angles of geodesic triangles of the same side lengths in the round sphere of curvature $K_0$, one obtain the angles $0 < \angle_{q_i}(p, q_1) < \frac{\pi}{2}$, a contradiction. $\square$

For given $T > a_m$, we choose a partition $u_0 = 0 < u_1 < ... < u_N = T$ such that $\{u_i\}_{0 \leq i \leq m}$ is a subset of $\{u_j\}$; and $\Omega_{u_j} \subset U_{\delta_0(T)}(\Omega_{u_{j-1}})$.

**Definition 1.4.** (The Cheeger-Gromoll broken geodesic) Let $\{\Omega_u\}$ be a Cheeger-Gromoll convex exhaustion. For each $T > a_m$, let $\delta_0(T)$ and the partition $\{u_j\}$ be as above. If $\mathcal{P}_{j-1} : \Omega_{u_j} \to \Omega_{u_{j-1}}$ is the nearest point projection, then for each $x \in \Omega_T$, we let $x_N = x$, $x_{N-1} = \mathcal{P}_{N-1}(x)$, ..., $x_{j-1} = \mathcal{P}_{j-1}(x_j)$ for $j = N, N - 1, ..., 1$. The broken geodesic $\sigma = \{\sigma_j\}$ joining $x_0, x_1, ..., x_N = x$ is called a Cheeger-Gromoll broken geodesic from $x_0 \in S$ to $x_N = x$.

In Definition 1.4 above, the points $\{x_j\}$ are not necessarily distinct.

**b. The monotone property of tangent cones of the Cheeger-Gromoll exhaustion**

Suppose that $\sigma = \{\sigma_j\}$ is a Cheeger-Gromoll broken geodesic from the soul to $x$ as above. Assume $x_{i-1} \neq x_i$ for some $i$. We consider the geodesic segment $\sigma_i : [0, \ell_i] \to M^n$ from $x_{i-1}$ to $x_i$. If $x_i \in \partial \Omega_{w_i}$ for some $w_i > 0$, we let $u_i(t) = w_i - d(\sigma_i(t), \partial \Omega_{w_i})$. In this case, we have $\sigma_i(t) \in \partial \Omega_{u_i(t)}$. 


In what follows, we let $T^+_y(\Omega)$ be the tangent cone of $\Omega$ at $y$:

$$T^+_y(\Omega) = \{ \bar{v} \in T_y(M^n) \mid \limsup_{t \to 0^+} \frac{d(Exp_y(t\bar{v}), \Omega)}{t} = 0 \}.$$ 

The following monotone theorem plays an important role in the proof of Theorem 1.1.

**Theorem 1.5.** Let $\mathbb{P}_\sigma$ be the parallel transport along a Cheeger-Gromoll broken geodesic $\sigma = \{\sigma_j\}$ and $u_i(t)$ be as above. Then the tangential cones of $\{\Omega_u\}$ are monotone under the parallel transport along a Cheeger-Gromoll broken geodesic:

$$\mathbb{P}_\sigma[T^+_\sigma_{t_0}(\Omega_{u_i(t_0)})] \subset T^+_\sigma_{t_1}(\Omega_{u_i(t_1)})$$

for any pair $0 \leq t_0 \leq t_1 \leq \ell_i$.

For the proof of Theorem 1.5, it is sufficient to consider the normal cones of $\Omega_u$ instead.

**Definition 1.6.** (1) Let $\Omega$ be a convex subset of $M^n$ and $\sigma : [0, \ell] \to M^n$ be a geodesic with $\sigma(0) \in \Omega$. The geodesic $\sigma$ is called at least normal to $\Omega$ if the angle between $\sigma'(0)$ and any vector tangent to $\Omega$ is at least $\pi/2$.

(2) The normal cone of $\Omega$ in $M^n$ is defined by

$$N^+(\Omega, M^n) = \{(p, \bar{v}) \mid p \in \Omega, d(Exp_p(t\bar{v}), \Omega) = t|\bar{v}|, \text{ for } 0 \leq t|\bar{v}| < \delta_\Omega \}.$$ 

Similarly, one can define the relative normal cone $N^+(\Omega_u, \text{int}(\Omega_u + \epsilon))$.

(3) (Minimal normal vector) Let $\Omega_u$, $\Omega_u + \epsilon$ and $N^+(\Omega_u, \text{int}(\Omega_u + \epsilon))$ be as above. Let $\sigma_{(p, \bar{v})} : [0, \epsilon] \to M^n$ be a geodesic given by $\sigma_{(p, \bar{v})}(t) = Exp_p(t\frac{\bar{v}}{|\bar{v}|})$, where $\bar{v} \neq 0$. If $\sigma_{(p, \bar{v})}$ is a length-minimizing geodesic from $p \in \Omega_u$ to $\partial \Omega_{u+\epsilon}$, then $\bar{v}$ is called a minimal normal vector in $N^+_p(\Omega_u, \text{int}(\Omega_u + \epsilon))$.

**Proof of Theorem 1.5.** We need to choose $\epsilon$ sufficiently small so that (1) there is a nearest point projection $\mathcal{P} : \text{int}(\Omega_{u+\epsilon}) \to \Omega_u$; and (2) $\Omega_u = \{x \in \Omega_{u+\epsilon} \mid d(x, \partial \Omega_{u+\epsilon}) \geq \epsilon\}$ holds. We first find $j$ so that $a_j \leq u < a_{j+1}$ for some $0 \leq j \leq m$. Let $T = u + a_m + 1$ and $\delta_0(T)$ be given by Lemma 1.3. It follows from a result of Yim that there is a constant $C_T$ such that, for $0 \leq a < b \leq T$, we have

$$\max\{d(x, \Omega_a) \mid x \in \Omega_b\} \leq C_T(b - a), \quad (1.3)$$

see [Y2, Theorem A.5(3)]. In what follows, we always choose

$$0 < \epsilon = \epsilon_u < \min\{[a_{j+1} - u], \frac{\delta_0(T)}{2C_T}\}. \quad (1.4)$$
With such a choice of $\epsilon = \epsilon_u$ by (1.4), the geometry of $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ is determined by its minimal normal vectors. In fact, it follows from Corollary 1.5 and Proposition 1.7 of [Y1] that $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ is contained in the convex hull of minimal normal vectors.

It is sufficient to verify the following assertion.

**Assertion 1.5.1.** Let $\bar{v}_0$ be a minimal normal vector at $\sigma(t_0) \in \Omega_{u(t_0)}$. Then the angle between $P_\sigma(\bar{v}_0)$ and $T_{\sigma(t_1)}(\Omega_{u(t_1)})$ is less than or equal to $\frac{\pi}{2}$.

We will identify the outward normal cone $\mathcal{N}^+(\Omega_{u(t_0)}, \Omega_{u(t_1)})$ with another cone as follows. Observe that the nearest projection map $P$ is not necessarily injective. We consider a subset $A = \Omega_{u(t_1)} \cap \mathcal{P}^{-1}(\sigma(t_0))$. It follows from a result of [Y1] that $B = T_{\sigma(t_0)}(W) = \mathcal{N}^+(\Omega_{u(t_0)}, \Omega_{u(t_1)}) = A$. For each minimal unit outward normal vector $\bar{v}_0$, we let $\psi(t) = \text{Exp}_{\sigma(t_0)}([t - t_0]v_0)$. Then $\psi : [t_0, t_1] \to \Omega_{t_1}$ is a length minimizing geodesic from $\sigma(t_0)$ to $\partial \Omega_{t_1}$.

Let $p_1 = \psi(t_1)$. We consider geodesic triangle $\triangle_{\sigma(t_0), \sigma(t_1), p_1}$. Since $\Omega_{u(t_1)}$ is convex, we see that the inner angle of $\triangle_{\sigma(t_0), \sigma(t_1), p_1}$ at $p_1$ is $\leq \frac{\pi}{2}$, i.e., $\angle_{p_1}(\sigma(t_1), \sigma(t_0)) \leq \frac{\pi}{2}$.

