THE SMOOTHNESS OF RIEMANNIAN SUBMERSIONS
WITH NONNEGATIVE SECTIONAL CURVATURE

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Abstract. Let \( M^n \) be a complete, non-compact and \( C^\infty \)-smooth Riemannian manifold with nonnegative sectional curvature. Suppose \( S \) is a soul of \( M^n \) given by the fundamental theory of Cheeger and Gromoll, and suppose that \( \Psi : M^n \to S \) is a distance non-increasing retraction from the whole manifold to the soul (e.g., the retraction given by Sharafutdinov). Then we show that the retraction \( \Psi \) above must give rise to a \( C^\infty \)-smooth Riemannian submersion from \( M^n \) to the soul \( S \).

Moreover, we derive a new flat strip theorem associated with the Cheeger-Gromoll convex exhaustion for the manifold above.

In this article, we study the smoothness of Riemannian submersions for open manifolds with non-negative sectional curvature. Suppose that \( M^n \) is a \( C^\infty \)-smooth, complete and non-compact Riemannian manifold with nonnegative sectional curvature. Cheeger-Gromoll [ChG] established a fundamental theory for such a manifold. Among other things, they showed that \( M^n \) admits a totally convex exhaustion \( \{ \Omega_u \}_{u \geq 0} \) of \( M^n \), where \( \Omega_0 = S \) is a totally geodesic and compact submanifold without boundary. Furthermore, \( M^n \) is diffeomorphic to the normal vector bundle of the soul \( S \).

Sharafutdinov found that there exists a distance non-increasing retraction \( \Psi : M^n \to S \) from the open manifold \( M^n \) of non-negative sectional curvature to its soul, (cf. [Sh], [Y1]). Perelman [Per] further showed that such a map \( \Psi \) is indeed a \( C^1 \)-smooth Riemannian submersion. Furthermore, \( \Psi[Exp_q(t\vec{v})] = q \) for any \( q \in S \) and \( \vec{v} \perp T_q(S) \). Therefore, the fiber \( F_q = \Psi^{-1}(q) \) is a \( k \)-dimensional submanifold, which is \( C^\infty \)-smooth almost everywhere, where \( k = \dim(M^n) - \dim(S) > 0 \).

Guijarro [Gu] proved that the fiber \( F_q \) is indeed a \( C^2 \)-smooth submanifold for each \( q \in S \). In this paper, we prove that the fibres are \( C^\infty \)-smooth.

**Theorem 1.** Let \( M^n \) be a complete, non-compact and \( C^\infty \)-smooth Riemannian manifold with nonnegative sectional curvature. Suppose \( S \) is a soul of \( M^n \). Then any distance non-increasing retraction \( \Psi : M^n \to S \) must give rise to a \( C^\infty \)-smooth Riemannian submersion.

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Consequently, if $\mathbb{R}^k = N_q(S, M^n)$ is the normal space of the soul $S$ in $M^n$ at $q$, then the fiber $F_q = \Psi^{-1}(q) = \text{Exp}_q(\mathbb{R}^k)$ is a $k$-dimensional $C^\infty$-smooth submanifold of $M^n$, for any $q \in S$.

Professor Wilking kindly informed us that he has recently obtained a similar result (cf. [Wi]). His method is completely independent of ours. Our proof of Theorem 1 uses a flat strip theorem associated with Cheeger-Gromoll exhaustion (cf. Theorem 4 below), an uniform estimate for cut-radii of convex subsets in [ChG] and a smooth extension theorem for ruled surfaces.

For each compact convex subset $\Omega \subset M^n$, we let $U_\epsilon(\Omega) = \{x \in M^n | d(x, \Omega) < \epsilon\}$. Its cut-radius is given by $\delta_\Omega = \sup \{|\epsilon| \therefore$ there is a unique nearest point projection $P_\Omega : U_\epsilon(\Omega) \rightarrow \Omega\}$.

For each $x \in M^n$, we let $\text{Inj}_{M^n}(x)$ be the injectivity radius of $M^n$ at $x$. Similarly, let $\text{Inj}_{M^n}(A) = \sup\{|\text{Inj}_{M^n}(x)|x \in A\}$.

