Traces of differential forms on Lipschitz domains, the boundary De Rham complex, and Hodge decompositions

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1 Introduction

A basic prerequisite in the study of boundary value problems associated with a differential operator in a domain $\Omega$ is the availability of suitable trace and extension theorems. In the context when the smoothness of the functions in question is measured on the scales of Sobolev (potential) spaces, $L^p_\alpha$, and Besov spaces, $B^{p,p}_\alpha$, two fundamental results in this regard are as follows.

**Theorem A.** If $\Omega$ is a bounded, Lipschitz domain in $\mathbb{R}^n$ and $1 < p < \infty$, $\alpha \in \mathbb{R}$, then the operator of restriction to $\Omega$, mapping $L^p_\alpha(\mathbb{R}^n)$ onto $L^p_\alpha(\Omega)$, has a linear, continuous, right inverse. That is, there exists a linear, bounded operator

$$E : L^p_\alpha(\Omega) \rightarrow L^p_\alpha(\mathbb{R}^n)$$

such that $(E u)|_\Omega = u$ for each $u \in L^p_\alpha(\Omega)$.

**Theorem B.** Assume that $\Omega$ is a bounded, Lipschitz domain in $\mathbb{R}^n$ and that $1 < p < \infty$, $\alpha \in (1/p, 1 + 1/p)$. Then the restriction to the boundary operator, originally defined from $C^\infty(\overline{\Omega})$ to $\text{Lip}(\partial \Omega)$ extends to a bounded mapping

$$\text{Tr} : L^p_\alpha(\Omega) \rightarrow B^{p,p}_{\alpha-1/p}(\partial \Omega).$$

Furthermore, this operator is onto; indeed, it has a bounded, linear right inverse.

These results have a rich history and have received a great deal of attention in the literature. Here we would like to mention that some of the pioneering work has been done by E. Gagliardo [6], and A.P. Calderón [3], in the 50’s and early 60’s; see also the excellent monographs [31] by E.M. Stein and [12] by A. Jonsson and H. Wallin. Other versions and extensions, as well as more references, can be found D. Jerison and C. Kenig [11], S. Mayboroda and M. Mitrea [18], V.G. Maz’ya, M. Mitrea and T. Shaposhnikova [19], and S. Rychkov [28].

In this paper we are concerned with proving suitable versions of Theorems A-B in the case when differential forms are considered in place of scalar functions. Carrying out this

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*The authors have been supported in part by NSF
2000 Mathematics Subject Classification. Primary 46E35, 58J10, 35N10; Secondary 14F40, 35F05.
Key words: Differential forms, Lipschitz domains, Sobolev and Besov spaces, traces, extensions, cohomology
program is justified given that a great many boundary value problems in mathematical physics (Maxwell equations, elasticity, hydrodynamics, etc.) involve working with vector fields or, more generally, differential forms; the reader is referred to the monographs \[4\], \[5\], \[8\], \[15\], \[16\], \[33\] and \[36\]. Another area of mathematics where such a theory plays a significant role is the study of the $\bar{\partial}$-Neumann problem in several complex variables and its real counterpart, the $d$-Neumann problem. Note that these problems, as well as many others, are naturally formulated in the framework of differentiable manifolds. For maximum applicability, it is therefore important to consider, as we do in the present paper, differential forms of arbitrary degrees in a Lipschitz subdomain $\Omega$ of a given Riemannian manifold $M$.

One distinctive feature, naturally inherent to this setting, is that the definitions of Sobolev-like and Besov-like smoothness spaces must be adapted to the exterior derivative operator $d$ on the manifold $M$, much as the standard, scalar Sobolev and Besov spaces in the Euclidean context are adapted to the gradient operator. For example, one natural analogue of the scalar Sobolev (potential) spaces $L^p_s(\Omega)$ is

\[
D_\ell(d; L^p_s(\Omega)) := \ell\text{-forms } u \text{ with coefficients in } L^p_s(\Omega) \text{ for which } du \text{ has also coefficients in } L^p_s(\Omega). \tag{1.3}
\]

Since the operator of restriction to $\Omega$ maps $D_\ell(d; L^p_s(\Omega))$ into $D_\ell(d; L^p_s(\Omega))$, the issue arises whether this mapping is onto. In this regard, we shall prove the following.

**Theorem C.** If $\Omega$ is a Lipschitz subdomain of $M$ and $1 < p < \infty$, $-1 + 1/p < s < 1/p$, then there exists a linear, bounded operator

\[
\mathcal{E} : D_\ell(d; L^p_s(\Omega)) \longrightarrow D_\ell(d; L^p_s(M)) \tag{1.4}
\]

such that $(\mathcal{E} u)|_\Omega = u$ for each $u \in D_\ell(d; L^p_s(\Omega))$.

The trace operator naturally associated with the space (1.3) is the assignment $u \mapsto \nu \wedge u$, where $\nu$ is the unit conormal to $\partial \Omega$ and wedge stands for the exterior product of forms. It is then possible to check that, with $\delta$ denoting the formal adjoint of $d$, any element $\xi$ in the range of this map satisfies

\[
\xi \in B^{s, 1}_p(\partial \Omega, \Lambda^{\ell+1}) \text{ and there exists } \eta \in B^{s, \ell+2}_p(\partial \Omega, \Lambda^{\ell+2}) \tag{1.5}
\]

such that $\langle \xi, (\delta f)|_{\partial \Omega} \rangle = \langle \eta, f|_{\partial \Omega} \rangle$, $\forall f \in C^\infty(M, \Lambda^{\ell+2})$.

Let $NB^{s, \ell}_p(\partial \Omega, \Lambda^{\ell})$ denote the space of all forms $\xi$ as in (1.5). One of our main results states that this is the smallest space in which the trace map $D_\ell(d; L^p_s(\Omega)) \ni u \mapsto \nu \wedge u \in NB^{s, \ell}_p(\partial \Omega, \Lambda^{\ell})$ takes values. More precisely, we have:

**Theorem D.** Assume that $\Omega$ is a Lipschitz subdomain of the Riemannian manifold $M$ and that $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Then

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is well-defined, linear, bounded and onto. Furthermore, it has a bounded, linear right inverse.

Theorem D is the key ingredient in the proof of Theorem C. Indeed, our strategy is to extend a given \( u \in D_\ell(d; L^p_s(\Omega)) \) to \( M \) by constructing a differential form in \( M \setminus \bar{\Omega} \) whose trace (in the sense of Theorem D) coincides with that of \( u \). More specifically, we take

\[
\mathcal{E}(u) := \begin{cases} u \text{ in } \Omega, \\ \text{Ex}(\nu \wedge u) \text{ in } M \setminus \bar{\Omega}, \end{cases}
\]

where \( \text{Ex} : NB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^\ell) \to D_\ell(d; L^p_s(M \setminus \bar{\Omega})) \) is a right-inverse for the trace operator \( \nu \wedge : D_\ell(d; L^p_s(M \setminus \bar{\Omega})) \to NB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^\ell) \).

Going further, a result similar in spirit to Theorem D is valid for the map

\[
\nu \vee : D_\ell(\delta; L^p_s(\Omega)) \to TB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1})
\]

(see the body of the paper for the relevant definitions). This and the observation that the restriction to the boundary of any differential form \( u \in C^\infty(M, \Lambda^\ell) \) can be decomposed as \( u|_{\partial\Omega} = \nu \vee (\nu \wedge u) + \nu \wedge (\nu \vee u) \) allow us to finally define a global trace map

\[
\text{tr} : D_\ell(d; L^p_s(\Omega)) \cap D_\ell(\delta; L^p_s(\Omega)) \to \left( TB^{p',p'}_{s-\frac{1}{p'}}(\partial\Omega, \Lambda^{\ell}) \oplus NB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^\ell) \right)^*
\]

\[
\langle \text{tr} u, f \oplus g \rangle := \langle \nu \vee (\nu \wedge u), f \rangle + \langle \nu \wedge (\nu \vee u), g \rangle,
\]

which we prove (cf. Theorem 5.6) to be well-defined, linear, bounded, and onto whenever \( 1 < p, p' < \infty, 1/p + 1/p' = 1, -1 + 1/p < s < 1/p \). Again, this has a linear, bounded, right inverse.

In the context of (1.5), define \( B^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^\ell) \ni \xi \mapsto d_\partial \xi := \eta \in B^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell+1}) \) and note that \( d_\partial \circ d_\partial = 0 \). This gives rise to a sequence of homomorphisms – referred to in this paper as the boundary De Rham complex –

\[
\cdots \xrightarrow{d_\partial} NB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1}) \xrightarrow{d_\partial} NB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^\ell) \xrightarrow{d_\partial} NB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell+1}) \xrightarrow{d_\partial} \cdots
\]

in which the image of each arrow is contained in the kernel of the subsequent one. We study its cohomology and establish that for each \( \ell = 1, 2, \ldots, n - 1 \),

\[
\left\{ \xi \in NB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell+1}) : d_\partial \xi = 0 \right\} \cong H^\ell_s(\partial\Omega; \mathbb{R}),
\]

\[
\left\{ d_\partial \zeta : \zeta \in NB^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^\ell) \right\}
\]

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the $\ell$-th singular homology group of $\partial \Omega$ over the field of real numbers. The proof of (1.9) relies on elements of sheaf theory, suitably adapted to the current setting. In this regard, the crux of the matter is proving that the aforementioned complex is locally exact, i.e.

$$\text{if } \xi \in NB^{\frac{p}{s-p}}(\partial \Omega, \Lambda^{\ell+1}) \text{ is such that } d_{\partial} \xi = 0 \text{ near } x_o \in \partial \Omega$$

$$\implies \exists \zeta \in NB^{\frac{p}{s-p}}(\partial \Omega, \Lambda^{\ell}) \text{ such that } d_{\partial} \zeta = \xi \text{ near } x_o.$$ \hspace{1cm} (1.10)

It is precisely at this stage that the ontoness of the trace (1.6) in Theorem D is most useful, as it allows us to shift focus from differential forms defined on $\partial \Omega$ to differential forms defined in $\Omega$. See the proof of Theorem 6.2 for details.

Another notable corollary of Theorem D is that the family of spaces $NB^{\frac{p}{s-p}}(\partial \Omega, \Lambda^{\ell})$, $1 < p < \infty$, $-1 + 1/p < s < 1/p$, is stable under complex interpolation. This observation then allows us to deduce, for the first time, Hodge decompositions for differential forms with coefficients in $L^p s(\Omega)$, for an arbitrary Lipschitz domain $\Omega \subset M$, when $p \in (2-\varepsilon, 2+\varepsilon)$ and $s \in (-\varepsilon, \varepsilon)$, extending work in [21] where the case $s = 0$ has been considered.

The particular case of Theorem C has been proved in [1] when $M = \mathbb{R}^n$, $\ell = n - 1$, $p = 2$ and $s = 0$, by reducing matters to solving a suitable boundary value problem for the Laplacian. Another approach, which eliminates all the above restrictions on the indices involved with the exception of $s = 0$, has been developed in [21]. This requires (locally) flattening the boundary of $\Omega$ via bi-Lipschitz maps. We would now like to briefly comment on the nature of this approach and explain why its applicability is limited to the $L^p$ scale alone. Specifically, for a differential form $f$ of degree $\ell$,

$$f = \sum_{|I| = \ell} f_I \, dx^I, \quad f_I : \mathcal{O} \to \mathbb{R},$$ \hspace{1cm} (1.11)

where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, and a Lipschitz map $\Phi = (\Phi_1, \ldots, \Phi_n) : \mathcal{O}' \to \mathcal{O}$ where $\mathcal{O}'$ is another open subset of $\mathbb{R}^n$, the pull-back of $f$ by $\Phi$ is given by

$$\Phi^* f = \sum_{I = (i_1, \ldots, i_\ell)} (f_I \circ \Phi) \, d\Phi_{i_1} \land d\Phi_{i_2} \land \cdots \land d\Phi_{i_\ell}$$

$$= \sum_{I = (i_1, \ldots, i_\ell)} \sum_{J = (j_1, \ldots, j_\ell)} (f_I \circ \Phi) \frac{D(\Phi_{i_1}, \Phi_{i_2}, \ldots, \Phi_{i_\ell})}{D(x_{j_1}, x_{j_2}, \ldots, x_{j_\ell})} \, dx_{j_1} \land dx_{j_2} \land \cdots \land dx_{j_\ell}.$$ \hspace{1cm} (1.12)

Since, in general, the determinants $\frac{D(\Phi_{i_1}, \Phi_{i_2}, \ldots, \Phi_{i_\ell})}{D(x_{j_1}, x_{j_2}, \ldots, x_{j_\ell})}$ are merely $L^\infty$ functions, it is only the space of forms with $L^p$ coefficients which is stable under pull-back via Lipschitz maps. This observation, which has been employed in, e.g., [10], [27], [35], [34], allows one to do analysis on a Lipschitz manifold $\Sigma$ in the context of differential forms with $L^p$ coefficients. When $\Sigma$ a Lipschitz submanifold of codimension one of a smooth manifold $M$, it is possible to take advantage of the smooth ambient structure of $M$ in order to define classes of differential
forms on $\Sigma$ whose coefficients are smoother than just $L^p$. In particular, this is the case when $\Sigma = \partial \Omega$ where $\Omega$ is a Lipschitz subdomain of $M$. However, much as before, these classes are not invariant under Lipschitz pull-back.