On the other hand, the sectional curvature is nonnegative, by the classical angle comparison theorem of Alexandroff and Toponogov for $K \geq 0$, we know that the sum of inner angles of $\triangle_{\sigma(t_0), \sigma(t_1), p_1}$ is greater than or equal to $\pi$, (e.g., see [Pe] ). It follows that

$$\angle_{\sigma(t_1)}(p_1, \sigma(t_0)) + \angle_{\sigma(t_0)}(p_1, \sigma(t_1)) \geq \pi - \angle_{p_1}(\sigma(t_1), \sigma(t_0)) \geq \frac{\pi}{2}.$$ 

We consider the Fermi coordinate system around the geodesic $\sigma$. Namely, we choose an orthonormal frame $\{e_1, e_2, ..., e_m\}$ of $T_{\sigma(t_0)}(\Omega_{u(t_0)})$ such that $e_m = \sigma'(0)$. Suppose that $\{\vec{E}_1(t), ..., \vec{E}_m(t)\}$ is a parallel transport of $\{e_1, e_2, ..., e_m\}$ along $\sigma$. The Fermi coordinate system is given by the following map:

$$F : \mathbb{R}^{m-1} \times [0, \ell] \to M^n$$

$$(x_1, ..., x_{m-1}, x_m) \to \text{Exp}_{\sigma(x_m)}[\sum_{k=1}^{m-1} x_k \vec{E}_k(s)]$$

It is well-known that the derivative of $F$ at the zero section is equal to identity, i.e.,

$$F_*|_{(0, ..., 0, x_m)} = \text{id}$$

for all $x_m$. Let $\mathbb{R}^2_0 \subset T_{\sigma(t_0)}(M^n)$ be a 2-dimensional subspace spanned by $\sigma'(\ell)$ and the vector $\Psi'(t_0)$. Similarly, if $\phi : [0, t_1 - t_0] \to M^n$ be a geodesic of unit speed from $\sigma(t_1)$ to $p_1$, then we consider a 2-plane $\mathbb{R}^2_1 \subset T_{\sigma(t_1)}(M^n)$ spanned by $\sigma'(\ell)$ and $\phi'(0)$. Then by (1.5) we have

$$\angle_{\sigma(t_1)}(\mathbb{R}^2_1, P_{\sigma}[\mathbb{R}^2_0]) \leq O(\ell^2).$$
where \( O(\ell^2) \) denotes a term of order 2 in \( \ell \). We also have \( \ell = d(\sigma(t_0), \sigma(t_1)) = O(|t_1 - t_0|) \) by Lemma 1.1 of [Y1] and its proof.

We consider \( \bar{v}(t) = \mathbb{P}_\sigma(\bar{v}_0) \) at \( \sigma(t) \) and an angle function
\[
\beta_v(t) = \angle_{\sigma(t)}(\bar{v}(t), T_{\sigma(t)}(\Omega_u(t))),
\]
where \( u(t) = u(t_1) - d(\sigma(t), \partial \Omega_{u(t_1)}) \) for \( t \in [t_0, t_1] \). It follows from that
\[
\beta_v(t_1) \leq \angle_{\sigma(t_1)}(\bar{v}(t_1), p_1)
= \pi - [\angle_{\sigma(t_1)}(\sigma'(t_1), \bar{v}(t_1)) + \angle_{\sigma(t_1)}(p_1, -\sigma'(t_1))] + O(\ell^2)
= \pi - [\angle_{\sigma(t_0)}(\sigma(t_1), p_1) + \angle_{\sigma(t_1)}(p_1, \sigma(t_0))] + O(\ell^2)
\leq \frac{\pi}{2} + O(\ell^2) = \frac{\pi}{2} + O(|t_1 - t_0|^2).
\]

Recall that \( \bar{v}(t_0) = v_0 \) is a minimal outer normal vector of \( \Omega_{t_0} \) and hence \( \beta_v(t_0) = \frac{\pi}{2} \). It follows that the one-sided derivative of angle function \( \beta_v \) is non-positive, i.e.,
\[
\frac{\partial^+ (\beta_v)}{\partial t} \bigg|_{t_0} \leq 0,
\]
for all minimal normal vector \( \bar{v}_0 \). This completes the proof of Assertion 1.5.1 as well as Theorem 1.5. \( \square \)

c. The proof of the broken flat strip theorem.

**Proof of Theorem 1.1.**

When the soul \( S \) has positive dimension, we choose a geodesic \( c_0 : \mathbb{R} \to S \) with \( c_0(0) = Q_0 \) and \( |c'_0(0)| = 1 \). Let \( \{V_1(s)\} \) be a parallel vector field along \( c_0 \) such that \( V_1(0) = \sigma'_1(0) \). We now consider the following Fermi map.
\[
\phi_1 : \mathbb{R} \times [0, \ell_1] \to M^n
(s, t) \mapsto \text{Exp}_{c_0(s)}(t\bar{v}_0(s))
\]

Using the convexity of \( \{\Omega_u\} \) and the assumption of nonnegative sectional curvature, we will verify the following:

**Claim 1.6.** Let \( M^n, \{\Omega_u\}, \{P_j\}, Q_j, \sigma_j \) and \( \phi_1 \) be as in Definition 1.4 and Definition 1.6 above. Then

(1.6.1) The normal cone bundle \( N^+(S, \Omega_u) \) is invariant under parallel transport along the geodesic \( c_0 \);
(1.6.2) The map \( \phi_1 \) is a totally geodesic isometric immersion;
implies that the \( \gamma \) and \( v \) one computes the vector-valued the second fundamental form \( II \) a parametrization curvature \( \lambda \)

\[ \nabla_{\partial} \frac{\partial}{\partial s} \sigma \equiv \hat{\Sigma}(s, t) \]

we conclude that \( \frac{\partial}{\partial s} \sigma \) is as in Definition 1.2.

Assertion (1.6.1) is a direct consequence of a result of [Y1]. To verify (1.6.2)-(1.6.4), for each \( \hat{s} \in \mathbb{R} \), we let \( \{W_{\hat{s}}(t)\} \) be a parallel vector field along the vertical geodesic

\[ \sigma_{\hat{s}} : t \rightarrow \phi_{1}(\hat{s}, t) \]

with \( W_{\hat{s}}(0) = \frac{\partial \phi_{1}}{\partial s}(\hat{s}, 0) = c_{0}(\hat{s}). \)

It follows from Theorem 1.5 that if \( [\pm W_{\hat{s}}(0)] \in T_{\sigma_{\hat{s}}(0)}(\bar{\Omega}_{u(0)}) = T_{\sigma_{\hat{s}}(0)}(S) \) then \([\pm W_{\hat{s}}(t)] \in T_{\sigma_{\hat{s}}(t)}(\bar{\Omega}_{u(t)}) \) as well, for all \( t \in [0, \ell_{1}] \), where \( u(t) = a_{1} - d(\sigma(t), \partial\Omega_{a_{1}}) \) and \( a_{1} \) is as in Definition 1.2.

Recall that by Cheeger-Gromoll’s convex exhaustion, the inward tangent cone \( T_{\sigma_{\hat{s}}(t)}(\bar{\Omega}_{u(t)}) \) is convex. By the convexity and the fact that \([\pm W_{\hat{s}}(t)] \in T_{\sigma_{\hat{s}}(t)}(\bar{\Omega}_{u(t)}) \), we conclude that \([\pm W_{\hat{s}}(t)] \in T_{\sigma_{\hat{s}}(t)}(\partial\Omega_{u(t)}) \).

Since \([\pm W_{\hat{s}}(t)] \in T_{\sigma_{\hat{s}}}(\partial\Omega_{u(t)}) \), we can consider the horizontal Fermi map

\[ \Psi_{1, \hat{s}} : [0, \ell_{1}] \times \mathbb{R} \rightarrow M^{n} \]

\[ (s, t) \rightarrow \text{Exp}_{\sigma_{\hat{s}}(t)}[sW_{\hat{s}}(t)], \]

which is foliated by an 1-parameter family of curves \( \{\gamma_{t}\} \) which we now describe.

Let \( \hat{\Sigma}_{1, \hat{s}} = \Psi_{1, \hat{s}}([0, \ell_{1}] \times (-\varepsilon, \varepsilon)) \) be an immersed surface. Suppose that the curve \( \gamma_{u} = \hat{\Sigma}_{1, \hat{s}} \cap \partial\Omega_{u(t)} \) is the intersection curve passing through \( \sigma_{\hat{s}}(t) \). Let us choose a parametrization \( \gamma_{t} : (-\varepsilon, \varepsilon) \rightarrow \hat{\Sigma}_{1, \hat{s}} \) such that \( \gamma_{t}(0) = \sigma_{\hat{s}}(t) \). The argument above implies that the \( \gamma_{t} \) has the tangential vector \( W(t) \) at \( \gamma_{t}(0) \).