A subset $\Omega$ of a complete Riemannian manifold $M^n$ is said to be totally convex if for any pair of points $\{p, q\} \subset \Omega$ and for any geodesic segment $\sigma$ joining $p$ and $q$, the geodesic segment $\sigma$ is contained in $\Omega$. There is a totally convex exhaustion $\{\Omega_u\}_{u \geq 0}$ of $M^n$ given in [ChG]. By comparing the inner angles of geodesic triangles, we have the following semi-global estimate for cut-radius.

**Lemma 2.** (Lemma 2.4 of [ChG], [CaS]) Let $A \subset \Omega_T$ be a connected, convex and compact subset in a Riemannian manifold $M^n$ with nonnegative curvature, let $K_0 = \max\{K(x)|x \in \Omega_{T+1}\}$ be the upper bound of sectional curvature on $\Omega_{T+1}$, $\text{Inj}_{M^n}(\Omega_T)$ and $S$ be as above. Suppose that $\dim(\Omega_T) = n$. Then the subset $A$ has cut-radius bounded below by

$$
\delta_A \geq \delta_0(T) = \frac{1}{4} \min\{\text{Inj}_{M^n}(\Omega_T), \frac{\pi}{\sqrt{K_0}}, 1\},
$$

where $\delta_0(T)$ is independent of choices of $A$ with $A \subset \Omega_T$.

Let us briefly recall the Cheeger-Gromoll convex exhaustion. According to [ChG], there is a partition $a_0 = 0 < a_1 < ...a_m < a_{m+1} = \infty$ of $[0, \infty)$ and an exhaustion $\{\Omega_u\}_{u \geq 0}$ of $M^n$ such that the following holds:

(1) $M^n = \bigcup_{u \geq 0} \Omega_u$. If $u > a_m$ then $\dim[\Omega_u] = n$. If $u \leq a_m$, then $\dim[\Omega_u] < n$.

(2) $\Omega_0 = S$ is the soul of $M^n$, which is a totally geodesic $C^\infty$-smooth compact submanifold without boundary.

(3) If $u > 0$, $\Omega_u$ is a totally convex, compact subset of $M^n$ and hence $\Omega_u$ is a compact submanifold with a $C^\infty$-smooth relative interior. Furthermore, $\dim(\Omega_u) = k_u > 0$ and $\Omega_u$ has a non-empty $(k_u - 1)$-dimensional relative boundary $\partial\Omega_u$.

(4) For any $u_0 \in [a_j, a_{j+1}]$ and $0 \leq t \leq u_0 - a_j$, the family $\{\Omega_{u_0-t}\}_{t \in [0, u_0-a_j]}$ is given by the inward equidistant evolution:

(2.1) $\Omega_{u_0-t} = \{x \in \Omega_{u_0} | d(x, \partial\Omega_{u_0}) \geq t\}$. 

(5) If \( u > a_m \) then \( u - a_m = \max\{d(x, \partial \Omega_u)|x \in \Omega_u}\}. If \( 0 \leq j \leq m - 1 \) then \( a_{j+1} - a_j = \max\{d(x, \partial \Omega_{a_{j+1}})|x \in \Omega_{a_{j+1}}\} \) and hence \( \dim[\Omega_{a_j}] < \dim[\Omega_{a_{j+1}}] \) for \( j \geq 0 \).

Assume that \( k = \dim[M^n] - \dim[S] = \dim(F_q) \) for all \( q \in S \). Since \( M^n = \cup_{T \geq 0} \Omega_T \), it is sufficient to verify that the subset \( [U_{\delta_0}(T) \cap F_q] \) has a \( k \)-dimensional \( C^\infty \)-smooth interior, where \( \delta_0(T) \) is given by Lemma 2 and \( T > a_m \).