As already alluded to, the main results proved in this paper are particularly relevant in the context of boundary value problems involving vector fields in Lipschitz domains, such as those arising in electromagnetic scattering by rough obstacles. See, e.g., the monographs [4] by M. Cessenat and [5] R. Dautray and J.-L. Lions, as well as the articles [2] by A. Buffa, M. Costabel and C. Schwab, [26] by L. Paquet, and [32] by L. Tartar. Here we only want to remark that [32] contains a proof of the case Theorem D for $M = \mathbb{R}^3$, $\ell = 1$, $s = 0$ and $p = 2$, via an argument which requires flattening the boundary (cf. also §7 in [2] for a discussion), and that [26] proves Theorem D for subdomains with a $C^\infty$ boundary of smooth manifolds, when $p = 2$, $s = 0$, via Fourier methods.

The organization of the remainder of the paper is as follows.

§2 The geometrical setting
§3 Sobolev and Besov spaces on Lipschitz domains
§4 Differential forms with Sobolev-Besov coefficients
§5 Traces of differential forms
§6 The boundary De Rham complex
§7 Extending differential forms from $\Omega$ to $M$
§8 Hodge decompositions

Acknowledgments. This work has been initiated while the second named author was visiting University of Notre Dame during the Winter of 2005. He gratefully acknowledges the hospitality of this institution.

2 The geometrical setting

Let $M$ be a smooth, compact, oriented manifold of real dimension $n$, equipped with a smooth metric tensor, $\sum_{j,k} g_{jk} dx_j \otimes dx_k$. Denote by $TM$ and $T^* M$ the tangent and cotangent bundles to $M$, respectively. We shall frequently identify $T^* M \equiv \Lambda^1$ canonically, via the metric. Set $\Lambda^\ell$ for the $\ell$-th exterior power of $TM$. Sections in this latter vector bundle are $\ell$-differential forms. The Hermitian structure on $TM$ extends naturally to $T^* M := \Lambda^1$ and, further, to $\Lambda^\ell$. We denote by $\langle \cdot, \cdot \rangle$ the corresponding (pointwise) inner product. The volume form on $M$, $dV$, is the unique unitary, positively oriented differential form of maximal degree on $M$. In local coordinates, $dV := [\det (g_{jk})]^{1/2} dx_1 \wedge dx_2 \wedge ... \wedge dx_n$.

Going further, we denote by $*: \Lambda^\ell \to \Lambda^{n-\ell}$ the Hodge star operator. The interior product between a 1-form $\nu$ and an $\ell$-form $u$ is then defined by

$$\nu \lrcorner u := (-1)^{\ell(n+1)} (* (\nu \wedge * u)).$$

Let $d$ stand for the (exterior) derivative operator and denote by $\delta$ its formal adjoint (with respect to the metric introduced above). For further reference some basic properties of these objects are summarized below.
Proposition 2.1 For arbitrary 1-form $\nu$, $\ell$-form $u$, $(n-\ell)$-form $v$, and $(\ell+1)$-form $w$, the following are true:

1. $\langle u, \ast v \rangle = (-1)^{\ell(n-\ell)} \langle \ast u, v \rangle$ and $\langle \ast u, \ast v \rangle = \langle u, v \rangle$. Also, $\ast \ast u = (-1)^{\ell(n-\ell)} u$;

2. $\langle \nu \wedge u, w \rangle = \langle u, \nu \vee w \rangle$;

3. $\ast(\nu \wedge u) = (-1)^{\ell} \nu \vee (\ast u)$ and $\ast(\nu \vee u) = (-1)^{\ell+1} \nu \wedge (\ast u)$;

4. $\ast d^* = (-1)^{\ell} d^*, \delta^* = (-1)^{\ell+1} \ast d^*$, and $\delta = (-1)^{n(\ell+1)+1} \ast d^*$ on $\ell$-forms.

Let $\Omega$ be a Lipschitz subdomain of $M$. That is, $\partial \Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions (cf., e.g., [25]). Then the unit conormal $\nu \in T^*M$ is defined a.e., with respect to the surface measure $d\sigma$, on $\partial \Omega$. For any two sufficiently well-behaved differential forms (of compatible degrees) $u, w$ we then have the integration by parts formula

\[
\int_{\Omega} \langle du, w \rangle dV = \int_{\Omega} \langle u, \delta w \rangle dV + \int_{\partial \Omega} \langle \nu \wedge u, w \rangle d\sigma
\]

\[
= \int_{\Omega} \langle u, \delta w \rangle dV + \int_{\partial \Omega} \langle u, \nu \vee w \rangle d\sigma.
\]  

(2.2)

We conclude with a brief discussion of a number of notational conventions used throughout the paper. By $C^k(\Omega)$, $k \in \mathbb{N}_0 \cup \{\infty\}$, we shall denote the space of functions of class $C^k$ in $\Omega$, and by $C^\infty_c(\Omega)$ the subspace of $C^\infty(\Omega)$ consisting of compactly supported functions. When viewed as a topological space, the latter is equipped with the usual inductive limit topology and its dual, i.e. the space of distributions in $\Omega$, is denoted by $D'(\Omega) := (C^\infty_c(\Omega))^\prime$. We also denote by Lip$(\partial \Omega)$ the class of real-valued Lipschitz functions defined on $\partial \Omega$ and set $C^k(\Omega, \Lambda^\ell) := C^k(\Omega) \otimes \Lambda^\ell$, Lip$(\Omega, \Lambda^\ell) :=$ Lip$(\Omega) \otimes \Lambda^\ell$, etc. Finally, we would like to alert the reader that, besides denoting the pointwise inner product of forms, $\langle \cdot, \cdot \rangle$ is also used as a duality bracket between a topological space and its dual (in each case, the spaces in question should be clear from the context).

3 Sobolev and Besov spaces on Lipschitz domains

The Sobolev (potential) scale in $\mathbb{R}^n$ can be defined as

\[
L^p_s(\mathbb{R}^n) := (I - \Delta)^{-s/2} L^p(\mathbb{R}^n), \quad 1 < p < \infty, \ s \in \mathbb{R}.
\]  

(3.1)

The Besov spaces can then be introduced via real interpolation, i.e.

\[
B^{s,q}_p(\mathbb{R}^n) := (L^p_{s_0}(\mathbb{R}^n), L^p_{s_1}(\mathbb{R}^n))_{\theta,q}
\]  

(3.2)

if $1 < p, q < \infty$, $s \in \mathbb{R}$, and $1 < p_j < \infty$, $s_j \in \mathbb{R}$, $j = 0, 1$, $\theta \in (0,1)$ are such that $1/p = (1-\theta)/p_0 + \theta/p_1$, $s = (1-\theta)s_0 + \theta s_1$. Above, $\langle \cdot, \cdot \rangle_{\theta,q}$ stands for the real interpolation bracket.
Next, the classes $L^p_s(M)$, $B^{p,q}_s(M)$, $1 < p, q < \infty$, $s \in \mathbb{R}$, are obtained by lifting the corresponding Euclidean scales to $M$ via a $C^\infty$ partition of unity and pull-back. Given an arbitrary open subset $\Omega$ of $M$, we denote by $f|_\Omega \in D'(\Omega)$ the restriction of a distribution $f$ on $M$ to $\Omega$. For $1 < p, q < \infty$ and $s \in \mathbb{R}$ we then set

$$L^p_s(\Omega) := \{ f \in D'(\Omega) : \exists F \in L^p(M) \text{ such that } F|_\Omega = f \},$$

$$\|f\|_{L^p_s(\Omega)} := \inf \{ \|F\|_{L^p_s(M)} : F \in L^p_s(M), F|_\Omega = f \}, \quad f \in L^p_s(\Omega),$$

and

$$B^{p,q}_s(\Omega) := \{ f \in D'(\Omega) : \exists F \in B^{p,q}(M) \text{ such that } F|_\Omega = f \},$$

$$\|f\|_{B^{p,q}_s(\Omega)} := \inf \{ \|F\|_{B^{p,q}(M)} : F \in B^{p,q}(M), F|_\Omega = f \}, \quad f \in B^{p,q}_s(\Omega).$$

For the remainder of this section we assume that $\Omega$ is a Lipschitz subdomain of $M$. In this case, according to [28], there exists a universal linear extension operator. More specifically, we have

**Proposition 3.1** If $\Omega$ is a Lipschitz subdomain of $M$, then there exists a linear operator $E$ mapping $C^\infty_c(\Omega)$ into distributions on $M$, and such that for any $1 < p < \infty$ and $s \in \mathbb{R}$,

$$E : L^p_s(\Omega) \longrightarrow L^p_s(M)$$

boundedly, and

$$E(f)|_\Omega = f, \quad \forall f \in L^p_s(\Omega). \quad (3.5)$$

Other properties of interest are summarized in the propositions below; cf., e.g., [37] for proofs.

**Proposition 3.2** Assume that $1 < p_j < \infty$, $s_j \in \mathbb{R}$, $j \in \{0,1\}$, $\theta \in (0,1)$ and that $1/p = (1-\theta)/p_0 + \theta/p_1$, $s = (1-\theta)s_0 + \theta s_1$. Then

$$[L^{p_0}_{s_0}(\Omega), L^{p_1}_{s_1}(\Omega)]_\theta = L^p_s(\Omega), \quad (3.7)$$

where $[\cdot, \cdot]_\theta$ stands for the complex interpolation bracket.

**Proposition 3.3** If $p \in (1, \infty)$, $1/p + 1/p' = 1$, then

$$(L^p_s(\Omega))^* = L^{p'}_{-s}(\Omega), \quad \forall s \in (-1 + 1/p, 1/p). \quad (3.8)$$

Furthermore, for each $s \in \mathbb{R}$ and $1 < p < \infty$ the space $L^p_s(\Omega)$ is reflexive.
Proposition 3.4  If $1 < p < \infty$ and $-1 + 1/p < s < 1/p$ then

$$C_c^\infty(\Omega) \hookrightarrow L^p_s(\Omega) \text{ densely.}$$  \hfill (3.9)

Also, for the same range of indices, the operator $C_c^\infty(\Omega) \ni u \mapsto \tilde{u} \in C^\infty(M)$, where $\tilde{u}$ is the extension of $u$ to $M$ by zero outside $\Omega$, extends to a linear, bounded, one-to-one operator

$$\tilde{\cdot} : L^p_s(\Omega) \longrightarrow L^p_s(M).$$  \hfill (3.10)

In fact, (3.10) has a left-inverse, given by the restriction operator

$$\cdot| : L^p_s(M) \longrightarrow L^p_s(\Omega).$$  \hfill (3.11)

The operators (3.10)-(3.11) are further related by

$$\langle \tilde{u}, w \rangle = \langle u, w| \rangle, \quad \forall u \in L^p_s(\Omega), \; \forall w \in L^p_s(M).$$  \hfill (3.12)

Turning to spaces defined on Lipschitz boundaries, assume $1 < p, q < \infty$, $0 < s < 1$, and that $\Omega$ is the unbounded region in $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. We then define $B^{p,q}_s(\partial \Omega)$ as the space of locally integrable functions $g$ for which the assignment $\mathbb{R}^{n-1} \ni x \mapsto g(x', \varphi(x'))$ belongs to $B^{p,q}_s(\mathbb{R}^{n-1})$. The above definition readily adapts to the case of a Lipschitz subdomain of the manifold $M$, via a standard partition of unity argument. The resulting space is reflexive. Having defined Besov spaces on $\partial \Omega$ with a positive, sub-unitary amount of smoothness, we then set

$$B^{p,q}_s(\partial \Omega) := \left( B^{p',q'}_{1/p}(\partial \Omega) \right)^*, \quad 1 < p, q < \infty, \; 1/p + 1/p' = 1/q + 1/q' = 1, \; 0 < s < 1.$$  \hfill (3.13)

The next result is proved in [11], [12].