Because of the convexity of \( \partial\Omega_{u(t)} \) with respect to the outward normal vector \( \sigma'_{\hat{s}}(t) \), the curve \( \gamma_{t} \) is convex at \( \gamma_{t}(0) \) as well. Thus, \( \gamma_{t} \) has nonnegative geodesic curvature \( \lambda(t) \geq 0 \) in subsurface \( \hat{\Sigma}_{1, \hat{s}} \) at the point \( \gamma_{t}(0) = \sigma_{\hat{s}}(t) \).

By our construction, along the curve \( \sigma_{\hat{s}} \), \( \hat{\Sigma}_{1, \hat{s}} \) is totally geodesic. (To see this, one computes the vector-valued the second fundamental form \( II \) of the surface \( \hat{\Sigma}_{1, \hat{s}} \) along the curve \( \sigma_{\hat{s}} \) as follows. It is easy to see that \( \nabla_{\partial} \frac{\partial}{\partial s} \sigma = \nabla_{\partial} \frac{\partial}{\partial s} \sigma_{\hat{s}}(0, t) = \nabla_{\partial} \frac{\partial}{\partial s} \sigma_{\hat{s}}(0, t) = 0 \). Hence we have \( II(\sigma_{\hat{s}}(t), X, Y) = 0 \) for all \( t \in [0, \ell_{1}] \)). It follows that, along the curve \( \sigma_{\hat{s}} \), the intrinsic curvature \( K_{\hat{\Sigma}_{1, \hat{s}}} \) of \( \hat{\Sigma}_{1, \hat{s}} \) is equal to its extrinsic curvature:

\[ K(t) = K_{\hat{\Sigma}_{1, \hat{s}}}(0, t) = K_{\hat{\Sigma}_{1, \hat{s}}}(\sigma'_{\hat{s}}, W_{\hat{s}}(t)) = K_{M^{n}}(\sigma'_{\hat{s}}(t), W_{\hat{s}}(t)) \geq 0. \]
Notice that the geodesic segment $\sigma_\ell$ is orthogonal to the 1-parameter family of curves $\{\gamma_t\}_{0 \leq t \leq \ell_1}$. However, $\{\gamma_t\}_{0 \leq t \leq \ell_1}$ is not necessarily an equidistant family of curves in the surface $\hat{\Sigma}_{1, \ell}$. To overcome this difficulty, we will replace $\{\gamma_t\}$ by another family of “upper barrier” curves $\{\hat{\gamma}_t\}$. (This “upper barrier curve” method has been inspired by the pioneering work of Calabi [Ca] on the barrier function in geometry). Let us compare two distance functions:

$$r(x) = d_{\mathcal{M}^n}(x, c_0(\mathbb{R})) \text{ and } \hat{r}(x) = d_{\hat{\Sigma}_{1, \ell}}(x, c_0(\mathbb{R}))$$

for $x \in \hat{\Sigma}_{1, \ell}$. There is an 1-parameter family of equidistant curves $\{\hat{\gamma}_t\}_{0 \leq t \leq \ell_1}$, where $\hat{\gamma}_t = \Sigma_{1, \ell} \cap \hat{r}^{-1}(\{t\})$ for $0 \leq t \leq \ell_1$.

Clearly, $\hat{r}(x) \geq r(x)$ for $x \in \hat{\Sigma}_{1, \ell}$. Moreover, $\hat{r}(x) = r(x)$ when $x$ is on the geodesic $\sigma_\ell$. Let $\hat{\lambda}(t) = \text{Hees}_{\hat{\Sigma}}(\hat{r})(W_\ell(t), W_\ell(t))|_{\sigma_\ell(t)}$. In terms of barrier functions, by Calabi’s definition [Ca] of the second derivatives, we have

$$\hat{\lambda}(t) = \text{Hees}_{\hat{\Sigma}}(\hat{r})(W_\ell(t), W_\ell(t)) \geq \text{Hees}_{\hat{\Sigma}}(r)(W_\ell(t), W_\ell(t)) = \lambda(t). \quad (1.7)$$

Applying the Riccati equation to the 1-parameter family of equidistant curves $\{\hat{\gamma}_t\}_{0 \leq t \leq \ell_1}$ along the orthogonal geodesic $\sigma_\ell$, we obtain the following equation:

$$\frac{\partial \hat{\lambda}(t)}{\partial t} + [\hat{\lambda}(t)]^2 + K = 0. \quad (1.8)$$

Recall that by assumption and (1.7) we have

$$\hat{\lambda}(0) = 0 \quad \text{and} \quad \hat{\lambda}(t) \geq 0. \quad (1.9)$$

It follows from (1.8)-(1.9) that $\hat{\lambda}'(t) \leq 0$ and $\hat{\lambda}(t) \equiv 0$ for all $t \in [0, \ell_1]$. Using (1.8) and the fact that $\hat{\lambda}(t) = 0$ for all $t$, we conclude

$$K_{\mathcal{M}^n}(\sigma_\ell'(t), W_\ell(t)) = K_{\Sigma_{1, \ell}}(\sigma_\ell'(t), W_\ell(t)) = 0, \quad (1.10)$$

for all $t \in [0, \ell_1]$.

It remains to show that $W_\ell(t) = \frac{\partial \phi_\ell}{\partial \sigma_\ell}(s, t)$. For this purpose, we observe that the bi-linear symmetric curvature form $R_{(s, t)} : (X, Y) \rightarrow \langle R_{\mathcal{M}^n}(\sigma_\ell'(t), X)\sigma_\ell'(t), Y \rangle$ is nonnegative semi-definite in $\{X, Y\}$. By (1.10), we know that $\{W_\ell(t)\}$ is an eigenvector of the symmetric bi-linear form $R_{(s, t)}$ with eigenvalue 0:

$$R_{\mathcal{M}^n}(\sigma_\ell'(t), W_\ell(t))\sigma_\ell'(t) = 0, \quad (1.10)$$

for all $u$.

Because $\{W_\ell(t)\}$ is parallel along the geodesic $\sigma_\ell$, by (1.10) we know that the vector field $\{W_\ell(t)\}$ is parallel Jacobian field along the geodesic $\sigma_\ell$ as well.
It is well-known that any variation vector field of 1-family of geodesics is a Jacobian field. By our construction, the variation field $\partial \phi_1/\partial s(\hat{s}, t)$ is also a Jacobian field along the geodesic $\sigma_{\hat{s}}$. Moreover, $J_{\hat{s}}(t) = \partial \phi_1/\partial s(\hat{s}, t)$ has the same initial conditions as $W(t)$ does. Namely, we have $W_{\hat{s}}(0) = \partial \phi_1/\partial s(\hat{s}, 0) = c'_0(\hat{s})$ and $W'_{\hat{s}}(0) = 0 = \nabla_{\sigma'}(0) \partial \phi_1/\partial s(\hat{s}, 0)$. Therefore, by the uniqueness of Jacobi field with the given initial conditions, we arrive at $W_{\hat{s}}(t) = \partial \phi_1/\partial s(\hat{s}, u)$.

Consequently, $\{\partial \phi_1/\partial s(\hat{s}, t)\}$ is a parallel Jacobian field along each geodesic $\sigma_{\hat{s}}$ for any $\hat{s} \in \mathbb{R}$. This completes the proof of Claim 1.6.

By induction on $j$, we let $c_{j-1}(s) = \phi_{j-1}(s, \ell_{j-1})$ for $j \geq 2$ and let $\{V_j(s)\}$ be a parallel vector field along the geodesic $c_{j-1}$ with initial condition $V_j(0) = \sigma'_j(0)$.

Finally, we set

$$\phi_j : \mathbb{R} \times [0, \ell_j] \to M^n$$

$$(s, t) \to \exp_{c_{j-1}(s)}[tV_j(s)].$$

**Claim A.j.** The normal cone bundle $\mathcal{N}^+(\Omega_{u_{j-1}}, \Omega_{u_{j-1}+\epsilon})$ is invariant under parallel transport along the geodesic $c_{j-1}$. The map $\phi_j$ is a totally geodesic isometric immersion.

When $j = 2$, we see that $c_1(\mathbb{R})$ is a geodesic in a tubular neighborhood of the soul. Hence, $c_1(\mathbb{R})$ is contained in a compact totally convex set $\Omega_\hat{T}$ for some large $\hat{T}$. It follows from Theorem 5.1 of [ChG] that $c_1(\mathbb{R}) \subset \partial \Omega_\lambda$ for some $\lambda \leq u_1$. Applying a result of Yim [Y1], we see that the normal cone bundle $\mathcal{N}^+(\Omega_{u_1}, \Omega_{u_1+\epsilon})$ is invariant under parallel transport along the geodesic $c_1$. Similarly, by the proof of Claim 1.6, we can show that each $\phi_2$ is a totally geodesic isometric immersion.