For this purpose, we need to study the geometry of the equidistant hypersurfaces from \( \partial \Omega \). Federer [Fe, p435] has studied the smoothness of the outward equidistant hypersurfaces \( \partial [U_\epsilon(\Omega)] \) for \( 0 < \epsilon < \delta_\Omega \). Following his approach, we consider the outward normal cone of \( \Omega \) as follows:

\[
\mathcal{N}^+(\Omega, M^n) = \{(p, \vec{v})|p \in \Omega, d(Exp_p(t\vec{v}), \Omega) = t|\vec{v}|, \quad \text{for } 0 \leq t|\vec{v}| < \delta_\Omega\}.
\]

If \( \{\Omega_u\} \) is the Cheeger-Gromoll convex exhaustion as above and \( u > 0 \), then the relative boundary \( \partial \Omega_u \) is not necessarily smooth. When \( u > 0 \), we let \( \text{int}(\Omega_u) \) be the relative interior of the convex subset \( \Omega_u \). We are going to study the corresponding decomposition of \( \mathcal{N}^+(\Omega, M^n) \):

\[
(2.2) \quad \mathcal{N}^+_p(\Omega_u, M^n) \subset [\mathcal{N}^+_p(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \oplus \mathcal{N}^+_p(\text{int}(\Omega_{u+\epsilon}), M^n)],
\]

where \( \mathcal{N}^+_p(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \) is defined by

\[
\mathcal{N}^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) = \{(p, \vec{v})|p \in \Omega_u, d(Exp_p(t\vec{v}), \Omega_u) = t|\vec{v}|, \quad \text{for } 0 \leq t|\vec{v}| < \delta_{\Omega_u}, \quad \text{Exp}_p(t\vec{v}) \in \text{int}(\Omega_{u+\epsilon})\}.
\]

Our next step is to choose \( \epsilon \) sufficiently small so that (1) there is a nearest point projection \( \mathcal{P} : \text{int}(\Omega_{u+\epsilon}) \to \Omega_u \), and (2) \( \Omega_u = \{x \in \Omega_{u+\epsilon}|d(x, \partial \Omega_{u+\epsilon}) \geq \epsilon\} \) holds. We first find \( j \) so that \( a_j \leq u < a_{j+1} \) for some \( 0 \leq j \leq m \). Let \( T = u + a_m + 1 \) and \( \delta_0(T) \) be given by Lemma 2. It follows from a result of Yim that there is a constant \( C_T \) such that, for \( 0 \leq u_1 < u_2 < T \), we have

\[
(2.3) \quad \max\{d(x, \Omega_{u_1})|x \in \Omega_{u_2}\} \leq C_T(u_2 - u_1),
\]

see [Y2, Theorem A.5(3)]. In what follows, we always choose

\[
(2.4) \quad 0 < \epsilon = \epsilon_u < \min\{[a_{j+1} - u], \frac{\delta_0(T)}{2C_T}\},
\]

where \( u \in [a_j, a_{j+1}] \), \( T = u + a_m + 1 \) and \( \delta_0(T) \) is given by Lemma 2.

With such a choice of \( \epsilon = \epsilon_u \) by (2.4), the geometry of \( \mathcal{N}^+_p(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \) is determined by its minimal normal vectors which we now describe.
Definition 3. (Minimal normal vector) Let $\Omega_u$, $\Omega_{u+\epsilon}$ and $\mathcal{N}^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ be as above. Let $\sigma_{(p,\vec{v})} : [0, \epsilon) \to M^n$ be a geodesic given by $\sigma_{(p,\vec{v})}(t) = \text{Exp}_p(t\vec{v})$, where $\vec{v} \neq 0$. If $\sigma_{(p,\vec{v})}$ is a length-minimizing geodesic from $p \in \Omega_u$ to $\partial \Omega_{u+\epsilon}$, then $\vec{v}$ is called a minimal normal vector in $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$.

It is known that any other normal vector $\vec{w} \in \mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ can be expressed as a linear combination of minimal normal vectors at $p$. Moreover, the convex hull of minimal normal vectors at $p$ is equal to $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$, (cf. [Y1, Proposition 1.7]).

For each $p \in M^n$, we let $\mathcal{V}_p = T_p(F_{\Psi(p)})$ and $\mathcal{H}_p = [\mathcal{V}_p]^\perp$. A geodesic $\alpha : [a, b] \to M^n$ is said to be horizontal, if $\alpha'(t) \perp F_{\Psi(\alpha(t))}$ for all $t \in [a, b]$. We need the following flat strip theorem for the proof of Theorem 1.