Proposition 3.5  The restriction to the boundary, $C^\infty(\overline{\Omega}) \ni u \mapsto u|_{\partial \Omega} \in \text{Lip}(\partial \Omega)$, extends to a bounded, linear trace operator

$$\text{Tr} : L^p_s(\Omega) \longrightarrow B^{p,p}_{s-1/p}(\partial \Omega),$$  \hfill (3.14)

whenever $1 < p < \infty$ and $1/p < s < 1 + 1/p$. This operator is also onto and, in fact, has a bounded, linear, right-inverse

$$\text{Ext} : B^{p,p}_{s-1/p}(\partial \Omega) \longrightarrow L^p_s(\Omega).$$  \hfill (3.15)

Finally, for the same range of indices,

$$C_c^\infty(\Omega) \hookrightarrow \{ u \in L^p_s(\Omega) : \text{Tr} u = 0 \} \text{ densely.}$$  \hfill (3.16)
4 Differential forms with Sobolev-Besov coefficients

In this paper we shall work with certain nonstandard smoothness spaces which are naturally adapted to the type of differential operators we intend to study. Specifically, for $1 < p < \infty$ and $s \in \mathbb{R}$, we consider the spaces

$$D_\ell(d; L^p_s(\Omega)) := \{ u \in L^p_s(\Omega, \Lambda^\ell) : du \in L^p_s(\Omega, \Lambda^{\ell+1}) \},$$

(4.1)

$$D_\ell(\delta; L^p_s(\Omega)) := \{ u \in L^p_s(\Omega, \Lambda^\ell) : \delta u \in L^p_s(\Omega, \Lambda^{\ell-1}) \},$$

(4.2)

equipped with the natural graph norms. Throughout the paper, all derivatives are taken in the sense of distributions.

Let us now assume (as we shall do for the remainder of this section) that $\Omega \subseteq M$ is an arbitrary Lipschitz domain with outward unit conormal $\nu \in T^*M \equiv \Lambda^1$, and that

$$1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1, \text{ and } -1 + \frac{1}{p} < s < \frac{1}{p}.$$

Let $\ell \in \{0, 1, \ldots, n\}$. Inspired by (2.2), for each $u \in D_\ell(d; L^p_s(\Omega))$ we can define $\nu \wedge u$ as a functional on $\partial \Omega$ by setting

$$\langle \nu \wedge u, \psi \rangle := \langle du, \Psi \rangle - \langle u, \delta \Psi \rangle$$

(4.3)

whenever $\psi \in B^{p', p' - s}_{1 - 1/p'}(\partial \Omega, \Lambda^{\ell+1})$ is arbitrary and $\Psi \in L^{p'}_{1 - s}(\Omega, \Lambda^{\ell+1})$ is so that $\text{Tr} \, \Psi = \psi$.

**Proposition 4.1** The normal trace operator

$$\nu \wedge \cdot : D_\ell(d; L^p_s(\Omega)) \longrightarrow B^{p, p}_{s - \frac{1}{p}}(\partial \Omega, \Lambda^{\ell+1})$$

(4.4)

introduced above is well-defined, linear and bounded.

**Proof.** Given $u \in D_\ell(d; L^p_s(\Omega))$, we first note that $\xi := \nu \wedge u$ is unambiguously defined as an element in $B^{p, p}_{s - \frac{1}{p}}(\partial \Omega, \Lambda^{\ell+1})$.

To see this, assume that, for a given $\psi \in B^{p', p' - s}_{1 - 1/p'}(\partial \Omega, \Lambda^{\ell+1})$, two differential forms $\Psi_1, \Psi_2 \in L^{p'}_{1 - s}(\Omega, \Lambda^{\ell+1})$ are selected such that $\text{Tr} \, \Psi_1 = \text{Tr} \, \Psi_2 = \psi$. We aim to show that

$$\langle du, \Psi_1 \rangle - \langle u, \delta \Psi_1 \rangle = \langle du, \Psi_2 \rangle - \langle u, \delta \Psi_2 \rangle.$$

(4.5)

With this in mind, observe that $\Psi := \Psi_1 - \Psi_2 \in L^{p'}_{1 - s}(\Omega, \Lambda^{\ell+1})$ satisfies $\text{Tr} \, \Psi = 0$ so that, by (3.16), there exists a sequence $\Psi_j \in C^\infty_c(\Omega)$ which converges to $\Psi$ in $L^{p'}_{1 - s}(\Omega, \Lambda^{\ell+1})$. This and the distributional definition of $d$ then imply that

$$\langle du, \Psi \rangle - \langle u, \delta \Psi \rangle = \lim_j \left( \langle du, \Psi_j \rangle - \langle u, \delta \Psi_j \rangle \right) = 0,$$

from which the desired conclusion follows.

Next, we notice that the estimate

$$\|\xi\|_{B^{p, p}_{s - \frac{1}{p}}(\partial \Omega, \Lambda^{\ell+1})} \leq C \left( \|u\|_{L^p_s(\Omega, \Lambda^\ell)} + \|du\|_{L^p_s(\Omega, \Lambda^{\ell+1})} \right)$$

(4.5)

follows from (3.13), (4.3), (3.8), plus the fact that, given $\psi \in B^{p', p' - s}_{1 - 1/p'}(\partial \Omega, \Lambda^{\ell+1})$, one is able to choose some $\Psi \in L^{p'}_{1 - s}(\Omega, \Lambda^{\ell+1})$ such that $\text{Tr} \, \Psi = \psi$ with the additional property that
\[ \|\Psi\|_{L_{1-s}^{p'}(\Omega,\Lambda^\ell+1)} \leq C \|\psi\|_{B_{s-1/p'-s}^{p',p'}(\partial\Omega,\Lambda^\ell+1)} \] where \( C = C(\Omega, p, s) > 0 \). See Proposition 3.5 for the latter claim.

An immediate corollary of the above proposition and definition (4.3) is the following useful integration by parts formula.

**Corollary 4.2** For any \( u \in D_\ell(d; L_s^p(\Omega)) \) and \( \Psi \in L_{1-s}^{p'}(\Omega, \Lambda^\ell+1) \), there holds

\[ \langle du, \Psi \rangle = \langle u, \delta \Psi \rangle + \langle \nu \wedge u, \text{Tr} \Psi \rangle. \] (4.6)

Finally, a similar set of results are valid for *tangential* traces of differential forms. Below, we record the main result in this regard.

**Proposition 4.3** There exists a linear, bounded, tangential trace operator

\[ \nu \vee \cdot : D_\ell(\delta; L_s^p(\Omega)) \rightarrow B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell-1}) \] (4.7)

such that

\[ \langle u, d\Phi \rangle = \langle \delta u, \Phi \rangle + \langle \nu \vee u, \text{Tr} \Phi \rangle \] (4.8)

for any \( u \in D_\ell(\delta; L_s^p(\Omega)) \) and any \( \Phi \in L_{1-s}^{p'}(\Omega, \Lambda^{\ell-1}) \).

## 5 Traces of differential forms

Assume that

\[ 1 < p < \infty, \quad -1 + 1/p < s < 1/p, \quad 1/p + 1/p' = 1, \] (5.1)

and consider the space

\[ NB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell) := \left\{ \xi \in B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell) : \exists \eta \in B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell+1}) \text{ such that} \right\} \] (5.2)

equipped with the natural graph norm, i.e.

\[ \|\xi\|_{NB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell)} := \|\xi\|_{B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell)} + \|\eta\|_{B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell+1})}. \] (5.3)

Our first result shows that there is no ambiguity in defining the norm (5.3) and that the space \( NB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell) \) depends exclusively on \( \partial\Omega \) and not on \( \Omega \) itself.
Proposition 5.1 The above definition is meaningful and, in fact,

\[ NB_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^\ell) = \{ \xi \in B_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^\ell) : \exists \eta \in B_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^{\ell+1}) \text{ such that} \] \[ \langle \xi, (\delta f)|_{\partial\Omega} \rangle = \langle \eta, f|_{\partial\Omega} \rangle, \quad \forall f \in C^\infty(M, \Lambda^{\ell+1}) \}. \] (5.4)

Proof. We note that, in the context of (5.2), the differential form \( \xi \) determines \( \eta \) uniquely. Indeed, if \( \eta_1, \eta_2 \in B_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^{\ell+1}) \) are such that

\[ \langle \eta_1, \text{Tr} f \rangle = \langle \eta_2, \text{Tr} f \rangle, \quad \forall f \in D_{\ell+1}(\delta; L^p_{1-s}(\Omega)), \] (5.5)

then \( \eta := \eta_1 - \eta_2 \in B_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^{\ell+1}) \) satisfies

\[ \langle \eta, f|_{\partial\Omega} \rangle = 0 \quad \forall f \in C^\infty(\Omega, \Lambda^{\ell+1}). \] (5.6)

Since \( \text{Tr} : C^\infty(\Omega, \Lambda^{\ell+1}) \to B_{-s+1-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^{\ell+1}) \) has dense range, it follows that \( \eta = 0 \), i.e., \( \eta_1 = \eta_2 \), as claimed.

Second, the equality (5.4) is a consequence of a density result, proved in [24], to the effect that if \( f \in D_{\ell+1}(\delta; L^p_{1-s}(\Omega)) \), then there exists a sequence \( f_j \in C^\infty(\Omega, \Lambda^{\ell+1}), \ j = 1, 2, \ldots, \) such that

\[ f_j \to f \quad \text{in } L^p_{1-s}(\Omega, \Lambda^{\ell+1}) \quad \text{and} \quad \delta f_j \to \delta f \quad \text{in } L^p_{1-s}(\Omega, \Lambda^\ell) \quad \text{as } j \to \infty. \] (5.7)

This concludes the proof of the proposition. \( \square \)

Remarks. (i) It follows from (5.2) that \( NB_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^\ell) \) can be identified with a closed subspace of \( B_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^\ell) \oplus B_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^{\ell+1}) \) and, hence, this space is reflexive.

(ii) The membership of a differential form \( \xi \in B_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^\ell) \) to \( NB_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^\ell) \) ensures that \( \xi \) is normal to \( \partial\Omega \) in a weak sense, i.e. it satisfies

\[ \langle \xi, \text{Tr}(\delta f) \rangle = 0, \quad \forall f \in D_{\ell+1}(\delta; L^p_{1-s}(\Omega)) \ \text{with} \ \text{Tr} f = 0. \] (5.8)

In particular, whenever \( \xi \in B_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^\ell) \) is such that (5.8) holds, the functional

\[ C^\infty(M, \Lambda^{\ell+1})|_{\partial\Omega} \ni f|_{\partial\Omega} \mapsto \langle \xi, (\delta f)|_{\partial\Omega} \rangle \in \mathbb{R} \] (5.9)

is well-defined.

After this preamble, we are now ready to state and prove the first major result of this section.

Theorem 5.2 The trace operator

\[ \nu \wedge \cdot : D_{\ell}(d; L^p_s(\Omega)) \to NB_{s-\frac{1}{p}}^{b,p}(\partial\Omega, \Lambda^{\ell+1}) \] (5.10)

is well-defined, linear, bounded, and onto. In fact, there exists a bounded, linear operator
Ex : $\mathcal{N}B_{\frac{s-\frac{1}{p}}{p}}(\partial \Omega, \Lambda^{\ell+1}) \longrightarrow D_\ell(d; L^p_s(\Omega))$ \hspace{1cm} (5.11)

such that

$$\nu \wedge (\text{Ex } \xi) = \xi, \quad \forall \xi \in \mathcal{N}B_{\frac{s-\frac{1}{p}}{p}}(\partial \Omega, \Lambda^{\ell+1}).$$ \hspace{1cm} (5.12)

**Proof.** We shall proceed in a series of steps, starting with

**Step I.** The operator (5.10) is well-defined, linear and bounded.

If $u \in D_\ell(d; L^p_s(\Omega))$ then $du \in D_{\ell+1}(d; L^p_s(\Omega))$ and, hence, $\eta := -\nu \wedge (du) \in B_{\frac{s-\frac{1}{p}}{p}}(\partial \Omega, \Lambda^{\ell+2})$ and

$$\|\eta\|_{B_{\frac{s-\frac{1}{p}}{p}}(\partial \Omega, \Lambda^{\ell+2})} \leq C \|du\|_{L^p_s(\Omega)},$$ \hspace{1cm} (5.13)

by Proposition 4.1. Furthermore, if $u \in D_\ell(d; L^p_s(\Omega))$ then $\xi := \nu \wedge u \in B_{\frac{s-\frac{1}{p}}{p}}(\partial \Omega, \Lambda^{\ell+1})$ by Proposition 4.1. In addition, for each $f \in D_{\ell+2}(\delta; L^{p'}_{1-s}(\Omega))$ we may write, based on repeated integrations by parts (cf. Corollary 4.2):

$$\langle \xi, \text{Tr} (\delta f) \rangle = \langle \nu \wedge u, \text{Tr} (\delta f) \rangle = \langle du, \delta f \rangle = -\langle \nu \wedge (du), \text{Tr} f \rangle = \langle \eta, \text{Tr} f \rangle$$ \hspace{1cm} (5.14)

which shows that $\xi \in \mathcal{N}B_{\frac{s-\frac{1}{p}}{p}}(\partial \Omega, \Lambda^{\ell+1})$, as desired.