By induction on $j$, one can verify Claim (A.j) for all $j \geq 2$. This completes the proof of Theorem 1.1. □

### §2. A new proof of Takeuchi Theorem

In this subsection, we study pseudoconvex domains in a Kähler manifold instead of convex domains.

**Definition 2.1.** Let $J$ be the complex structure of a Kähler manifold $M^{2n}$ and let $\Omega$ be compact $C^2$-smooth domain with boundary $\partial \Omega$ in $M^{2n}$. Suppose that $\nabla \rho$ is the outward unit normal vector of $\Omega$ along its boundary $\partial \Omega$. If

$$\langle \nabla_X(\nabla \rho), X \rangle + \langle \nabla_JX(\nabla \rho), JX \rangle \geq 0$$

(2.1)

holds for all $X \perp \{\nabla \rho, J(\nabla \rho)\}$ with $X \in T_p(\partial \Omega)$ and all $p \in \partial \Omega$, then the domain $\Omega$ is said to be pseudoconvex.

A convex sub-domain of a Kähler manifold is necessarily pseudoconvex. Conversely, pseudoconvex domains of a Kähler manifold are not necessarily convex. For
instance, any compact $C^2$-smooth domain $\Omega$ of $\mathbb{C}$ is pseudoconvex, because there is no non-zero tangential vectors $X$ with $X \perp \{\nabla \rho, J(\nabla \rho)\}$ in $\mathbb{C}$.

Inspired by Cheeger-Gromoll’s work [ChG] for real Riemannian manifolds, we also study the inward equidistant evolution for a pseudoconvex domain $\Omega$:

$$
\Omega(-t) = \{x \in \Omega | d(x, \partial \Omega) \geq t\}
$$

**Theorem A.** (Takeuchi [Ta], [Su]) Let $\Omega$ be a pseudoconvex domain with $C^2$-smooth boundary in a Kähler manifold $M^{2n}$ and $r = d(x, \partial \Omega)$. Suppose that the Kähler manifold $M^{2n}$ has holomorphic bisectional curvature $\geq 1$. Then the second fundamental form of $\partial \Omega(-t)$ satisfies:

$$
i\partial{\bar{\partial}}(-r)(\zeta, \bar{\zeta}) \geq r\|\zeta\|^2
$$

for all $\zeta \in T^{1,0}_x(\partial \Omega(-t))$. Moreover, we have the curved version of Oka inequality:

$$
i\partial{\bar{\partial}}(-\log r)(\zeta, \bar{\zeta}) \geq \frac{1}{2}\|\zeta\|^2
$$

for any $\zeta \in T^{1,0}_x(\partial \Omega)$ and $x \in \Omega$.

For any $C^2$ smooth function $f$ and a complex vector $\tau$ of $(1, 0)$-type, the Levi form and complex Hessian are related as follows:

$$
{\mathcal{L}}f(\tau, \bar{\tau}) = 4 \sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \tau_j \bar{\tau}_k = 2 \sqrt{-1} (\partial{\bar{\partial}}f)(\tau, \bar{\tau}),
$$

where $\tau = \sum_{j=1}^n \tau_j \frac{\partial}{\partial z_j} \in T^{1,0}(M^{2n})$. Notice that the complex Hessian $\sqrt{-1}(\partial{\bar{\partial}}f)$ is independent of the choice of the metrics on $M^n$.

When $M^{2n}$ admits a Kähler metric $g = \langle \cdot, \cdot \rangle$, both the Levi form $\mathcal{L}f$ and $\sqrt{-1}(\partial{\bar{\partial}}f)$ are related to the real Hessian of $f$ which we now recall.

Since the Kähler metric $g$ is a Hermitian metric, it preserves the complex structure $J$, i.e., $|JX|^2 = \langle JX, JX \rangle = \langle X, X \rangle = |X|^2$ for any real vector $X \in [T(M^{2n})]_{\mathbb{R}}$. There is a natural isometry between $T(M^{2n})_{\mathbb{R}}$ and $T^{(1,0)}(M^{2n})$ over the real numbers. The map

$$
u \mapsto \tilde{\nu} = \frac{1}{\sqrt{2}}(\nu - \sqrt{-1}Ju)
$$

is a linear isomorphism from $[T(M^{2n})]_{\mathbb{R}}$ to $T^{(1,0)}(M^{2n})$.

Recall that, if the metric is Kähler, for $\tilde{\nu} = \frac{1}{\sqrt{2}}(\nu - \sqrt{-1}Ju)$, we have

$$
\sqrt{-1}\partial{\bar{\partial}}f(\tilde{\nu}, \bar{\nu}) = \text{Hess}(f)(\nu, \nu) + \text{Hess}(f)(Ju, Ju)
$$

see [GW1], where $\text{Hess}(f)(X, Y) = XY f - (\nabla_X Y)f = \langle \nabla_X (\nabla f), Y \rangle$ and $\nabla$ is the covariant derivative (the induced connection) determined by the Kähler metric $g$. 
§2.a. Estimates for the complex Hessian of distance functions

When \( f \) has the property \(|\nabla f| = |df| = 1\), it is easy to check integral curves of the gradient flow are geodesics of unit speed. Therefore, \( \nabla \nabla f(\nabla f) = 0 \) and \( \text{Hess}(f)(\nabla f, Y) = \langle \nabla \nabla f(\nabla f), Y \rangle = 0 \) for any \( Y \in [T(M^{2n})]_\mathbb{R} \). In particular, if \( f(x) = r(x) = d(x, \partial \Omega) \) is a distance function, we have

\[
\text{Hess}(r)(\nabla r, Y) = 0, \tag{2.2}
\]

for any \( Y \in [T(M^{2n})]_\mathbb{R} \).

It is sufficient to estimate \( \text{Hess}(r) \) when it is restricted to the tangential subspace \([T(\partial \Omega_{(-r)})]_\mathbb{R}\), where \( \Omega_{(-r)} = \{ x \in \Omega | d(x, \partial \Omega) \geq r \} \). The real Hessian \( \text{Hess}(r)|_{[T(\partial \Omega_{(-r)})]_\mathbb{R}} \) is exactly the so-called second fundamental form of \( \partial \Omega_{(-r)} \) in the Kähler manifold \( M^{2n} \). It is well-known that the tangential part of \( \text{Hess}(r) \) satisfies the Riccati equation:

\[
\nabla \nabla_r \text{Hess}(r) + [\text{Hess}(r)]^2 + \mathcal{R} = 0,
\]

where \( \mathcal{R} \) is a bilinear form related to sectional curvatures of the Kähler metric \( g \).

The following result was proved by the variational method (e.g., see Takeuchi [Ta] or Siu [Siu1]). We shall use the Riccati equation to give a new simple proof.

**Theorem 2.1.** Let \( M^{2n} \) be a Kähler manifold with bisectional curvature \( \geq 1 \). Let \( \Omega \subset M^{2n} \) be a pseudoconvex domain with \( C^2 \) boundary \( \partial \Omega = \Sigma \) and let \( \rho(x) = -d(x, \Sigma) \) for \( x \in \Omega \). Then

\[
\mathcal{L}(\rho)(\tau, \bar{\tau}) = 2\sqrt{-1}(\partial \bar{\partial} \rho)(\tau, \bar{\tau}) \geq |\rho||\tau|^2, \tag{2.3}
\]

for any \( \tau \in T^{(1,0)}(\partial \Omega_{(-|\rho|)}) \) and \( |\rho| \leq \epsilon_0 \), where \( \epsilon_0 \) is sufficiently small.

**Proof.** Let \( Q_0 \in \partial \Omega \) and \( \exp_{Q_0} \) denote the exponential map from \( T_{Q_0}(M^{2n}) \) to \( M^{2n} \). Let \( \sigma : [-t_0, t_0] \to M^{2n} \) be the geodesic given by

\[
\sigma(t) = \exp_{Q_0}(t \nabla \rho), \tag{2.4}
\]

for small \( t_0 > 0 \). We will study how the Levi form \( \mathcal{L}(\rho) \) changes along \( \sigma \). By (0.2), it suffices to analyze \( \text{Hess}(\rho) \) along \( \sigma(t) \). Recall that \( \sigma'(t) = \nabla \rho|_{\sigma(t)} \) and

\[
\text{Hess}(\rho)(\nabla \rho, \xi) = \langle \bar{\mathcal{D}}_{\nabla \rho}(\nabla \rho), \xi \rangle \equiv 0
\]

since \( \sigma \) is geodesic. It remains to discuss \( \text{Hess}(\rho)(\xi, \eta) \) for \( \xi \perp \nabla \rho \), or equivalently, \( \xi \in T(\Sigma_{(-s)}) \) where \( \Sigma_{(-s)} = \{ x \in X | \rho(x) = -s \} = \partial \Omega_{(-s)} \) for some small number \( s > 0 \). Notice that the second fundamental form of \( \Sigma_{(-s)} \) is equal to the \( \text{Hess}(\rho) \) restricted to the tangent space \( T(\Sigma_{(-s)}) \), i.e.,

\[
\text{Hess}(\rho)(\xi, \eta) = \langle \nabla_{\xi} \nabla \rho, \eta \rangle = \Pi_{\Sigma_{(-s)}}(\xi, \eta)
\]
for $\xi, \eta \in T(\Sigma_{(-s)})$.