Theorem 4. Let $\{\Omega_u\}$ be the Cheeger-Gromoll totally convex exhaustion of $M^n$ as above. Suppose that $\Psi : M^n \to S$ be a distance non-increasing retraction and $F_q = \Psi^{-1}(q)$ be a fibre for some $q \in S$. Then for $p \in F_q \cap \Omega_u$ and any $(p, \vec{v}) \in \mathcal{N}^+(\Omega_u, M^n)$, we have

$$\Psi(\text{Exp}_p(\|\vec{v}\|)) = \Psi(p) = q.$$  

Moreover, if $\dim(S) \geq 1$ and if $\vec{w} \in \mathcal{H}_p$ has $|\vec{w}| = 1 = |\vec{v}|$, then the surface $\Sigma_{\vec{v},\vec{w}}^2 = \text{Exp}_p[\mathbb{R}\{\vec{v}\} \oplus \mathbb{R}\{\vec{w}\}]$ is totally geodesic immersed flat plane in $M^n$.

A result similar to Theorem 4 was proved in [CaS] via a totally different method.

Proof of Theorem 4. Theorem 4 was proved by Perelman [Per] for the case of $\Omega_0 = S$. Applying Perelman’s argument for the case of $p \not\in S$, Guijarro [Gu1] found the following sufficient condition for (4.1).

(4.2) $\vec{v} \in \mathcal{V}_p$ stays vertical under parallel transport along any horizontal broken geodesic.

Guijarro showed that (4.1) follows from (4.2). Moreover, if (4.2) holds and if $\vec{w} \in \mathcal{H}_p$ has $|\vec{w}| = 1 = |\vec{v}|$, then the surface $\Sigma_{\vec{v},\vec{w}}^2 = \text{Exp}_p[\mathbb{R}\{\vec{v}\} \oplus \mathbb{R}\{\vec{w}\}]$ is totally geodesic immersed flat plane in $M^n$, (cf. Theorem 3.1 of [Gu1]).

In order to see that $\mathcal{N}^+(\Omega_u, M^n) \subset \mathcal{V}_p$ holds, we recall that any horizontal geodesic $\alpha$ is contained a tubular neighborhood of the soul $S$, by Perelman’s theorem [Per]. Hence, $\alpha$ is contained in a compact totally geodesic subset $\Omega_T$ for a sufficiently large $T$. It follows from Theorem 5.1 of [ChG] that $\alpha \subset \partial \Omega_u$ for some $u$, (cf. [Gu2]).

(4.3) Any horizontal geodesic $\alpha$ with $\alpha(0) \in \partial \Omega_\lambda$ must be entirely contained in $\partial \Omega_\lambda$. Since $\Omega_u = S \cup [\bigcup_{\lambda \leq u} (\partial \Omega_\lambda)]$, we have $\mathcal{H}_p \subset T^-_p(\partial \Omega_\lambda) \subset T^-_p(\Omega_u)$, where $T^-_p(\partial \Omega_\lambda)$ is the tangent cone of $\partial \Omega_\lambda$ at $p$. 

Recall that \( \text{int}(\Omega) \) is the relative interior of the convex subset \( \Omega \). If \( p \in \text{int}(\Omega_u) \) and if \( \vec{v} \in \mathcal{N}_p^+(\Omega_u, M^n) \), Guijarro [Gu1, Corollary 3.2] showed that \( \vec{v} \) satisfies (4.2), because \( \text{int}(\Omega_u) \) is totally geodesic and (4.3) holds.

It remains to consider the case when \( p \in \partial\Omega_u \). Recall that by (2.2) we have

\[
\mathcal{N}_p^+(\Omega_u, M^n) \subset [\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \oplus \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n)].
\]

For \( \vec{v} \) in either \( \mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \) or \( \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n) \), we will show that such a \( \vec{v} \) satisfies (4.2).

It follows from Theorem 1.10 of [ChG] (or Corollary 1.4 of [Y1]) that any minimal normal vector \( \vec{v} \) of \( \mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \) stays minimal under parallel transport along any geodesic in \( \partial\Omega_u \). Since the convex hull of minimal normal vectors is equal to the outward normal cone (cf. [Y1, Proposition 1.7]), the bundle \( \mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \) is invariant under parallel transport along any geodesic in \( \partial\Omega_u \). This together with (4.3) implies that if \( \vec{v} \in \mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \) then \( \vec{v} \) satisfies (4.2).