**Step II.** Localization.

Set

$$\Omega_+ := \Omega, \quad \Omega_- := M \setminus \overline{\Omega}.$$ \hspace{1cm} (5.15)

Since $\mathcal{N}B_{\frac{s-\frac{1}{p}}{p}}(\partial \Omega, \Lambda^\ell)$ is a module over $C^\infty(M)$, there is no loss of generality in assuming that we seek to extend forms $\xi \in \mathcal{N}B_{\frac{s-\frac{1}{p}}{p}}(\partial \Omega, \Lambda^\ell)$ with the additional property that $\text{supp } \xi \subset \mathcal{O}$ where $\mathcal{O}$ is an open coordinate chart on $M$ such that, when viewed as a subset of the Euclidean space, $\mathcal{O} \cap \Omega_-$ becomes a bounded Lipschitz domain which is star-like with respect to a ball.

If $\Delta_\ell := -\delta d - d\delta$ is the Hodge Laplacian on $\ell$-forms, then

$$\Delta_\ell - 1 : L^2_\ell(M, \Lambda^\ell) \longrightarrow L^2_{\ell-1}(M, \Lambda^\ell)$$ \hspace{1cm} (5.16)

has an inverse, $(\Delta_\ell - 1)^{-1}$, for each $\ell \in \{0, \ldots, n\}$ whose Schwartz kernel, $\Gamma_\ell(x, y)$, is a symmetric double form of bidegree $(\ell, \ell)$. The commutation relations $d\Delta_\ell = \Delta_{\ell+1}d$ and $\delta \Delta_\ell = \Delta_{\ell-1}\delta$ translate into

$$\delta_x \Gamma_{\ell+1}(x, y) = d_y \Gamma_\ell(x, y), \quad d_x \Gamma_\ell(x, y) = \delta_y \Gamma_{\ell+1}(x, y).$$ \hspace{1cm} (5.17)
Next, denote by $S^\pm_\ell$ the single layer potential operators associated with $\Omega_{\pm}$, i.e.,

$$S^\pm_\ell f(x) := \langle \Gamma_\ell(x, \cdot), f \rangle \quad f \in B^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^\ell), \quad x \in \Omega_{\pm}. \tag{5.18}$$

Note that $(\Delta_\ell - 1)S^\pm_\ell f = 0$ in $\Omega_{\pm}$. Mapping properties for these operators have been established in Theorem 7.1 of [23], where it has been proved that

$$S^\pm_\ell : B^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^\ell) \to L^p_{s+1}(\Omega_{\pm}, \Lambda^\ell), \quad 1 < p < \infty, \quad -1 + 1/p < s < 1/p, \tag{5.19}$$

boundedly. Let us also set

$$S_\ell := \text{Tr} \circ S^+_\ell = \text{Tr} \circ S^-_\ell : B^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^\ell) \to B^{p,p}_{s+1-\frac{1}{p}}(\partial \Omega, \Lambda^\ell), \tag{5.20}$$

where Tr is the trace on $\partial \Omega$. In particular,

$$S_\ell f := S^\pm_\ell f \quad \text{in} \quad \Omega_{\pm} \implies S_\ell : B^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^\ell) \to L^p_{s+1}(M, \Lambda^\ell), \tag{5.21}$$

boundedly, whenever $1 < p < \infty, \quad -1 + 1/p < s < 1/p$. Finally, define the Newtonian (volume) potential

$$\Pi_\ell u(x) := \langle \Gamma_\ell(x, \cdot), u \rangle, \quad x \in M, \tag{5.22}$$

which is a classical pseudo-differential operator of order $-2$. Then

$$\Pi_\ell : L^p_{s}(M, \Lambda^\ell) \to L^p_{s+2}(M, \Lambda^\ell), \quad 1 < p < \infty, \quad s \in \mathbb{R}, \tag{5.23}$$

$$(\Delta_\ell - 1)\Pi_\ell = I, \quad \text{the identity operator.} \tag{5.24}$$

**Step III.** Fix an arbitrary $\xi \in NB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell+1})$ and let $\eta \in B^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell+2})$ be the differential form associated with $\xi$ as in (5.4). Finally, define

$$u^\pm := \delta S^\pm_{\ell+1} \xi \quad \text{in} \quad \Omega_{\pm}. \tag{5.25}$$

Then $u^\pm \in D_\ell(d; L^p_{s}(\Omega_{\pm}))$ and

$$du^\pm = -\delta S^\pm_{\ell+2} \eta - S^\pm_{\ell+1} \xi, \quad \text{in} \quad \Omega_{\pm}. \tag{5.26}$$

Indeed, we have

$$du^\pm = d\delta S^\pm_{\ell+1} \xi = -\delta dS^\pm_{\ell+1} \xi - S^\pm_{\ell+1} \xi, \tag{5.27}$$

and for each fixed $x \in \Omega_{+}$ we may write

13
\[(dS^+_{\ell+1}\xi)(x) = \langle d_x \Gamma_{\ell+1}(x, \cdot), \xi \rangle = \langle \delta_y \Gamma_{\ell+2}(x, y), \xi(y) \rangle = \langle \text{Tr} (\delta \Gamma_{\ell+2}(x, \cdot)), \xi \rangle = \langle \Gamma_{\ell+2}(x, \cdot), \eta \rangle = S^+_{\ell+1}\eta, \quad (5.28)\]

Since \(\Gamma_{\ell+2}(x, \cdot) \in C^\infty(\Omega, \Lambda^{\ell+2})\). The case when the superscript + is replaced by − is analogous.

**Step IV.** With \(\xi, \eta, u^+, \) and \(u^-\) as in Step III,
\[\nu \wedge u^+ - \nu \wedge u^- = \xi. \quad (5.29)\]

To justify (5.29), for an arbitrary \(\Phi \in C^\infty(M, \Lambda^{\ell+1})\) we write (using the fact that the outward unit normal for \(\Omega_-\) is \(-\nu\)):
\[
\langle \nu \wedge u^+ - \nu \wedge u^-, \text{Tr} \Phi \rangle = \langle du^+, \Phi \rangle - \langle u^+, \delta \Phi \rangle + \langle du^-, \Phi \rangle - \langle u^-, \delta \Phi \rangle = I + II + III + IV. \quad (5.30)
\]

Next, (5.26) and (4.7) yield
\[
I = -\langle \delta S^+_{\ell+2}\eta, \Phi |_{\Omega_-} \rangle - \langle S^+_{\ell+1}\xi, \Phi |_{\Omega_-} \rangle = -\int_{\Omega_-} \langle S^+_{\ell+2}\eta, d\Phi \rangle dV - \int_{\Omega_-} \langle S^+_{\ell+1}\xi, \Phi \rangle dV + \int_{\partial\Omega_-} \langle \nu \wedge S_{\ell+2}\eta, \Phi \rangle d\sigma, \quad (5.31)
\]
\[
III = -\langle \delta S^-_{\ell+2}\eta, \Phi |_{\Omega_-} \rangle - \langle S^-_{\ell+1}\xi, \Phi |_{\Omega_-} \rangle = -\int_{\Omega_-} \langle S^-_{\ell+2}\eta, d\Phi \rangle dV - \int_{\Omega_-} \langle S^-_{\ell+1}\xi, \Phi \rangle dV - \int_{\partial\Omega_-} \langle \nu \wedge S_{\ell+2}\eta, \Phi \rangle d\sigma. \quad (5.32)
\]

Consequently, since the boundary integrals in (5.32) and (5.31) have opposite signs, we may use Fubini’s Theorem, (5.17) and the definition of \(\eta\) in order to write
\[
I + III = -\int_M \langle S_{\ell+2}\eta, d\Phi \rangle dV - \int_M \langle S_{\ell+1}\xi, \Phi \rangle dV = -\langle \eta, \text{Tr} \Pi_{\ell+2}(d\Phi) \rangle - \langle \xi, \text{Tr} \Pi_{\ell+1}\Phi \rangle = -\langle \xi, \text{Tr} (\delta \Pi_{\ell+2}(d\Phi)) \rangle - \langle \xi, \text{Tr} \Pi_{\ell+1}\Phi \rangle = -\langle \xi, \text{Tr} (\delta d\Pi_{\ell+1}\Phi + \Pi_{\ell+1}\Phi) \rangle. \quad (5.33)
\]

Going further,
\[ II = -\langle \delta S^+_{\ell+1} \xi, \delta \Phi |_{\Omega_+} \rangle = - \int_{\Omega_+} \langle S^+_{\ell+1} \xi, d\delta \Phi \rangle \, dV + \int_{\partial \Omega} \langle \nu \lor S_{\ell+1} \xi, \delta \Phi \rangle \, d\sigma, \quad (5.34) \]
\[ IV = -\langle \delta S^-_{\ell+1} \xi, \delta \Phi |_{\Omega_-} \rangle = - \int_{\Omega_-} \langle S^-_{\ell+1} \xi, d\delta \Phi \rangle \, dV - \int_{\partial \Omega} \langle \nu \lor S_{\ell+1} \xi, \delta \Phi \rangle \, d\sigma, \quad (5.35) \]

so that

\[ II + IV = - \int_M \langle S_{\ell+1} \xi, d\delta \Phi \rangle \, dV = -\langle \xi, \text{Tr} (d\delta \Pi_{\ell+1} \Phi) \rangle. \quad (5.36) \]

Since

\[ d\delta \Pi_{\ell} + \delta d\Pi_{\ell} + \Pi_{\ell} = -I \quad \text{on} \quad M, \quad (5.37) \]

it follows from (5.30)-(5.37) that

\[ \langle \nu \lor u^+ - \nu \lor u^-, \text{Tr} \Phi \rangle = \langle \xi, \text{Tr} \Phi \rangle. \quad (5.38) \]

Finally, (5.29) follows from this and the fact that \( \text{Tr} : C^\infty_c(M, \Lambda_\ell) \to B^{p,p\quad-s-1}_1(\partial \Omega, \Lambda_{\ell-1}) \) has a dense range. This completes the proof of Step IV.

**Step V.** There exists \( w \in L^p_{s+1}(M, \Lambda_\ell) \) with \( dw = du^- \) in \( \mathcal{O} \cap \Omega_- \).

Under the current assumptions on \( \mathcal{O} \cap \Omega_- \) it has been proved in §4 of [24] that there exist \( \theta \in C^\infty_c(\Omega) \) and a family of linear operators

\[ K_\ell : \left( C^\infty_c(\mathcal{O} \cap \Omega_, \Lambda_\ell) \right)' \longrightarrow \left( C^\infty_c(\mathcal{O} \cap \Omega_, \Lambda_{\ell-1}) \right)' \quad 1 \leq \ell \leq n, \quad (5.39) \]

such that

\[ \forall w \in \left( C^\infty_c(\mathcal{O} \cap \Omega_, \Lambda_\ell) \right)' \implies w = \begin{cases} K_1(dw) + \langle w, \theta \rangle & \text{if } \ell = 0, \\ d(K_\ell w) + K_{\ell+1}(dw) & \text{if } 1 \leq \ell \leq n - 1, \\ d(K_n w) & \text{if } \ell = n. \end{cases} \quad (5.40) \]

and for which

\[ K_\ell : L^p_s(\mathcal{O} \cap \Omega_, \Lambda_\ell) \longrightarrow L^p_{s+1}(\mathcal{O} \cap \Omega_, \Lambda_{\ell-1}) \quad (5.41) \]

in a bounded fashion. Thus, we may define the differential form \( w \) as the extension of \( K_\ell(dw^-) \in L^p_{s+1}(\mathcal{O} \cap \Omega_, \Lambda_{\ell-1}) \) to an element in \( L^p_{s+1}(M, \Lambda_{\ell-1}) \).