The second fundamental forms of $\Sigma_{(-s)}$ along $\sigma(s)$ satisfy the Riccati equation. We choose an orthonormal frame $e_1(0), \ldots, e_{2n}(0)$ of $T_{Q_0}(M^{2n})$, where $\tilde{e}_k = \frac{1}{\sqrt{2}}[e_{2k} - \sqrt{-1}Je_{2k}]$, $k = 1, \ldots, n$. We require that $\tilde{e}_1(0), \ldots, \tilde{e}_{n-1}(0)$ span $T_{Q_0}^{1,0}(\Sigma)$ and that $e_{2k-1} = -Je_{2k}$. We also choose

$$e_{2n}(0) = \sigma'(0) = \nabla|_{Q_0}$$

Let $\{E_k(t)\}$ be a parallel vector field along $\sigma(t)$ with initial condition $E_k(0) = e_k(0)$. Since $M$ is Kähler, we have

$$E_{2n}(t) = \nabla|_{\sigma(t)}, \quad E_{2n-1} = -J(\nabla),$$
$$E_{2j-1}(t) = -J(E_{2j}(t)), \quad j = 1, \ldots, n - 1$$

for all $0 \leq t \leq \epsilon$.

For each $k = 1, \ldots, 2n - 1$, we consider the Jacobi field $\xi_k$ with initial condition

$$\begin{cases}
\xi_k(0) = E_k(0), \\
\xi'_k(0) = \nabla_{E_k(0)}(\nabla). 
\end{cases}$$

For any Jacobi field $\xi(s)$, we have

$$\text{Hess}(\rho)(\xi, \xi) = \Pi_{\Sigma_{(-s)}}(\xi, \xi)$$
$$= \langle \nabla \xi \nabla \rho, \xi \rangle = \langle \nabla \nabla \rho \xi, \xi \rangle$$
$$= \langle \xi'(s), \xi(s) \rangle.$$ (2.7)

Let $A(s) = (a_{jk}(s))$ be the matrix-valued function defined by

$$\xi_k(s) = \sum_{j=1}^{2n-1} a_{jk}(s) E_j(s)$$

and the curvature matrix $R(s) = (R_{ij}(s))$ defined by

$$R(\sigma', E_i)\sigma' = \sum_{j=1}^{2n-1} R_{ij} E_j.$$ (2.8)

With the notation above, we have using the Jacobi equation

$$0 = \xi''_k + R(\sigma', \xi_k)\sigma'$$
$$= \sum_{j=1}^{2n-1} a''_{jk} E_j(s) + \sum_{i,j=1}^{2n-1} R_{ij} a_{ik} E_j.$$
Thus we have the matrix expression of the Jacobi equation

$$A''(s) + R(s)A(s) = 0.$$  
(2.8)

Let

$$B(s) = A'(s)A^{-1}(s) = (b_{ij}(s)).$$

Then

$$\Pi_{\Sigma_{(-s)}}(E_i, E_j) = b_{ij}(s).$$

Using (2.7), we get

$$\Pi_{\Sigma_{(-s)}}(-s(E_i, E_j)) = b_{ij}(s).$$  
(2.9)

Thus $B(s)$ is the matrix representation of the second fundamental form $\Pi_{\Sigma_{(-s)}}$ with respect to the orthonormal basis $E_1(s), \ldots, E_{2n-1}(s)$. It follows from (2.8) and (2.9) that

$$0 = A''A^{-1} + R = B' + B^2 + R,$$  
(2.10)

or equivalently,

$$\Pi' + \Pi^2 + R = 0.$$  
(2.11)

We now apply the above Riccati equation (2.11) to prove Theorem 2.1. If $\tau(s) \in T^{(1,0)}(\Sigma_{(-s)})$, then

$$\tau(s) = \xi(s) - \sqrt{-1}J(\xi(s)),$$

where $\xi = \sum_{k=1}^{2n-2} c_k E_k(s)$ for some $\zeta = (c_1, \ldots, c_{2n-2}) \in \mathbb{R}^{2n}$.

Let

$$\lambda_\xi(s) = \Pi(\xi(s), \xi(s))$$

and let

$$\mu_\tau(s) = \mathcal{L}(\rho)(\tau(s), \bar{\tau}(s))$$

be the Levi form in the $\tau$ direction. From the assumption that $\Omega$ is pseudoconvex, we have

$$\mu_\tau(0) \geq 0.$$  

Using (2.11), we get

$$\lambda'_\xi(s) = \langle B'(s)\zeta, \zeta \rangle$$

$$= \langle -B^2C, \zeta \rangle - \langle RC, \zeta \rangle$$

$$= -\|BC\|^2 - \langle RC, \zeta \rangle$$

$$\leq -\|R(s', \xi)s', \xi\rangle,$$

where we have used that the second fundamental form is symmetric and $B(s)$ is a symmetric matrix. Similarly, we have

$$\lambda'_{J\xi}(s) \leq -\langle R(s', J\xi)s', J\xi \rangle.$$  
(2.13)
It follows from (2.12) and (2.13) that

\[ \mu'(\tau)(s) \leq - (\langle R(\sigma', \xi)\sigma', \xi \rangle + \langle R(\sigma', J\xi)\sigma', J\xi \rangle). \]

The term \((\langle R(\sigma', \xi)\sigma', \xi \rangle + \langle R(\sigma', J\xi)\sigma', J\xi \rangle)\) is equal to the bisectional curvature (see e.g. Zheng [Zh]) in the direction of \(\tilde{e}_n, \tau\). Thus from our assumption, the bisectional curvature is greater or equal to one. Hence, we have

\[ \mu'(\tau)(s) \leq -1. \] (2.14)

Using

\[ \mu(\tau)(0) - \mu(\tau)(-\epsilon) = \int_{-\epsilon}^{0} \mu'(\tau)(s)ds \]

and (2.14), we have that

\[ \mu(\tau)(-\epsilon) = \mu(\tau)(0) - \int_{-\epsilon}^{0} \mu'(\tau)(s)ds \geq 0 - (-1)\epsilon = \epsilon \]

for any \(0 < \epsilon < t_0\). Thus

\[ \mathcal{L}(\rho)|_{\sigma(-\epsilon)}(\tau, \bar{\tau}) \geq \epsilon \]

for any \(0 < \epsilon \leq t_0\) with \(\tau \in T^{(1,0)}(\Sigma_{(-\epsilon)})\). This proves (2.3) and Theorem 2.1. \(\square\)

We would also like to extend the inequality (2.3) to the subset of full measure in domain \(\Omega\), not just near the boundary \(\partial \Omega\). To do this, we need to recall the definition of cut loci in Riemannian geometry.

**Definition 2.2.** (Cut loci) Let \(\Omega \subset M^m\) be a compact domain with \(C^2\)-smooth boundary in a Riemannian manifold \((M^m, g)\). Suppose that \(\sigma : [0, \ell] \to \Omega\) is a geodesic of unit speed such that \(\sigma(0) \in \partial \Omega\) and \(\sigma'(0)\) is orthogonal to \(\partial \Omega\) at \(\sigma(0)\).

1. The above geodesic segment \(\sigma\) is said to be length-minimizing from \(\partial \Omega\) if \(d(\sigma(t), \partial \Omega) = t\) for any \(t \in [0, \ell]\);

2. Suppose that the above geodesic segment \(\sigma\) is length-minimizing from \(\partial \Omega\). The endpoint \(Q = \sigma(\ell)\) is said to be a cut point of \(\partial \Omega\) in \(\Omega\) if \(d(\sigma(\ell + \epsilon), \partial \Omega) < \ell + \epsilon\) for any \(\epsilon > 0\).

3. The subset of all cut points \(Q\) described in (2) is called the cut-loci of \(\partial \Omega\) in \(\Omega\), denoted by \(Cut_\Omega(\partial \Omega)\).