For \( \vec{v} \in \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n) \), the assertion (4.2) follows from Corollary 3.2 of [Gu1]. In fact, since \( \text{int}(\Omega_{u+\epsilon}) \) is totally convex and totally geodesic, both \( T(\text{int}(\Omega_{u+\epsilon})) \) and \( \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n) \) are invariant under parallel transport along any geodesic in \( \text{int}(\Omega_{u+\epsilon}) \). This together with (4.3) implies that (4.2) holds for any vector \( \vec{v} \in \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n) \).

Therefore, (4.2) holds for any \( \vec{v} \in \mathcal{N}_p^+(\Omega_u, M^n) \). This completes the proof of Theorem 4. \( \square \)

In order to see that Theorem 4 implies Theorem 1, we need to establish a bootstrap argument for the smoothness of ruled surfaces. A \( C^1 \)-smooth one-parameter family of a straight lines in \( \mathbb{R}^3 \) gives rise to a ruled surface. Suppose that \( \{\beta(s), \vec{v}(s)\} \) are \( C^1 \)-smooth vector valued functions with \( [\beta'(s) + t\vec{v}'(s)] \wedge \vec{v}(s) \neq 0 \) for all \( (s, t) \in (a, b) \times (c, d) \). Then we have a corresponding \( C^1 \)-smooth immersed ruled surface.

\[
F: \quad (a, b) \times (c, d) \rightarrow \mathbb{R}^3 \\
(s, t) \mapsto \beta(s) + t\vec{v}(s)
\]

Our bootstrap argument is motivated by the following observation.

**Lemma 5.** (The smooth extension for ruled surfaces in \( \mathbb{R}^3 \)) Let \( F((a, b) \times (c, d)) = \Sigma^2 \) be an embedded ruled surface in \( \mathbb{R}^3 \) and let \( F: \quad (a, b) \times (c, d) \rightarrow \mathbb{R}^3 \) be a \( C^{1,1} \)-smooth embedding map as above. Suppose that a subset \( \hat{\Sigma}^2_\epsilon = F((a, b) \times (\epsilon_1, \epsilon_2)) \) is a \( C^\infty \)-smooth embedded surface of \( \mathbb{R}^3 \), where \( (\epsilon_1, \epsilon_2) \subset (c, d) \). Then the whole ruled surface \( \Sigma^2 \) is a \( C^\infty \)-smooth surface of \( \mathbb{R}^3 \).

**Proof.** By our assumption, \( F \) is an embedding map, and hence the surface \( \hat{\Sigma}^2_\epsilon = F((a, b) \times (\epsilon_1, \epsilon_2)) \) is foliated by straight lines. Because the surface \( \hat{\Sigma}^2_\epsilon \) and each orbit (each straight line) are \( C^\infty \), the quotient space \( Q = [\hat{\Sigma}^2_\epsilon/\sim] \) is a \( C^\infty \)-smooth 1-dimensional space as well, where \( \sim \) is the equivalent relation induced by the orbits.
(the ruling straight lines). Thus, we have a fibration \((\epsilon_1, \epsilon_2) \to \hat{\Sigma}_\epsilon^2 \to Q\). We may assume that \(Q = (0, 1)\). Let \(\pi : \hat{\Sigma}_\epsilon^2 \to Q\) be the quotient map. Because the fibration is topologically trivial, we can find two disjoint \(C^\infty\)-smooth cross-sections

\[
h_i : Q \to \hat{\Sigma}_\epsilon^2 \\
\mu \to h_i(\mu)
\]

for \(i = 0, 1\), where \(\pi(h_i(\mu)) = 0\). (Since the fibre is 1-dimensional, we may assume that the graph of the cross-section \(h_1\) lies above that of \(h_0\).) Because \(h_0(Q)\) and \(h_1(Q)\) are disjoint, we obtain a new \(C^\infty\)-smooth parametrization of the ruled surface

\[
G : Q \times \mathbb{R} \to \mathbb{R}^3 \\
(u, \lambda) \to h_0(u) + \lambda \frac{[h_1(u) - h_0(u)]}{\|h_1(u) - h_0(u)\|}
\]