**Step VI.** There exists \( \omega \in L^p_{s+1}(M, \Lambda_{\ell-1}) \) such that \( d\omega = u^- - w \) in \( \mathcal{O} \cap \Omega_- \).
This time, take \( \omega \) to be the extension of \( K_{\ell}(u^- - w) \in L^{p}_{s+1}(\O \cap \O_-, \Lambda^{\ell-1}) \) to an element in \( L^{p}_{s+1}(M, \Lambda^{\ell-1}) \).

**Step VII.** Retaining the notation used in the previous steps, we have

\[
v := w + d\omega \in D_{\ell}(d; L^{p}_{s}(M)).
\]

Moreover, if \( \psi \in C^\infty_c(\O) \) is such that \( \psi \equiv 1 \) on \( \text{supp} \xi \), then \( u := \psi (u^+ - v)|_{\O} \) satisfies \( u \in D_{\ell}(d; L^{p}_{s}(\O)) \) and \( \nu \wedge u = \xi \).

To see this, we compute

\[
\nu \wedge (\psi v|_{\O_+}) = \nu \wedge (\psi v|_{\O_-}) = \psi \nu \wedge (w + d\omega)|_{\O_-} = \psi \nu \wedge u^-.
\]

Thus,

\[
\nu \wedge u = \psi \nu \wedge u^+ - \psi \nu \wedge (v|_{\O_+}) = \psi (\nu \wedge u^+ - \nu \wedge u^-) = \psi \xi = \xi,
\]

by Step IV.

This concludes Step VII and finishes the proof of the theorem. \( \square \)

Under the assumptions (5.1), we shall also consider the space

\[
TB^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell}) := \left\{ \xi \in B^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell}) : \exists \eta \in B^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell+1}) \text{ such that} \right. \\
\left. \langle \xi, \text{Tr}(df) \rangle = \langle \eta, \text{Tr} f \rangle, \quad \forall f \in D_{\ell-1}(d; L^{p}_{1-s}(\O)) \right\}
\]

(5.45)

once again, equipped with the natural graph norm. As with the space \( NB^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell}) \), this definition is unambiguous and the following alternative description holds:

\[
TB^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell}) := \left\{ \xi \in B^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell}) : \exists \eta \in B^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell+1}) \text{ such that} \right. \\
\left. \langle \xi, \text{Tr}(df) \rangle = \langle \eta, \text{Tr} f \rangle, \quad \forall f \in C^\infty(M, \Lambda^{\ell-1}) \right\}.
\]

(5.46)

Moreover,

\[
* : NB^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell}) \rightarrow TB^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{n-\ell})
\]

(5.47)

is an isomorphism.

**Corollary 5.3** Let \( 1 < p < \infty \) and \( -1 + 1/p < s < 1/p \). Then the operator

\[
\nu \vee : D_{\ell}(\delta; L^{p}_{s}(\O)) \rightarrow TB^{p, p}_{s-\frac{1}{p}}(\partial \O, \Lambda^{\ell-1})
\]

(5.48)

is well-defined, linear, bounded, and has a linear, bounded right inverse. In particular, it is onto.
show that it is also onto. To this end, pick an arbitrary this map is bounded, thanks to (3.8), as well as linear and one-to-one. There remains to result, proved in [24], according to which if for each u for defined by

\[ NB_{s-\frac{1}{p}}^{\partial p} (\partial \Omega, \Lambda^{\ell+1})^* \] (5.49)

defined by

\[ \langle \nu \wedge f, g \rangle := \langle u, dw \rangle - \langle \delta u, w \rangle \] (5.50)

for u ∈ D_{\ell+1}(\delta; L_p^0(\Omega)), with w ∈ D_{\ell+1}(\Omega) \) with g = \nu \wedge w is well-defined, in fact an isomorphism, for each \ell ∈ \{0, 1, ..., n\}. Furthermore, the adjoint of (5.49)-(5.50) is the operator

\[ \nu \vee \cdot : NB_{s-\frac{1}{p}}^{\partial p} (\partial \Omega, \Lambda^{\ell+1}) \rightarrow \left( \text{TB}_{s-\frac{1}{p}}^{\partial p} (\partial \Omega, \Lambda^{\ell}) \right)^* \] (5.51)
defined as

\[ \langle \nu \vee f, g \rangle := \langle du, w \rangle - \langle u, dw \rangle \] (5.52)

for u ∈ D_{\ell}(d; L_{-s}^p(\Omega)), with w ∈ D_{\ell+1}(\delta; L_p^0(\Omega)) with g = \nu \vee w.

Proof. Proving that the map (5.49)-(5.50) is well-defined comes down to checking the following claim: if u ∈ D_{\ell+1}(\delta; L_p^0(\Omega)) and w ∈ D_{\ell}(d; L_{-s}^p(\Omega)) such that \nu \wedge u = 0 or \nu \wedge w = 0, then \langle u, dw \rangle - \langle \delta u, w \rangle = 0. In turn, this is an easy consequence of a density result, proved in [24], according to which if w ∈ D_{\ell}(d; L_p^0(\Omega)) has \nu \wedge w = 0, then there exists a sequence w_j ∈ C_c^\infty(\Omega, \Lambda^\ell), j = 1, 2, ..., such that

\[ w_j \rightarrow w \text{ in } L_p^0(\Omega, \Lambda^\ell) \text{ and } dw_j \rightarrow dw \text{ in } L_p^0(\Omega, \Lambda^{\ell+1}) \text{ as } j \rightarrow \infty. \] (5.53)

Having established the well-definiteness of the map (5.49)-(5.50), we next remark that this map is bounded, thanks to (3.8), as well as linear and one-to-one. There remains to show that it is also onto. To this end, pick an arbitrary \theta ∈ \left( NB_{s-\frac{1}{p}}^{\partial p} (\partial \Omega, \Lambda^{\ell+1}) \right)^* and consider \hat{\theta} : D_{\ell}(d; L_{-s}^p(\Omega)) \rightarrow \mathbb{R} defined by \hat{\theta}(u) := \theta(\nu \wedge u). Regarding D_{\ell}(d; L_{-s}^p(\Omega)) as a (closed) subspace of L_{-s}^p(\Omega, \Lambda^\ell) \oplus L_{-s}^p(\Omega, \Lambda^{\ell+1}) via the identification u \mapsto (u, du), the Hahn-Banach theorem in concert with (3.8) allow us to conclude that there exist v_1 ∈ L_p^0(\Omega, \Lambda^{\ell+1}) and v_2 ∈ L_s^p(\Omega, \Lambda^{\ell}) such that

\[ \hat{\theta}(u) = \langle v_1, du \rangle - \langle v_2, u \rangle, \quad \forall u \in D_{\ell}(d; L_{-s}^p(\Omega)). \] (5.54)

Note that choosing u ∈ C_c^\infty(\Omega, \Lambda^\ell) yields \delta v_1 = v_2. In particular v_1 ∈ D_{\ell}(\delta; L_p^0(\Omega)). Utilizing this identity back in (5.54) gives that \theta(\nu \wedge u) = \langle \nu \wedge (\nu \vee v_1), \nu \wedge u \rangle, for each
\[ u \in D_{\ell}(d; L_{s}^{\ell}(\Omega)) \] which, in turn, entails \( \nu \wedge (\nu \vee \nu) = \theta \). Hence, the map (5.49) is onto and this finishes the proof of the claims made in the first part of the proposition.

Finally, the claim that (5.51) is the adjoint of the operator (5.49) follows by comparing (5.52) with (5.50).

The above analysis allows us to deduce a rather general integration by parts formula for differential forms on Lipschitz domains.

**Corollary 5.5** Under the assumptions (5.1), the following identities hold:

\[
\langle du, w \rangle = \langle u, \delta w \rangle + \langle \nu \wedge u, \nu \wedge (\nu \vee w) \rangle, \quad (5.55)
\]

\[
\langle du, w \rangle = \langle u, \delta w \rangle + \langle \nu \vee (\nu \wedge u), \nu \wedge w \rangle, \quad (5.56)
\]

for any \( u \in D_{\ell}(d; L_{s}^{\ell}(\Omega)) \) and \( w \in D_{\ell+1}(\delta; L_{s}^{\ell}(\Omega)) \).

The above corollary is an immediate consequence of Proposition 5.4. Here we only want to point out that, in the context of (5.55), \( \nu \vee w \in TB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell+1}) \) by (5.48) so that, further, \( \nu \wedge (\nu \vee w) \in \left( NB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell+1}) \right)^{\ast} \) by Proposition 5.4. In particular, since \( \nu \wedge u \) belongs to \( NB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell+1}) \), the last pairing in the right-hand side of (5.55) is well-defined. Of course, the same considerations apply to (5.56).

Finally, we are now in a position to discuss global traces of differential forms in the space \( D_{\ell}(d; L_{s}^{\ell}(\Omega)) \cap D_{\ell}(\delta; L_{s}^{\ell}(\Omega)) \).

**Theorem 5.6** The assignments

\[
D_{\ell}(d; L_{s}^{\ell}(\Omega)) \ni u \mapsto u_{\text{tan}} := \nu \vee (\nu \wedge u) \in \left( TB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell}) \right)^{\ast}, \quad (5.57)
\]

\[
D_{\ell}(\delta; L_{s}^{\ell}(\Omega)) \ni u \mapsto u_{\text{nor}} := \nu \wedge (\nu \vee u) \in \left( NB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell}) \right)^{\ast}, \quad (5.58)
\]

are linear, bounded and onto. In fact, each has a bounded, linear, right inverse. Furthermore, they are compatible with the mappings

\[
\text{Tr} : L_{s+1}^{p}(\Omega, \Lambda^{\ell}) \rightarrow B_{s+1-1/p}^{p,p}(\partial\Omega) = \left( B_{s-1/p}^{p,p}(\partial\Omega) \right)^{\ast} \hookrightarrow \left( TB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell}) \right)^{\ast} \quad (5.59)
\]

\[
\text{Tr} : L_{s+1}^{p}(\Omega, \Lambda^{\ell}) \rightarrow B_{s+1-1/p}^{p,p}(\partial\Omega) = \left( B_{s-1/p}^{p,p}(\partial\Omega) \right)^{\ast} \hookrightarrow \left( NB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell}) \right)^{\ast} \quad (5.60)
\]

in the sense that \( L_{s+1}^{p}(\Omega, \Lambda^{\ell}) \hookrightarrow D_{\ell}(d; L_{s}^{\ell}(\Omega)) \cap D_{\ell}(\delta; L_{s}^{\ell}(\Omega)) \) and the actions of (5.57), (5.58) agree with those of (5.59) and (5.60), respectively.

Finally, the global trace map

\[
18
\]
the expression in the right-hand side of (5.64) can be further expressed as

\[ \text{tr} : D_\ell(d; L^p_\delta(\Omega)) \cap D_\ell(\delta; L^p_\delta(\Omega)) \rightarrow \left( TB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell) \oplus NB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell) \right)^* \]

(5.61)

\[ (\text{tr} u, f \oplus g) := \langle u_{\text{tan}}, f \rangle + \langle u_{\text{nor}}, g \rangle, \]

(5.62)

\[ \forall u \in D_\ell(d; L^p_\delta(\Omega)) \cap D_\ell(\delta; L^p_\delta(\Omega)), \forall f \in TB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell), \forall g \in NB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell) \]

is linear, bounded, and has a linear, bounded right inverse. In particular, it is onto.

Furthermore, the action of the map (5.61)-(5.62) is compatible with that of

\[ \text{Tr} : L^p_{s+1}(\Omega, \Lambda^\ell) \rightarrow B^{p,p}_{s+1-\frac{1}{p}}(\partial\Omega, \Lambda^\ell) \]

\[ \hookrightarrow \left( TB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell) \oplus NB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell) \right)^* \]

(5.63)

where the above inclusion acts according to (u(\xi), f \oplus g) := \langle \xi, f \rangle + \langle \xi, g \rangle, in the sense that

\[ L^p_{s+1}(\Omega, \Lambda^\ell) \rightarrow D_\ell(d; L^p_\delta(\Omega)) \cap D_\ell(\delta; L^p_\delta(\Omega)) \text{ and the action of (5.63) agrees with that of (5.61)-(5.62).} \]

Hence, from this point of view, the global trace map for differential forms introduced in (5.61)-(5.62) can be regarded as an extension of the ordinary componentwise trace map from (3.14).

Proof. The claim that the mappings (5.57), (5.58) are well-defined, linear, bounded and that each has a linear, bounded, right inverse, follows from Proposition 5.4 and Theorem 5.2.