We need to use the following geometric properties of the cut-loci.
Proposition 2.3. ([CE] p99, [Pe]) Let \( \Omega \subset M^{2n} \) be a compact domain with \( C^2 \)-smooth boundary in a \( C^2 \)-smooth Riemannian manifold \((M^m, g)\). Then

1. The cut-loci of \( \partial \Omega \) in \( \Omega \) is a closed subset of zero measure;
2. There is a nearest point projection: \( P_{\partial \Omega} : [\overline{\Omega} - \text{Cut}_\Omega(\partial \Omega)] \to \partial \Omega \); i.e., for each \( Q \notin \text{Cut}_\Omega(\partial \Omega) \), there exists the unique nearest point \( P_Q = P_{\partial \Omega}(Q) \in \partial \Omega \) such that \( d(Q, \partial \Omega) = d(Q, P_Q) \).

The proof of Theorem 2.1 also implies the following:

Corollary 2.4. Let \( M^{2n} \) be a Kähler manifold with holomorphic bisectional curvature \( \geq 1 \). Suppose that \( \Omega \subset M^{2n} \) is a pseudoconvex domain with \( C^2 \) boundary \( \partial \Omega = \Sigma \). Let \( \Omega_{(-t)} = \{ x \in \Omega \mid d(x, \partial \Omega) = d(x, \Sigma) \geq |t| \} \) and \( \rho(x) = -r(x) = -d(x, \Sigma) \). Then

\[
\mathcal{L}(\rho)|_Q (\tau, \bar{\tau}) \geq |\rho||\tau|^2
\]

for any \( \tau \in \mathcal{T}^{(1,0)}_Q(\partial \Omega(\rho)) \) and \( Q \notin \text{Cut}_\Omega(\partial \Omega) \).

Notice that neither Theorem 2.1 nor Corollary 2.4 has estimates of complex Hessian \( \mathcal{L}(-r) \) on complex normal directions. In fact, we already have \( \text{Hess}(r)(\nabla r, Y) = 0 \) for any \( Y \). Furthermore, one can also construct an example of pseudoconvex domain \( \Omega \subset \mathbb{C}^n \), for which the signed distance function \( \rho \) has the property \( \text{Hess}(\rho)(J\nabla \rho, J\nabla \rho)|_Q < 0 \) for some \( Q \in \Omega \). In such an example, we have \( i\partial \bar{\partial}(\rho)(\nabla \rho, \nabla \rho)|_Q < 0 \) for some \( Q \in \Omega \).

In order to find a plurisubharmonic function \( f \) (i.e., \( \mathcal{L}f \geq 0 \) on \( \Omega \)), Oka considers \( [-\log r] \) instead of the signed distance function \( \rho \). Therefore, in next subsection, we estimate \( \mathcal{L}(-\log r)(\tau, \bar{\tau}) = 2i\partial \bar{\partial}[-\log r](\tau, \bar{\tau}) \). It will be shown that \( \mathcal{L}(-\log r) \) is strictly positive definite in all directions.

§2.b. The estimates for \( i\partial \bar{\partial}(-\log r) \) in all directions for domains in \( \mathbb{C}P^n \)

Compact Kähler manifolds with nonnegative holomorphic bisectional curvature have been classified, see [Mok]. In particular, Siu and Yau showed that any compact Kähler manifolds with positive holomorphic bisectional curvature must be biholomorphic to \( \mathbb{C}P^n \). Hence, we first consider the case of \( \mathbb{C}P^n \) with the Fubini-Study metric.

Our goal of this subsection is to show the following

Theorem 2.5. Let \( \Omega \) be a pseudoconvex domain with \( C^2 \) boundary \( \partial \Omega = \Sigma \) in \( \mathbb{C}P^n \) with the Fubini-Study metric of sectional curvature \( 1 \leq K \leq 4 \) and let \( r = d(x, \Sigma) \) be the distance function from \( x \in \Omega \) to \( \partial \Omega = \Sigma \). Then

\[
\mathcal{L}(-\log r)(\zeta, \bar{\zeta}) = 2i\partial \bar{\partial}(-\log r)(\zeta, \bar{\zeta}) \geq \frac{1}{2}||\zeta||^2
\]
for any $\zeta \in T_{x}^{(1,0)}(\Omega)$ and $x \in \Omega$.

Before we provide the proof of Theorem 2.5, we need to recall two elementary but useful facts, which will be used in the proof. The first one is related to the definition of Hessian of a continuous function by the barrier functions:

**Fact 2.6** (Calabi [Ca]) Let $U$ be an open disk of $\mathbb{C} = \mathbb{R}^2$, $f : U \to \mathbb{R}$ be a real-valued continuous function and $Q_0 \in U$. If there is another $C^2$-smooth function $h : U \to \mathbb{R}$ such that (1) $h \leq f$ on $U$, (2) $f(Q_0) = h(Q_0)$ and $\triangle h(Q_0) \geq C$, then we have $\triangle f(Q_0) \geq C$.

**Fact 2.7.** Suppose that $\Omega_{(-\epsilon)}$ is strongly pseudoconvex at $P$. Then there exists a small neighborhood $W_\epsilon$ of $P$ and a complex hypersurface $S_{(-\epsilon)} \subset W_\epsilon$ such that

1. $S_{(-\epsilon)}$ intersects with $\partial \Omega_{(-\epsilon)}$ at $P$ tangentially, i.e., $[T_P(S_{(-\epsilon)})]\mathbb{R} \subset T_P(\partial \Omega_{(-\epsilon)})$;
2. $S_{(-\epsilon)}$ lies outside of $\Omega_{(-\epsilon)}$.

For proof of Fact 2.7, see page 46 of [CS]. It was proved in the previous subsection that $\bar{\Omega}_{(-\epsilon)}$ is strongly pseudoconvex for any $\epsilon > 0$, see Theorem 2.1.

**Proof of Theorem 2.5.** Let us now apply Fact 2.7. By Theorem 2.1, $\Omega_{(-\epsilon)}$ is strongly pseudoconvex for each $\epsilon > 0$. Let us consider a family of functions $r_\epsilon(x) = d(x, \partial \Omega_{(-\epsilon)})$. It is easy to see that $\lim_{\epsilon \to 0} i \partial \bar{\partial} r_\epsilon = i \partial \bar{\partial} r$. Therefore, the verification of (2.15) can be reduced to the case when $\Omega$ is strictly pseudoconvex. In what follows, we may assume that $\partial \Omega$ is strongly pseudoconvex.

We first assume that $x \in U \cap \Omega$, where $U$ is a small neighborhood of $\partial \Omega$. It is easy to see that for any $C^2$ function $f$, we have

$$\text{Hess}(f)(\xi, \eta) = f'(\rho)\text{Hess}(\rho)(\xi, \eta) + f''(\rho) d\rho(\xi) \otimes d\rho(\eta).$$

Let $\rho = -r$. Then

$$\text{Hess}(-\log |\rho|)(\xi, \eta) = \frac{1}{-\rho} \text{Hess}(\rho)(\xi, \eta) + \frac{1}{\rho^2} d\rho(\xi) \otimes d\rho(\eta). \tag{2.16}$$

Using the same notation as in the proof of Theorem 2.1, by (2.3) and (2.16) we already have

$$\mathcal{L}(-\log |\rho|)(\tau, \bar{\tau}) \geq \|\tau\|^2, \quad \tau \in T^{(1,0)}(\partial \Omega_{(-\epsilon)}), \tag{2.17}$$

for $0 < \epsilon < t_0$.

When $\tau = aV_{n-1} + b\bar{\epsilon}_n$ with $ab \neq 0$ and $V_{n-1} \in T^{(1,0)}(\partial \Omega_{(-t)})$, we observe that

$$\text{Hess}(r)(\tau, \bar{\tau}) = |a|^2 \text{Hess}(r)(V_{n-1}, \bar{V}_{n-1})$$

$$+ 2\text{Re}\{ab \text{Hess}(r)(V_{n-1}, \bar{\epsilon}_n)\} + |b|^2 \text{Hess}(r)(\bar{\epsilon}_n, \bar{\epsilon}_n).$$
The term \( \text{Hess}(r)(V_{n-1}, \tilde{e}_n) \) is very difficult to handle. However, using Facts 2.6-2.7 we will avoid to estimate this term directly.