Clearly, \(G\) is a \(C^\infty\)-smooth map with \(\Sigma^2 \subset G(Q \times \mathbb{R})\). Because \(F\) is an embedding map, on the subset \(G^{-1}(\Sigma^2)\), one can check that \(G\) remains to be injective and with non-vanishing Jacobi \(G_u \wedge G_{\lambda} \neq 0\). Hence, \(G|_{G^{-1}(\Sigma^2)}\) is an embedding as well. Thus, \(\Sigma^2\) is a \(C^\infty\)-smooth embedded surface.

The proof of Lemma 5 can be applied to the proof of Theorem 1 as follows. Let \(\Omega_u\) be a totally convex subset as above. By a Theorem of Federer (cf. Theorem 4.8 (9), p435 of [Fe]), the hypersurface \(\partial[U_\epsilon(\Omega_u)]\) is \(C^{1,1}\)-smooth if the positive number \(\epsilon\) is less than the cut-radius of \(\Omega_u\). Assume that \(T > u\) and \(d = \delta_T - \epsilon > 0\). Let \(\tilde{v}(x)\) be the outward unit normal vector of \(\partial[U_\epsilon(\Omega)]\) at \(x\). There is an embedding:

\[
F : \partial[U_\epsilon(\Omega)] \times (c, d) \to M^n \\
(x, t) \to \text{Exp}_x[t\tilde{v}(x)]
\]

(6.1)

where \(c = -\epsilon\).

**Proposition 6.** (The smooth extension for the ruled sub-manifold) For each \(y \in \partial[U_\epsilon(\Omega)]\), we let \(B^k_{\epsilon}(y) \subset \partial[U_\epsilon(\Omega)]\) be a small \((k - 1)\)-dimensional ball around \(y\) which is \(C^1\)-diffeomorphic to \(B^k_{\epsilon}(0) \subset \mathbb{R}^{k-1}\) and let \(\Omega_u, \delta, \delta_T, c, d\) and \(F\) be as above. Suppose that \(F\) is an \(C^{1,1}\)-smooth embedding and that \(\hat{\Sigma}_\epsilon^k = F(B^k_{\epsilon}(y) \times (\epsilon_1, \epsilon_2))\) is a \(C^\infty\)-smooth embedded \(k\)-submanifold of \(M^n\), where \((\epsilon_1, \epsilon_2) \subset (c, d)\). Then the whole ruled submanifold \(\Sigma^k = F(B^k_{\epsilon}(y) \times (c, d))\) is a \(C^\infty\)-smooth submanifold of \(M^n\).

**Proof.** The proof of Proposition 6 is the same as above with minor modifications. By our assumption, \(\hat{\Sigma}_\epsilon^k\) is foliated by \(C^\infty\)-smooth open geodesic segments. The quotient space \(Q = [\hat{\Sigma}_\epsilon^k / \sim]\) is a \(C^\infty\)-smooth \((k - 1)\)-dimensional open manifold. Because the fibration \((\epsilon_1, \epsilon_2) \to \hat{\Sigma}_\epsilon^k \to Q\) is trivial, we can choose two disjoint
cross sections \( h_0 : Q \to \hat{\Sigma}^k \) for \( i = 0, 1 \). If \( \pi : \hat{\Sigma}^k \to Q \) is the quotient map, then \( \pi \circ h_i(u) = u \) for all \( u \in Q \). Since the two cross-sections are disjoint, we may assume that \( r(h_1(u)) > r(h_0(u)) \) for all \( u \in Q \), where \( r(y) = d(y, \partial[U_\delta(\Omega)]) \). For each \( u \in Q \), we consider the unit vector

\[
\tilde{\eta}(u) = \frac{\text{Exp}_{h_0(u)}^{-1}[h_1(u)]}{\|\text{Exp}_{h_0(u)}^{-1}[h_1(u)]\|}
\]

at the point \( h_0(u) \). Similarly, we consider a new \( C^\infty \)-smooth parametrization

\[
G : Q \times \mathbb{R} \to M^n
\]

\[
(u, \lambda) \to \text{Exp}_{h_0(u)}[\lambda\tilde{\eta}(u)].
\]

Clearly, we have \( \Sigma^k = F(B^1_k(y) \times (c, d)) \subset G(Q \times \mathbb{R}) \). This completes the proof. \( \square \)

With Lemma 2, Theorem 4 and Proposition 6, we are ready to prove Theorem 1.