Next, let u \in L^p_{s+1}(\Omega, \Lambda^\ell) \rightarrow D_\ell(d; L^p_\delta(\Omega)) and consider the action of the functional \nu \vee (\nu \wedge u) \in \left( TB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell) \right)^* on an arbitrary f \in TB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell), i.e.

\[ \langle \nu \vee (\nu \wedge u), f \rangle = \langle du, w \rangle - \langle u, \delta w \rangle \]

(5.64)

if f = \nu \vee w for some w \in D_\ell(\delta; L^p_{1-s}(\Omega)). Recalling the integration by parts formula (4.8), the expression in the right-hand side of (5.64) can be further expressed as

\[ \langle du, w \rangle - \langle u, \delta w \rangle = \langle \text{Tr} u, \nu \vee w \rangle = \langle \text{Tr} u, f \rangle. \]

(5.65)

Thus, all in all, \( u_{\text{tan}} = \text{Tr} u \) as functionals on \( TB_{s-\frac{1}{p}}^{\gamma'}(\partial\Omega, \Lambda^\ell) \hookrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^\ell) \), which proves that the mappings (5.57), (5.59) are compatible. The fact that (5.58) and (5.60) are also compatible is proved in a similar fashion.

Finally, the claims made about the map (5.61)-(5.62) follow from what we have proved so far in a straightforward manner; we omit the details. This finishes the proof of Theorem 5.6. \( \square \)
6 The boundary De Rham complex

Here we take a closer look at the mapping $\mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^\ell) \ni \xi \mapsto \eta \in \mathrm{B}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^{\ell+1})$ where $\xi$, $\eta$ are as in the description of (5.2).

Proposition 6.1 If (5.1) holds, then the operator

$$d_\partial : \mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^\ell) \longrightarrow \mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^{\ell+1})$$

(6.1)

given by

$$d_\partial \xi := \eta,$$

(6.2)

whenever $\xi$ and $\eta$ are as in (5.2), is well-defined, linear and bounded. Furthermore,

$$d_\partial \circ d_\partial = 0,$$

(6.3)

and

$$d_\partial (\nu \wedge u) = -\nu \wedge du, \quad \forall u \in D_{\ell}(d; L^p_s(\Omega)).$$

(6.4)

Proof. If $\xi$, $\eta$ are as in the right-hand side of (5.4), it follows that

$$\langle \eta, \text{Tr}(\delta f) \rangle = \langle \xi, \text{Tr}(\delta^2 f) \rangle = 0, \quad \forall f \in C^\infty(M, \Lambda^{\ell+2}),$$

(6.5)

which proves that $\eta \in \mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^{\ell+1})$. In particular, the operator (6.1)-(6.2) is well-defined. It is also implicit in the above argument that $d_\partial \eta = 0$ from which the identity (6.3) follows. Finally, (6.4) is implicit in Step I of the proof of Theorem 5.2.

The identity (6.3) suggests the consideration of the sequence of group homomorphisms

$$\cdots \longrightarrow \mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^{\ell-1}) \overset{d_\partial}{\longrightarrow} \mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^{\ell}) \overset{d_\partial}{\longrightarrow} \mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^{\ell+1}) \overset{d_\partial}{\longrightarrow} \cdots$$

(6.6)

in which the image of each arrow is contained in the kernel of the next. Our goal is to study the cohomology of the boundary De Rham complex (6.6). To begin with, it is clear from definitions that the kernel of the operator (6.1) is given by

$$\mathrm{N}_s \mathrm{B}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^\ell) := \{ \xi \in \mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^\ell) : d_\partial \xi = 0 \}$$

(6.7)

and that the image of $d_\partial$ acting on $\mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^\ell)$ is a subspace of $\mathrm{N}_s \mathrm{B}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^{\ell+1})$. The quotient group $\mathrm{N}_s \mathrm{B}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^{\ell+1})/d_\partial \left[ \mathrm{NB}_{s-\frac{1}{p}}^{p,p}(\partial\Omega,\Lambda^\ell) \right]$ is studied in the theorem below.

To state it, we first recall that for any reasonable topological space $\mathcal{X}$ one associates $H^\ell_{\text{sing}}(\mathcal{X}; \overline{\mathbb{R}})$, the classical $\ell$-th singular homology group of $\mathcal{X}$ over the reals; see, e.g., [17]. Generally speaking, we set $b_\ell(\mathcal{X}) := \dim H^\ell_{\text{sing}}(\mathcal{X}; \overline{\mathbb{R}})$ and refer to it as the $\ell$-th Betti number of $\mathcal{X}$.
Theorem 6.2 The operator
\[ d_0 : NB_{s}^{p,p}(\partial \Omega, \Lambda^\ell) \longrightarrow N_{s}B_{s}^{p,p}(\partial \Omega, \Lambda^{\ell+1}) \] (6.8)
has closed range and its cokernel is isomorphic to \( H_{\text{sing}}^{\ell}(\partial \Omega; \mathbb{R}) \), the \( \ell \)-th singular homology group of \( \partial \Omega \) over the reals, i.e.
\[ N_{s}B_{s}^{p,p}(\partial \Omega, \Lambda^{\ell+1}) / d_0[NB_{s-1/2}^{p,p}(\partial \Omega, \Lambda^\ell)] \cong H_{\text{sing}}^{\ell}(\partial \Omega; \mathbb{R}), \quad \ell = 1, 2, \ldots, n-1. \] (6.9)

In particular,
\[ d_0 : N_{s}B_{s}^{p,p}(\partial \Omega, \Lambda^\ell) \longrightarrow N_{s}B_{s}^{p,p}(\partial \Omega, \Lambda^{\ell+1}) \] (6.10)
is a Fredholm operator with index \( \beta_{\ell}(\partial \Omega) := \dim H_{\text{sing}}^{\ell}(\partial \Omega; \mathbb{R}) \), the \( \ell \)-th Betti number of \( \partial \Omega \).

Proof. We shall make use of a deep theorem of De Rham which we present below in an abstract form, well suited for our purposes.

De Rham’s Theorem Let \( X \) be a Hausdorff, para-compact topological space, and let \( L^0, L^1, \ldots \) be fine sheaves over \( X \) and, for \( \ell = 0, 1, \ldots \), let \( \vartheta_{\ell} : L^\ell \rightarrow L^{\ell+1} \) be sheaf homomorphisms such that the following is an exact complex:
\[ 0 \longrightarrow \text{LCF}_X \hookrightarrow L^0 \xrightarrow{\vartheta_0} L^1 \xrightarrow{\vartheta_1} L^2 \xrightarrow{\vartheta_2} \cdots \] (6.11)
(hereafter, \( \iota \) denotes inclusion). Then
\[ H_{\text{sing}}^{\ell}(X; \mathbb{R}) \cong \frac{\text{Ker}(\vartheta_{\ell} : L^\ell(X) \longrightarrow L^{\ell+1}(X))}{\text{Im}(\vartheta_{\ell-1} : L^{\ell-1}(X) \longrightarrow L^\ell(X))}, \quad \ell = 1, 2, \ldots \] (6.12)
See [38], Theorem 5.25, p. 185 for a proof; cf. also [9].

Next, we shall describe a setting in which this powerful machinery applies. To set the stage, we first need to define a local version of the space (5.2) (cf. also (5.4)) as well as of the operator \( d_0 \). More specifically, for \( U \) an arbitrary open subset of \( \partial \Omega \), we define \( NB_{s-1/2}^{p,p}(U, \Lambda^{\ell}; \text{loc}) \) as the space consisting of functionals \( f \in (\text{Lip}(\partial \Omega, \Lambda^{\ell}))' \) enjoying the following properties.

(i) for each \( \varphi \in \text{Lip}(\partial \Omega) \) with \( \text{supp}\, \varphi \subset U \), it follows that \( \varphi f \in B_{s-1/2}^{p,p}(\partial \Omega, \Lambda^{\ell}) \);

(ii) for each \( x \in U \), there exist \( W \), open neighborhood of \( x \) in \( M \) with \( W \cap \partial \Omega \subset U \) and \( u \in D_{\ell-1}(d; L^p_\ell(\Omega)) \) such that \( f|_{W \cap \partial \Omega} = (\nu \wedge u)|_{W \cap \partial \Omega} \).

Also, introduce
\[ d_0 : NB_{s-1/2}^{p,p}(U, \Lambda^{\ell}; \text{loc}) \longrightarrow NB_{s-1/2}^{p,p}(U, \Lambda^{\ell+1}; \text{loc}) \] (6.13)
by setting $d_0 f := -\nu \wedge du$ near $x$, if $f$ is locally given by $\nu \wedge u$ near $x$. Clearly, each $N^{s-1}_p((U, \Lambda^\ell; loc))$ is an additive Abelian group and also a module over the algebra Lip$(\partial \Omega)$.

It follows that the family $N^{s-1}_p(U, \Lambda^\ell; loc)$ indexed by open subsets in $\partial \Omega$, is a fine sheaf on the topological space $\partial \Omega$.

Going further, we observe that the operator $d_\partial$ induces a natural sequence of sheaf morphisms

$$
0 \longrightarrow \text{LCF} \longrightarrow N^1_s p \longrightarrow N^2_s p \longrightarrow N^3_s p \longrightarrow \cdots
$$

where LCF stands for the sheaf of germs of locally constant functions on $\partial \Omega$ and the embedding works according to $f \mapsto f\nu$. Since $d_\partial \circ d_\partial = 0$, the above is a complex. In fact, so we claim, (6.14) provides a fine resolution of the sheaf LCF. The essential ingredient in the proof of this claim is the acyclicity of the complex (6.14). Granted this, the so-called abstract De Rham theorem applies to our context and gives (6.9). With (6.9) in hand, all the claims made in the statement of the theorem follow easily.

Next, we aim to prove the acyclicity of the sheaf (6.14). It is not hard to see that this is equivalent to proving that, for each $1 \leq \ell \leq n$, the following claim is true:

$$
\forall x_o \in \partial \Omega \text{ and } \forall u \in D_{\ell}(d; L^p_s(\Omega)) \text{ with } \nu \wedge du = 0 \text{ near } x_o,
$$

$$
\exists v \in D_{\ell-1}(d; L^p_s(\Omega)) \text{ such that } \nu \wedge u = \nu \wedge dv \text{ near } x_o.
$$

Since the case $\ell = n$ is trivial, below we shall focus on the proof of this claim when $1 \leq \ell \leq n - 1$. For starters, fix $u \in D_{\ell}(d; L^p_s(\Omega))$ such that $\nu \wedge du = 0$ near $x_o$ and let $\mathcal{O} \subset M$ be an open neighborhood of $x_o$ with the following properties:

(i) $\mathcal{O}$ is a Lipschitz domain contained in a coordinate patch and such that, when viewed in local, Euclidean coordinates, $\mathcal{O}$ is star-like with respect to some ball $B \subset \mathcal{O} \setminus \overline{\Omega}$;

(ii) $\mathcal{O} \cap \Omega$ is a Lipschitz domain for which $b_{\ell}(\mathcal{O} \cap \Omega) = 0$, $\ell = 1, 2, \ldots, n$;

(iii) $\nu \wedge du = 0$ on $\mathcal{O} \cap \partial \Omega$.