Our strategy is as follows: For any given \( Q_0 \in \Omega - \text{Cut}_\Omega(\Omega) \), we choose a small neighborhood \( W \) around \( Q_0 \) and an upper barrier distance function \( \tilde{r} \geq r \). It follows that \( -\log r \geq -\log \tilde{r} \) and hence \( \text{Hess}(-\log r)|_{Q_0} \geq \text{Hess}(-\log \tilde{r})|_{Q_0} \). When \( \tilde{r}(x) = d(x, S) \) for some holomorphic submanifold \( S \) of complex dimension \( (n-1) \), the Hessian of \( \tilde{r} \) has the property that \( J\nabla \tilde{r} \) is an eigenvector of \( \text{Hess}(\tilde{r}) \).

Recall that if \( \tilde{r} \) is the distance function, then \( \nabla \tilde{r} \) is eigenvector of \( \text{Hess}(\tilde{r}) \). In fact, \( \text{Hess}(\tilde{r})(\nabla \tilde{r}, \cdot) = 0 \). Because \( \text{Hess}(\tilde{r}) \) is real and symmetric, there is an orthonormal eigenbasis. It follows that

\[
\text{Hess}(\tilde{r})(J\nabla \tilde{r}, V_{n-1}) = 0
\]

whenever \( V_{n-1} \) is orthogonal to \( J\nabla \tilde{r} \). The equation (2.18) will play crucial role in the proof presented below.

Let us now carry out the idea above in details.

Motivated by Fact 2.6, we choose \( f = -\log r \) and \( h(x) = -\log \tilde{r}_S(x) \) where \( S \) is a holomorphic submanifold of complex dimension \( (n-1) \) and \( \tilde{r}_S(x) = d(x, S) \). It remains to construct the complex submanifold \( S \) and verify (2.18). For any given \( Q_0 \in \Omega \) but \( Q_0 \notin \text{Cut}_\Omega(\partial \Omega) \), we let \( P_0 \in \partial \Omega \) be the nearest point with \( d(Q_0, P_0) = d(Q_0, \partial \Omega) = r_0 \). Let \( \sigma : [0, r_0] \to \Omega \) be the geodesic from \( P_0 \) to \( Q_0 \).

Using the polar coordinate system around \( \sigma(0) \) for the Fubini-Study metric, we see that \( \xi_{2n-1}(\tilde{r}) = \sin(2\tilde{r})(J\nabla \tilde{r}) \) is a Jacobi field along \( \sigma \). Since \( \nabla \nabla \tilde{r}_j \xi_{2n-1} = 2 \cos(2\tilde{r})(J\nabla \tilde{r}) \) is a scalar multiple of \( \xi_{2n-1} \), we have

\[
\nabla \xi_{2n-1} = \nabla \nabla \xi_{2n-1} = 2 \cot(2\tilde{r})\xi_{2n-1}.
\]

Therefore, the unit direction \( J(\nabla \tilde{r}) = \frac{\xi_{2n-1}}{|\xi_{2n-1}|} \) is an eigenvector of the real symmetric bilinear form \( \text{Hess}(\tilde{r})(X, Y) = \langle \nabla_X \nabla \tilde{r}, Y \rangle \) in \( X \) and \( Y \). Furthermore, we have

\[
\text{Hess}(\tilde{r})(J(\nabla \tilde{r}), Y) = 2 \cot(2\tilde{r})\langle J(\nabla \tilde{r}), Y \rangle
\]

for any tangent vector of \( T(\mathbb{C}P^n) \).

Let \( \Sigma_{(-s)}^{2n-1} = \{ x \in \Omega | d(x, S) = \tilde{r}(x) = s \} \) and we also let \( \Re\{\lambda\} \) denote the real part for any complex number \( \lambda \). It follows from (2.19) that if \( V_{n-1} \in T^{(1,0)}(\Sigma_{(-s)}^{2n-1}) \), then

\[
\text{Hess}(\tilde{r})(aV_{n-1} + bJ\nabla \tilde{r}, aV_{n-1} + bJ\nabla \tilde{r}) = |a|^2\text{Hess}(\tilde{r})(V_{n-1}, V_{n-1}) + 2\Re\{ab\text{Hess}(\tilde{r})(V_{n-1}, -J\nabla \tilde{r})\} + |b|^2\text{Hess}(\tilde{r})(J\nabla \tilde{r}, J\nabla \tilde{r}) = |a|^2\text{Hess}(\tilde{r})(V_{n-1}, V_{n-1}) + 0 + |b|^2\text{Hess}(\tilde{r})(J\nabla \tilde{r}, J\nabla \tilde{r}),
\]

where the midterm vanishes because of (2.2) and (2.20).
It remains to estimate other eigenvalues of \( \text{Hess}(\tilde{r}) \) in complex tangential direction \( V_{n-1} \), i.e., we need to estimate \( \text{Hess}(\tilde{r})(V_{n-1}, \bar{V}_{n-1}) \).

For this purpose, we use the Riccati equation and the same notation as in the proof of Theorem 2.1. Notice that, by the definition of \( P_0, \sigma \) and our upper barrier function \( \tilde{r} \), we see that \( \nabla \tilde{r} = \nabla r \) along the geodesic \( \sigma \) joining \( P_0 \) and \( Q_0 \). We choose an orthonormal frame \( \{-J e_2, e_2, \cdots, -J e_{2(n-1)}, e_{2(n-1)}\} \) of \( T_{P_0}(S) \), where \( S \) is the holomorphic hypersurface of complex dimension \( (n-1) \) given by Fact 2.7 above for \( \epsilon = 0 \).

In what follows, we let \( \tilde{\rho} = -\tilde{r} \). Let \( \{E_k(t)\} \) be a parallel vector field along \( \sigma(t) \) with initial condition \( E_k(0) = e_k \). Recall that \( e_{2n-1} = -J(\nabla \tilde{\rho}) \) and \( e_{2n} = \nabla \tilde{\rho} \).

Suppose that \( \tilde{B}(s) \) is the matrix representation of the second fundamental form \( \Pi_{\tilde{\Sigma}(-s)} \) with respect to the orthonormal basis \( E_1(s), \cdots, E_{2n-1}(s) \), where \( \tilde{\Sigma}(-s) = \{ x \in \Omega | d(x, S) = s \} \). Using the same argument as before, we obtain that

\[
0 = \tilde{B}' + \tilde{B}^2 + R.
\]

Observe that the proof of Theorem 2.1 is independent of the \((2n-1)\)-th column and the \((2n-1)\)-row of \( B(s) \). Since the complex hypersurface \( S \) is holomorphic, we have that, for \( p_0 \in S \),

\[
(i\partial\bar{\partial}\tilde{r})(\bar{\xi}, \bar{\xi})|_{P_0} = \langle \tilde{B}(0)\xi, \xi \rangle + \langle \tilde{B}(0)J\xi, J\xi \rangle = 0,
\]

for \( \xi \perp \{ \nabla \tilde{r}, J\nabla \tilde{r} \} \). Hence, we have the zero initial condition \( \langle [\tilde{B}(0) + J^{-1}\tilde{B}(0)]J\xi, \xi \rangle = 0 \) for \( \xi \perp \{ \nabla \tilde{r}, J\nabla \tilde{r} \} \).

Replacing the matrix-valued function \( B(s) \) by \( \tilde{B}(s) \) in the proof of Theorem 2.1, we obtain that if \( \tau = \frac{1}{\sqrt{2}}(\xi - iJ\xi) \in T^{(1,0)}(\tilde{\Sigma}(-s)) \), then

\[
\mathcal{L}(\tilde{\rho})(\tau, \bar{\tau}) = \text{Hess}(\tilde{\rho})(\xi, \xi) + \text{Hess}(\tilde{\rho})(J\xi, J\xi) \geq 2|\rho||\tau|^2, \tag{2.22}
\]

where \( \tilde{\Sigma}(-s) = \{ x \in \Omega | d(x, S) = s \} \).