**Proof of Theorem 1.** Let \( \{\Omega_u\} \) be a Cheeger-Gromoll convex exhaustion described as above. It is sufficient to verify that the subset \([U_{\delta_0(T)}(\Omega_T) \cap F_q] \) has a \( k \)-dimensional \( C^\infty \)-smooth interior for any given \( T > a_m \) and \( q \in S \), where \( \delta_0(T) \) is given by Lemma 2.

Fix \( T > a_m \) with \( \dim[\Omega_T] = n \). Let \( \delta_0(T) \) be given by Lemma 2 and \( C_T \) be given by (2.3). Choose a partition \( 0 = u_0 < u_1 < \ldots < u_N = T \) of \([0, T]\) such that \( u_j - u_{j-1} < 2\delta_0(T) \) for \( j = 1, \ldots, N \), where \( N = N_T \) is a number depending on \( T \).

We will prove the following assertion by induction on \( j = 0, 1, \ldots, N \).

**Assertion j.** The sub-level set \([U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q] \) has the \( k \)-dimensional \( C^\infty \)-smooth interior, where \( q \in S \) and \( k = \dim[F_q] \).

It follows from Perelman’s theorem or Theorem 4 that \( \text{Exp}_q[N^+_q(S, M^n)] \subset F_q \).

Since the soul \( S \) has the cut radius \( \geq \delta_0(T) \) and \( S \) is \( C^\infty \)-smooth, Assertion 0 holds.

Let \( \epsilon_1 = \frac{\delta_0(T)}{16} \) and \( \epsilon_2 = \frac{\delta_0(T)}{8} \). We consider

\[
A(\Omega_{u_j}, r_1, r_2) = \{ z \in F_q \mid 0 < r_1 < d(z, \Omega_{u_j}) < r_2 \}
\]

It is clear that \( A(\Omega_{u_1}, \epsilon_1, \epsilon_2) \subset U_{\delta_0(T)}(S) \). It follows from Assertion 0 that the subset \( \hat{\Sigma}^k = A(\Omega_{u_1}, \epsilon_1, \epsilon_2) \subset F_q \cap U_{\delta_0(T)}(S) \) is \( C^\infty \)-smooth \( k \)-dimensional open sub-manifold. By Theorem 4, we let \( \Sigma^k = A(\Omega_{u_1}, \frac{\delta_0(T)}{16}, \delta_0(T)) \) be the ruled \( k \)-dimensional submanifold. It follows from Proposition 6 (the smooth extension theorem for the ruled submanifold) that \( \Sigma^k \) is a \( C^\infty \)-smooth \( k \)-dimensional submanifold of \( M^n \). Observe that the subset \([U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q] \) is contained in the
union \{[U_{\delta_0(T)}(S) \cap F_q] \cup \Sigma^k_1\}. Since \Sigma^k_1 is a $C^\infty$-smooth, Assertion 1 follows from Assertion 0.

Similarly, using Theorem 4 and Proposition 6 we can verify that if Assertion (j-1) is true then Assertion j holds as well for $j \geq 2$. In fact, by induction we see that $A(\Omega_{u_j}, \epsilon_1, \epsilon_2) \subset [U_{\delta_0(T)}(\Omega_{u_{j-1}}) \cap F_q]$ is $C^\infty$-smooth. It follows from Theorem 4 and Proposition 6 that the ruled submanifold $\Sigma^k_j = A(\Omega_{u_j}, \delta_0(T))$ must be of $C^\infty$-smooth as well. Since $[U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q] \subset [U_{\delta_0(T)}(\Omega_{u_{j-1}}) \cap F_q] \cup \Sigma^k_j$, Assertion j follows. Theorem 1 follows from Assertion $N_T$ for any arbitrarily large T. □

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