In this context, it has been shown in §4 of [24], that there exists a family of linear operators

$$
K_{\ell} : \left(C_c^\infty(\mathcal{O}, \Lambda^\ell)\right) \longrightarrow \left(C_c^\infty(\mathcal{O}, \Lambda^{\ell-1})\right),
$$

such that (for some $\theta \in C_c^\infty(\mathcal{O} \setminus \overline{\Omega})$),

$$
\text{the homotopy formula (5.40) holds } \forall w \in \left(C_c^\infty(\mathcal{O}, \Lambda^\ell)\right),
$$

$$
K_{\ell} : L^p_s(\mathcal{O}, \Lambda^\ell) \longrightarrow L^{p+1}_s(\mathcal{O}, \Lambda^{\ell-1}) \text{ boundedly},
$$

$$
\text{supp } (K_{\ell}w) \subset \mathcal{O} \cap \overline{\Omega} \text{ whenever } \text{supp } w \subset \mathcal{O} \cap \overline{\Omega}.
$$
From (iii) above, it follows that \( du|_{\partial \Omega} \in L^p(\Omega, \Lambda^{\ell+1}) \) and \( d(du|_{\partial \Omega}) = 0 \) in \( \mathcal{O} \), where tilde denotes the extension by zero of forms in \( \mathcal{O} \cap \Omega \) to \( \mathcal{O} \). Thus,

\[
w := K_{\ell+1}(du|_{\partial \Omega}) \in L^p_{s+1}(\Omega, \Lambda^{\ell}), \quad dw = du|_{\partial \Omega} \text{ in } \mathcal{O}, \quad \text{and supp } w \subset \mathcal{O} \cap \Omega. \quad (6.20)
\]

Since the trace of \( w \) on \( \mathcal{O} \cap \partial \Omega \) vanishes, the above properties imply

\[
d(u - w) = 0 \text{ in } \mathcal{O} \cap \Omega \quad \text{and} \quad \nu \wedge w = 0 \text{ on } \mathcal{O} \cap \partial \Omega. \quad (6.21)
\]

Next, bring in the isomorphism

\[
\begin{align*}
\{ \omega \in D(\partial; L^s_p(U) : d\omega = 0) \} &\sim H^s_{\text{sing}}(U; \mathbb{R}), \quad 1 \leq \ell \leq n, \quad (6.22) \\
\{ \eta \in D_{\Lambda}(d; L^s_p(U)) \} &\sim H^s_{\text{sing}}(U; \mathbb{R}), \quad 1 \leq \ell \leq n,
\end{align*}
\]

proved in [24] for any Lipschitz subdomain \( U \) of \( M \). Thanks to property (ii), it follows from (6.22) with \( U := \mathcal{O} \cap \Omega \) that there exists \( \eta \in D_{\Lambda}(d; L^s_p(U)) \) such that \( u - w = dw \) in \( \mathcal{O} \cap \Omega \). In particular, on \( \mathcal{O} \cap \partial \Omega \), we have \( \nu \wedge dw = \nu \wedge (u - w) = \nu \wedge u \). Hence, if \( \varphi \in C^\infty(M) \) is a scalar function such that \( \text{supp } \varphi \subset \mathcal{O} \) and \( \varphi \equiv 1 \) near \( x_o \), then \( \nu := \varphi \eta \) (viewed as a form in \( \Omega \)) does the job advertised in (6.15). This finishes the proof of the theorem. \( \square \)

We continue to assume that \( 1 < p < \infty \) and \( -1 + 1/p < s < 1/p \). Upon recalling the space (5.45), we define the operator

\[
d_{\partial} : TB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell}) \longrightarrow TB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell-1}) \quad (6.23)
\]

by setting

\[
d_{\partial} \xi := \eta, \quad (6.24)
\]

whenever \( \xi \) and \( \eta \) are as in (5.45). Much as before, this is well-defined, linear and bounded. Let us also note here that, based on (6.4) and Hodge duality,

\[
d_{\partial} \nu \wedge u = -\nu \wedge du, \quad \forall u \in D(\partial; L^p_s(\Omega)). \quad (6.25)
\]

The operators \( d_{\partial}, \delta_{\partial} \), are further related via the following duality result.

**Proposition 6.3** For each \( f \in TB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell}) \) and \( g \in NB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell}) \), there holds

\[
\langle \delta_{\partial} f, \nu \wedge g \rangle = -\langle \nu \wedge f, d_{\partial} g \rangle. \quad (6.26)
\]

**Proof.** By Theorem 5.2, we can choose \( u \in D(\partial; L^p_s(\Omega)), \quad w \in D_{\Lambda}(d; L^p_s(\Omega)) \) such that \( f = \nu \wedge u, \quad g = \nu \wedge w \). Also, from (6.25), (6.4) we have \( \delta_{\partial} f = -\nu \wedge du \) and \( d_{\partial} g = -\nu \wedge dw \). Thus, Corollary 5.5 gives

\[
\langle \delta_{\partial} f, \nu \wedge g \rangle = -\langle \nu \wedge du, \nu \wedge (\nu \wedge w) \rangle = -\langle \delta u, dw \rangle. \quad (6.27)
\]
In a similar fashion one can prove that $\langle \nu \wedge f, d\partial g \rangle = \langle \delta u, dw \rangle$, so that (6.26) follows from this and (6.27).

Define

$$T_s B^{p,p}_s(\partial \Omega, \Lambda^\ell) := \{ \xi \in T_s B^{p,p}_s(\partial \Omega, \Lambda^\ell) : \delta \partial \xi = 0 \}. \quad (6.28)$$

**Corollary 6.4** The operator

$$\delta \partial : T_B^{p,p}(\partial \Omega, \Lambda^\ell) \to T_s B^{p,p}_s(\partial \Omega, \Lambda^{\ell-1}) \quad (6.29)$$

has closed range and its cokernel is isomorphic to $H_{\text{sing}}^{n-\ell}(\partial \Omega; \mathbb{R})$, i.e.

$$\frac{T_s B^{p,p}_s(\partial \Omega, \Lambda^\ell)}{\delta \partial \left[ T_B^{p,p}(\partial \Omega, \Lambda^\ell) \right]} \cong H_{\text{sing}}^{n-\ell}(\partial \Omega; \mathbb{R}), \quad \ell = 1, 2, \ldots, n. \quad (6.30)$$

In particular,

$$\delta \partial : \frac{T_s B^{p,p}_s(\partial \Omega, \Lambda^\ell)}{T_s B^{p,p}_s(\partial \Omega, \Lambda^\ell)} \to T_s B^{p,p}_s(\partial \Omega, \Lambda^{\ell-1}) \quad (6.31)$$

is a Fredholm operator with index $b_{n-\ell}(\partial \Omega)$.

**Proof.** This is a simple consequence of Theorem 6.2 and Hodge duality. \[\Box\]

**Corollary 6.5** If $b_{n-\ell}(\Omega) = 0$ then the operator

$$\nu \wedge \cdot : \{ w \in L^p_s(\Omega, \Lambda^\ell) : dw = 0 \} \to N_s B^{p,p}_s(\partial \Omega, \Lambda^{\ell+1}) \quad (6.32)$$

is onto. Likewise, if $b_{\ell}(\Omega) = 0$ then the operator

$$\nu \vee \cdot : \{ w \in L^p_s(\Omega, \Lambda^\ell) : \delta w = 0 \} \to T_s B^{p,p}_s(\partial \Omega, \Lambda^{\ell-1}) \quad (6.33)$$

is onto.

**Proof.** If $\xi \in N_s B^{p,p}_s(\partial \Omega, \Lambda^{\ell+1})$ is a given, arbitrary form, then Theorem 5.2 ensures that there exists $u \in D_\ell(d; L^p_s(\Omega))$ such that $\xi = \nu \wedge u$ and for which $\nu \wedge du = 0$. In order to continue, we now recall a general formula proved in [24], to the effect that

$$\dim \left[ \{ \eta \in D_\ell(d; L^p_s(\Omega)) : d\eta = 0 \text{ and } \nu \wedge \eta = 0 \} \right] = b_{n-\ell}(\Omega), \quad (6.34)$$

for each $\ell = 0, \ldots, n - 1$. In the present context, this guarantees that there exists some $\omega \in D_{\ell-1}(d; L^p_s(\Omega))$ satisfying $\nu \wedge \omega = 0$ and such that $du = \omega$. Then $\xi = \nu \wedge (u - \omega)$ and $u - \omega \in L^p_s(\Omega, \Lambda^\ell)$ has $d(u - \omega) = 0$. Hence, the map (6.32) is onto.

Finally, the claim about (6.33) is a consequence of this and Hodge duality. \[\Box\]
Corollary 6.6 The space $NB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell})$ is the completion of $\nu \wedge C^\infty(M, \Lambda^{\ell})|_{\partial \Omega}$ in the norm $f \mapsto \|f\|_{B^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell+1})} + \|d\nu \wedge f\|_{B^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell+1})}$. In particular, the space $NB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell})$ is reflexive.

Finally, similar results hold for the space $TB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell})$.

Proof. It has been proved in [24] that $C^\infty(M, \Lambda^{\ell})$ is a dense subspace of $D_{\ell}(d; L^p_s(\Omega))$. Granted this, the claim in the first part of the statement of the corollary is a direct consequence of Theorem 5.2. The last part of the corollary then follows from this and Hodge duality.

7 Extending differential forms from $\Omega$ to $M$

Throughout this section we shall assume that $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. Given a Lipschitz domain $\Omega \subset M$, we wish to study the restriction operator

$$D_{\ell}(d; L^p_s(M)) \ni u \mapsto u|_{\Omega} \in D_{\ell}(d; L^p_s(\Omega)),$$

(7.1)

In the above context, this is clearly well-defined, linear and bounded, and the issue arises whether this is onto as well. In this regard, we have the following.

Theorem 7.1 There exists a bounded, linear operator

$$E : D_{\ell}(d; L^p_s(\Omega)) \longrightarrow D_{\ell}(d; L^p_s(M))$$

(7.2)

such that $(Eu)|_{\Omega} = u$ for each $u \in D_{\ell}(d; L^p_s(\Omega))$. In particular, the operator (7.1) is onto.

Proof. Set $\Omega_+ := \Omega$, $\Omega_- := M \setminus \overline{\Omega}$, and let

$$Ex^+ : NB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell+1}) \longrightarrow D_{\ell}(d; L^p_s(\Omega_+)),$$

(7.3)

$$Ex^- : NB^{p,p}_{s-\frac{1}{p}}(\partial \Omega, \Lambda^{\ell+1}) \longrightarrow D_{\ell}(d; L^p_s(\Omega_-)),$$

(7.4)

be the extension operators constructed in Theorem 5.2, corresponding to $\Omega_\pm$. Then, given an arbitrary differential form $u^+ \in D_{\ell}(d; L^p_s(\Omega))$, set

$$u := \tilde{u}^+ + \tilde{u}^-,$$

(7.5)

where $u^- := Ex^-(\nu \wedge u^+)$ and tilde denotes the extension by zero outside $\Omega$. It follows that

$$u \in L^p_s(M, \Lambda^{\ell})$$

(7.6)

$$u|_{\Omega} = u^+.$$
\[ du = \tilde{du}^+ + \tilde{du}^- \in L^p_s(M, \Lambda^{\ell+1}). \]  

(7.7)

In order to justify this, pick an arbitrary \( \Phi \in C^\infty(M, \Lambda^\ell) \) and write

\[
\langle du, \Phi \rangle = \langle \tilde{u}^+, \delta\Phi \rangle + \langle \tilde{u}^-, \delta\Phi \rangle = \langle u^+, (\delta\Phi)|_{\Omega^+} \rangle + \langle u^-, (\delta\Phi)|_{\Omega^-} \rangle
\]

\[
= \langle \tilde{u}^+, \Phi|_{\Omega^+} \rangle + \langle \tilde{u}^-, \Phi|_{\Omega^-} \rangle
\]

where we have used the fact that, by design, \( \nu \wedge \tilde{u}^+ = \nu \wedge \tilde{u}^- \). This justifies (7.7) and finishes the proof of the theorem.

A similar result holds for spaces defined in connection with the operator \( \delta \). More specifically we have:

**Corollary 7.2** The restriction operator

\[
D_\ell(\delta; L^p_s(M)) \ni u \mapsto u|_{\Omega} \in D_\ell(\delta; L^p_s(\Omega))
\]

is well-defined, linear, bounded and onto. In fact, there exists a bounded, linear operator

\[
E': D_\ell(\delta; L^p_s(\Omega)) \longrightarrow D_\ell(\delta; L^p_s(M))
\]

such that \( (E'u)|_{\Omega} = u \) for each \( u \in D_\ell(\delta; L^p_s(\Omega)) \).

**Proof.** This is an immediate consequence of Theorem 7.1 and the properties of the Hodge star-isomorphism; cf. Proposition 2.1. \( \square \)

Our last result in this section is a version of Theorem 7.1 for closed forms.

**Theorem 7.3** Given a Lipschitz domain \( \Omega \subset M \) and a differential form \( u \in D_\ell(d; L^p_s(\Omega)) \) with \( du = 0 \) in \( \Omega \), there exist an open neighborhood \( \mathcal{O} \) of \( \overline{\Omega} \) and some \( v \in D_\ell(d; L^p_s(\mathcal{O})) \) such that \( dv = 0 \) in \( \mathcal{O} \) and \( v|_{\Omega} = u \).