Along the geodesic \( \sigma \), we have \( \tilde{r} = r \) and \( \nabla r = \sigma'(t) = \nabla \tilde{r} \). Let \( V_n = \frac{1}{\sqrt{2}}[\nabla r - iJ\nabla r] \). Observe that

\[
\frac{1}{r^2} - \frac{2}{r} \cot(2r) = 4[\frac{1}{2r}]^2[1 - 2r \cot(2r)] \geq 4 \frac{1}{2} = 2 \text{ by (2.20).}
\]

By (2.16), (2.19)-(2.20) and the above inequality \( \frac{1}{r^2} - \frac{2}{r} \cot(2r) \geq 2 \) above, we obtain

\[
[\text{Hess}(-\log \tilde{r})](V_n, \bar{V}_n) = \frac{1}{r^2} - \frac{2}{r} \cot(2r) \geq 2, \tag{2.23}
\]

where we used the fact that \( r \leq \text{Diam}(\mathbb{C}P^n) \) and the diameter \( \text{Diam}(\mathbb{C}P^n) \) of \( \mathbb{C}P^n \) is equal to \( \frac{\pi}{2} \).
Let $V_{n-1} \in T^{(1,0)}_{Q_0}(\partial \Omega(-r_0))$ with $|V_{n-1}| = 1$. Using (2.19)-(2.23) and Fact 2.6, we conclude that, for any $\tau = aV_{n-1} + bV_n \in T^{(1,0)}_{Q_0}(\mathbb{C}P^n)$, the following is true:

$$\mathcal{L}(-\log r)|_{Q_0}(\tau, \bar{\tau}) = \text{Hess}(-\log \bar{r})(aV_{n-1} + bV_n, aV_{n-1} + bV_n)$$

$$= |a|^2 \text{Hess}(\log \bar{r})(V_{n-1}, \bar{V}_{n-1}) + 2\Re\{ab\text{Hess}(-\log \bar{r})(V_{n-1}, \bar{V}_n)\}$$

$$+ |b|^2 \text{Hess}(-\log \bar{r})(V_n, \bar{V}_n)$$

$$= |a|^2 \text{Hess}(\log \bar{r})(V_{n-1}, \bar{V}_{n-1}) + |b|^2 \text{Hess}(-\log \bar{r})(V_n, \bar{V}_n)$$

$$\geq 2|a|^2 + |b|^2 \left[\frac{1}{r^2} - 2\cot(2r)\right] \geq 2|a|^2 + |b|^2 = 2|\tau|^2$$

This completes the proof of Theorem 2.5 away from the cut-locus. On the cut-locus $r$ is not $C^2$. However, it is well-known (see Proposition 2.3 above) that the cut-locus of $\partial \Omega$ has measure zero in $\Omega$. Observe that, on the cut-locus, the function $r(x) = d(x, \partial \Omega)$ remains to be continuous. By Fact 2.6, one can show that $[-\log r]$ is strictly subharmonic on any complex curve in $\Omega$. Hence, the function $[-\log r]$ is strictly plurisubharmonic in all of $\Omega$. □

§2.c. Proof of Theorem A

Proof of Theorem A. Let us now consider the general case when the holomorphic bisectional curvature of the Kähler manifold $M^{2n} \geq 1$.

By the Proof of Theorem 2.5, for each $Q_0 \notin \text{Cut}_\Omega(\partial \Omega)$, it is sufficient to construct an upper barrier function with the following property:

**Lemma 2.8.** For each $Q_0 \notin \text{Cut}_\Omega(\partial \Omega)$ and $P_0 \in \partial \Omega$ with $d(P_0, Q_0) = d(\partial \Omega, Q_0)$, there exists an upper barrier function $\ell$ defined on an open neighborhood $U$ of $Q_0$ such that the following holds

(i) $\ell(x) \geq r(x)$ for all $x \in U$, where $r(x) = d(x, \partial \Omega)$;

(ii) If $\sigma : [0, r_0] \to \Omega$ is the shortest geodesic of unit speed from $P_0$ to $Q_0$, then $\ell|\sigma = r|_\sigma$ and $(\nabla \ell)|_\sigma = (\nabla r)|_\sigma$;

(iii) If $X \in T_{\sigma(r)}(M^{2n})$ with $X \perp \nabla r, J(\nabla r)$, then $\text{Hess}(\ell)(X, X) = \text{Hess}(r)(X, X)$;

(iv) The direction $(J\nabla r)|_\sigma$ is an eigenvector of $\text{Hess}(\ell)$ along the geodesic $\sigma$ with $\nabla J\nabla r(\nabla \ell)|_\sigma = \lambda(r)J\nabla r|_\sigma$ and

$$\frac{1}{r^2} - \frac{\lambda(r)}{r} \geq \frac{1}{2}$$

for $r \in (0, r_0)$.  

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Theorem A will follow from the Proof of Theorem 2.5 and Lemma 2.8. It remains to verify Lemma 2.2.

As we pointed out in the proof of Theorem 2.5, it is sufficient to verify Theorem A when \( \Omega \) is strictly pseudoconvex. We may assume that \( \Omega \) is strictly pseudoconvex at \( P_0 \). By the proof of Corollary 3.3.2 of [CS, p43-46], there is a holomorphic coordinate system \( \{z_1, z_2, \ldots, z_n\} \) such that \( z_j(P_0) = 0 \) for \( j = 1, 2, \ldots, n \) and the distance function \( r \) in a small ball can be expressed as

\[
r(z) = \Re\{z_n\} - \sum_{j,k=1}^{2n-1} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k + O(|z|^3).
\]  

(2.25)

Both real hypersurface \( \Sigma^{2n-1} = \{(z_1, \ldots, z_n)|\Re\{z_n\} = 0, |z| < \epsilon\} \) and complex hypersurface \( S = \{(z_1, \ldots, z_n)|z_n = 0, |z| < \epsilon\} \) lie outside \( \Omega \), see [CS, p43]. If \( h(Q) = d(Q, \Sigma^{2n-1}) \) and \( \tilde{r}(Q) = d(Q, S) \), we will modify our functions \( h \) and \( \tilde{r} \) to obtain \( \ell \) in the following way. Let \( \{E_j(t)\}_{1 \leq j \leq 2n} \) be the parallel vector fields along \( \sigma \) as in (2.6). We choose a local coordinate system along the geodesic \( \sigma \) by the Fermi map:

\[
(t_1, t_2, \ldots, t_{2n-1}, t_{2n}) \rightarrow F(\tilde{t}) = \text{Exp}_{\sigma(t_{2n})}[\sum_{j=1}^{2n-1} t_j E_j(t_{2n})],
\]

where \( \tilde{t} = (t_1, t_2, \ldots, t_{2n-1}, t_{2n}) \). For each \( Q \in \Omega \) near the geodesic \( \sigma \), the shortest geodesic from \( Q \) to \( \Sigma^{2n-1} \) must pass through \( \partial \Omega \). It follows from (2.25) that there exists \( \epsilon_0 > 0 \) such that

\[
\tilde{r}(F(\tilde{t})) \geq h(F(\tilde{t})) \geq r(F(\tilde{t})) + \epsilon_0|\tilde{t}|^2,
\]

where \( \tilde{t} = (t_1, t_2, \ldots, t_{2n-1}) \) and \( |\tilde{t}| \) is sufficiently small.

Finally, we let \( \tilde{\Sigma} = \{F(0, \ldots, t_{2n-1}, t_{2n})||t_{2n-1}| < |\tilde{t}|^2, t_{2n} \in (-\epsilon, r_0 + \epsilon)\} \) and \( f(t_{2n-1}, t_{2n}) = d_{\tilde{\Sigma}}(F(0, \ldots, 0, t_{2n-1}, t_{2n}), P_0) \). Define

\[
\ell(F(\tilde{t})) = \tilde{r}(F(t'', 0, t_{2n}) + [f(t_{2n-1}, t_{2n}) - \tilde{r}(F(0, 0, t_{2n}))],
\]

where \( t'' = (t_1, t_2, \ldots, t_{2n-2}) \). It is easy to check that our barrier function satisfies Lemma 2.8(i)-(iii), when \( |\tilde{t}| \) is sufficiently small. Moreover, The direction \((J\nabla r)|_{\sigma}\) is an eigenvector of \( \text{Hess}(\ell) \) along the geodesic \( \sigma \) with \( \nabla J_{\nabla r}(\nabla \ell)|_{\sigma} = \lambda(r)J_{\nabla r}|_{\sigma} \). To estimate \( \lambda(r) \), we observe that \( \tilde{\Sigma} \) is totally geodesic along \( \sigma \) in \( M^{2n} \). Thus, the intrinsic curvature of \( \tilde{\Sigma} \) is equal to its extrinsic curvature along \( \sigma \). Recall that the holomorphic bisectional curvature of \( M^{2n} \geq 1 \). Hence \( K(r) \geq 1 \). By the standard comparison theorem, we have \( \lambda(r) \leq \cot(r) \). It follows that \( \frac{1}{r^2} - \frac{\lambda(r)}{r} \geq \frac{1}{r^2} - \frac{\cot(r)}{r} \geq \frac{1}{2} \). This completes the proof of Lemma 2.8 as well as Theorem A.

We remark that, using methods in our proof of Theorem 2.5, one can derive a new proof of the classical Oka Lemma.

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References


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