**Proof.** Fix a form \( u \) as in the statement of the theorem and note that the manifold \( M \) can be altered away from \( \overline{\Omega} \) as to ensure the existence of a Lipschitz domain \( \mathcal{O} \) for which

\[
\overline{\Omega} \subset \mathcal{O}, \quad b_{n-\ell-1}(\mathcal{O} \setminus \overline{\Omega}) = 0.
\]

(7.11)

Thanks to Theorem 7.1, there exists \( w \in D_\ell(d; L^p_s(M)) \) such that \( w|_{\Omega} = u \). Multiplying \( w \) by a function \( \psi \in C^\infty_c(\mathcal{O}) \) such that \( \psi \equiv 1 \) near \( \overline{\Omega} \), there is no loss of generality in assuming that \( w \equiv 0 \) near \( \partial\mathcal{O} \). In particular, \( \nu \wedge d(w|_{\mathcal{O}}) = 0 \) on \( \partial\mathcal{O} \). Since we also have

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\[ \nu \wedge (dw|_{\Omega})|_{\partial \Omega} = \nu \wedge (dw|_{\Omega}) = \nu \wedge du = 0, \]  
(7.12)

it ultimately follows that \( \nu \wedge d(w|_{\Omega}) = 0. \)

Granted the current assumptions on \( \Omega, \) formula (6.34) written with \( \Omega \setminus \overline{\Omega} \) in place of \( \Omega \) guarantees the existence of a differential form \( \omega \in D_\ell(d; L^p_s(\Omega \setminus \overline{\Omega})) \) such that

\[ d\omega = dw \text{ in } \Omega \setminus \overline{\Omega} \text{ and } \nu \wedge \omega = 0 \text{ on } \partial(\Omega \setminus \overline{\Omega}). \]  
(7.13)

Thus, by the nature of our construction,

\[ [\nu \wedge (w - \omega)|_{\partial \Omega} = (\nu \wedge w)|_{\partial \Omega} = \nu \wedge (w|_{\Omega}) = \nu \wedge u. \]  
(7.14)

Consequently, if \( \tilde{u} \) is the extension of \( u \) by zero to \( \Omega \) and \( \tilde{(w - \omega)}|_{\Omega \setminus \overline{\Omega}} \) is the extension of \( (w - \omega)|_{\Omega \setminus \overline{\Omega}} \) by zero to \( \Omega, \) then

\[ v := \tilde{u} + \tilde{(w - \omega)}|_{\Omega \setminus \overline{\Omega}} \in L^p_s(\Omega, \Lambda^{\ell}) \]  
(7.15)

satisfies \( v|_{\Omega} = u. \) Furthermore, thanks to (7.14), much as for the justification of (7.7), one can show that

\[ dv := \tilde{du} + \tilde{(dw - d\omega)}|_{\Omega \setminus \overline{\Omega}} = 0 \text{ in } \Omega. \]  
(7.16)

Thus, \( v \) does the job advertised in the statement of the theorem.

Of course, there is a natural Hodge dual version of Theorem 7.3; we omit the details. In closing, we only wish to point out that this result extends the work of T. Kato, M. Mitrea, G. Ponce and M. Taylor in [14] where the authors have dealt with the case \( M = \mathbb{R}^n, \ell = 1, s = 0, \) via a conceptually different proof which requires the domain \( \Omega \) to be suitably smooth if \( s \neq 0. \)

\section{Hodge decompositions}

We first discuss a key preliminary result, which can be viewed as the analogue of the well-known fact that the scalar Besov scale \( B^{s,p}_p(\partial \Omega) \) is stable under complex interpolation, at the level of differential forms.

\begin{theorem}
If \( 1 < p < \infty, -1 + 1/p < s < 1/p, 1 < p_j < \infty, -1 + 1/p_j < s_j < 1/p_j, j = 0,1 \) and \( \theta \in (0,1) \) is such that \( 1/p = (1 - \theta)/p_0 + \theta/p_1, s = (1 - \theta)s_0 + \theta s_1, \) then

\[ [NB^{p_0,p_0}_{s_0-1/p_0}(\partial \Omega, \Lambda^{\ell})], NB^{p_1,p_1}_{s_1-1/p_1}(\partial \Omega, \Lambda^{\ell})]|_{\partial \Omega} = NB^{p,p}_{s-1/p}(\partial \Omega, \Lambda^{\ell}), \]  
(8.1)

\[ [TB^{p_0,p_0}_{s_0-1/p_0}(\partial \Omega, \Lambda^{\ell})], TB^{p_1,p_1}_{s_1-1/p_1}(\partial \Omega, \Lambda^{\ell})]|_{\partial \Omega} = TB^{p,p}_{s-1/p}(\partial \Omega, \Lambda^{\ell}), \]  
(8.2)

for each \( 0 \leq \ell \leq n. \)
\end{theorem}
Proof. By Hodge duality, it suffices to establish (8.1) only. In turn, the proof of this identity is divided into three steps.

**Step I.** Let \(X_j, Y_j, Z_j, i = 0, 1\), be Banach spaces such that \(X_0 \cap X_1\) is dense in both \(X_0\) and \(X_1\), and similarly for \(Z_0, Z_1\). Suppose that \(Y_j \hookrightarrow Z_j, j = 0, 1\) and there exists a linear operator \(D : X_j \to Z_j\) boundedly for \(i = 0, 1\). Define the spaces

\[
X_j(D) := \{u \in X_j : Du \in Y_j\}, \quad j = 0, 1, \tag{8.3}
\]

equipped with the graph norm, i.e. \(\|u\|_X(D) := \|u\|_{X_j} + \|Du\|_{Y_j}, i = 0, 1\). Finally, suppose that there exist continuous linear mappings \(K : Z_j \to X_j\) and \(R : Z_j \to Y_j\) with the property \(D \circ K = I + R\) on the spaces \(Z_j\) for \(j = 0, 1\). Then

\[
[X_0(D), X_1(D)]_\theta = \{u \in [X_0, X_1]_\theta : Du \in [Y_0, Y_1]_\theta\}, \quad \theta \in (0, 1). \tag{8.4}
\]

This is due to J.-L. Lions and E. Magenes [16] (Theorem 14.3 on page 97); cf. also [11].

**Step II.** If \(p, p_0, p_1, s, s_0, s_1, \theta \in (0, 1)\) are as in the statement of the theorem, then

\[
[D_\ell(d; L_{s_0}(\Omega)), D_\ell(d; L_{s_1}(\Omega))]_\theta = D_\ell(d; L_s(\Omega)), \tag{8.5}
\]

for each \(0 \leq \ell \leq n\).

The problem localizes, so there is no loss of generality in assuming that \(\Omega\) is contained in a small coordinate patch such that, when viewed in local, Euclidean coordinates, \(\Omega\) is star-like with respect to some ball. In this context, we shall implement the abstract interpolation result from Step I twice. First, the goal is to show that

\[
\left[\{u \in L_{s_0}^p(\Omega, \Lambda^\ell) : du = 0\}, \{u \in L_{s_1}^p(\Omega, \Lambda^\ell) : du = 0\}\right]_\theta
= \{u \in L_s^p(\Omega, \Lambda^\ell) : du = 0\}, \quad \forall \ell \in \{0, 1, \ldots, n\}, \tag{8.6}
\]

so we find it convenient to take

\[
X_j := L_{s_j}^p(\Omega, \Lambda^\ell), \quad Y_j := 0, \quad j = 0, 1,
\]

\[
Z_j := \{u \in L_{s_j+1}^p(\Omega, \Lambda^{\ell+1}) : du = 0\}, \quad j = 0, 1,
\]

\[
D := d, \quad R := 0, \tag{8.7}
\]

and \(K := K_\ell\), the operator constructed in §4 of [24] where it has been shown that, if \(1 \leq \ell \leq n - 1\),

\[
K_\ell : L_{s-1}^p(\Omega, \Lambda^\ell) \to L_{s}^p(\Omega, \Lambda^{\ell-1}) \text{ boundedly, and} \tag{8.8}
\]

\[
u = dK_\ell u, \quad \forall u \in L_{s-1}^p(\Omega, \Lambda^\ell) \text{ with } du = 0 \text{ in } \Omega, \tag{8.9}
\]

whenever \(1 < p < \infty\) and \(s < 1 + 1/p\).
Since $X_j(D) = \{ u \in L^p_{s_j}(\Omega, \Lambda^\ell) : du = 0 \}, j = 0, 1$, the abstract result in Step I applies and yields (8.6), thanks to Proposition 3.2.

Our second implementation of the abstract interpolation result from Step I is when

$$X_j := L^p_{s_j}(\Omega, \Lambda^\ell), \quad j = 0, 1,$$

$$Y_j := \{ u \in L^p_{s_j}(\Omega, \Lambda^\ell+1) : du = 0 \}, \quad j = 0, 1,$$

$$Z_j := \{ u \in L^p_{s_j-1}(\Omega, \Lambda^\ell+1) : du = 0 \}, \quad j = 0, 1,$$

$$D := d, \quad R := 0, \quad K := K_\ell$$

as in (8.8)–(8.9).

Thanks to (8.6), this time $X_j(D) = D_\ell(d; L^p_{s_j}(\Omega)), j = 0, 1$, and an application of Step I yields (8.5).

**Step III. Proof of (8.1).**

This is now an immediate consequence of Step II, Theorem 5.2 and properties of retractions. \qed

The case $s_0 = s_1 = s = 0$ of Theorem 8.1 has been first established in [21] via an approach which requires flattening the boundary. As explained in the introduction, this method is confined to the $L^p$ scale, i.e., it does not allow the consideration of forms with coefficients in $L^p_s$ when the smoothness index satisfies $s \neq 0$.

Nonetheless, once Theorem 8.1 has been established, the rest of the approach in [21], originally developed for forms with coefficients in $L^p$, can be adapted to the case when the coefficients are from $L^p_s$, at least if $p$ is near $2$ and $s$ is sufficiently close to zero. Briefly, the genesis of the range $2 - \varepsilon < p < 2 + \varepsilon, -\varepsilon < s < \varepsilon$ as follows. The crux of the approach developed in [21] is deriving Hodge decompositions as corollary of the solvability of certain Poisson-type problems for the Hodge Laplacian $\Delta = -d\delta - \delta d$ in $\Omega$. In turn, these boundary value problems are reduced to the invertibility of a certain layer potential integral operator on the scale $\text{NB}_{p,p}^{s-\frac{1}{2}}(\partial \Omega, \Lambda^\ell)$. The case $s = 0, p = 2$ is special, as it naturally lends itself to Hilbert space methods. Once the invertibility has been established in this particular situation, perturbation methods yield a similar result for $|p - 2| < \varepsilon, |s| < \varepsilon$. Here, (8.1) is of paramount importance; see [13], [30].

In particular, all the main results from [21] have an analogue in this more general setting. For example, we have the following Hodge decomposition result for differential forms with coefficients in $L^p_s$:

**Theorem 8.2** Let $\Omega$ be an arbitrary Lipschitz subdomain of $M$. Then there exists some $\varepsilon = \varepsilon(\Omega) > 0$ with the following significance. For any $\ell \in \{0, 1, \ldots, n\}$, the space

$$\mathcal{H}(\Omega, \Lambda^\ell) := \{ u \in L^p_s(\Omega, \Lambda^\ell) : du = 0, \delta u = 0, \nu \wedge u = 0 \}$$

(8.12)

is independent of $p \in (2 - \varepsilon, 2 + \varepsilon)$ and $s \in (-\varepsilon, \varepsilon)$. Furthermore, for such $p$ and $s$, the dimension of (8.12) is $b_{n-\ell}(\Omega)$ and, if
\[ \mathcal{H}^\vee(\Omega, \Lambda^\ell) := *\mathcal{H}_\Lambda(\Omega, \Lambda^{n-\ell}) = \{ u \in L^p_s(\Omega, \Lambda^\ell) : \; du = 0, \; \delta u = 0, \; \nu \lor u = 0 \} \quad (8.13) \]

then

\[
L^p_s(\Omega, \Lambda^\ell) = \{ du : u \in D_{\ell-1}(d; L^p_s(\Omega)), \; \nu \land u = 0 \}
\]
\[
\oplus \{ \delta w : w \in D_{\ell+1}(\delta; L^p_s(\Omega)) \} \oplus \mathcal{H}_\Lambda(\Omega, \Lambda^\ell),
\]
\[
= \{ \delta u : u \in D_{\ell+1}(\delta; L^p_s(\Omega)), \; \nu \lor u = 0 \}
\]
\[
\oplus \{ dw : w \in D_{\ell-1}(d; L^p_s(\Omega)) \} \oplus \mathcal{H}^\vee(\Omega, \Lambda^\ell),
\]
\[
(8.14)
\]

where the direct sums are topological.

We leave to the interested reader the formulation of other related results from [21] in the more general setting considered in this paper. Here we only want to point out that, when \( n = 3 \), the sharp range of indices \( p, s \) for which (8.14)-(8.15) hold has been identified in [22]. What the optimal range is for \( n \geq 4 \) remains an open problem at the moment.

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