Analytic Alpha-Stable Noise Modeling in a Poisson Field of Interferers or Scatters

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Abstract—This paper addresses non-Gaussian statistical modeling of interference as a superposition of a large number of small effects from terminals/scatters distributed in the plane/volume according to a Poisson point process. This problem is relevant to multiple access communication systems without power control and radar. Assuming that the signal strength is attenuated over distance $r$ as $1/r^m$, we show that the interference/clutter could be modeled as a spherically symmetric $\alpha$-stable noise. A novel approach to stable noise modeling is introduced based on the LePage series representation. This establishes grounds to investigate practical constraints in the system model adopted, such as the finite number of interferers and nonhomogeneous Poisson fields of interferers. In addition, the formulas derived allow us to predict noise statistics in environments with lognormal shadowing and Rayleigh fading. The results obtained are useful for the prediction of noise statistics in a wide range of environments with deterministic and stochastic power propagation laws. Computer simulations are provided to demonstrate the efficiency of the $\alpha$-stable noise model in multiuser communication systems.

The analysis presented will be important in the performance evaluation of complex communication systems and in the design of efficient interference suppression techniques.

Index Terms—Random access systems, statistical modeling, wireless communications.

I. INTRODUCTION

A N IMPORTANT requirement for most signal processing problems is the specification for the corrupting noise distribution. The most widely used model is the Gaussian random process. However, in some environments, the Gaussian noise model may not be appropriate [1]. A number of models have been proposed for non-Gaussian phenomena, either by fitting experimental data or based on physical grounds. In the latter approach, we have to consider the physical mechanisms giving rise to these phenomena. The challenge in analytically deriving general noise models lies in attempts to characterize a random natural phenomenon in terms of a limited set of parameters. This is one of the main motivations for the research in this paper.

The most credited statistical-physical models have been proposed by Middleton [2]. Other common physically motivated model is based on the K-distribution [3]. The data fitting noise modeling [4], [5] using Weibull, lognormal, Laplacian, or generalized-Gaussian distributions is appropriate only to a narrow class of systems because data collected are limited to a finite number of conditions. Moreover, the current literature does not provide enough insight into the relation between the parameters of these distributions and environmental conditions in which noise occurs. Therefore, alternative models should be considered.

It has been suggested that among all the heavy-tailed distributions, the family of stable distributions provides a considerably accurate model for impulsive noise [6]. Stable interference modeling is used in many different fields, such as economics, physics, hydrology, biology, and electrical engineering [7], [8]. In communications, stable noise models have been verified experimentally in various underwater communications and radar applications [7]–[11].

Stable distributions share defining characteristics with the Gaussian distribution, such as the stability property and the generalized central limit theorem and, in fact, include the Gaussian distribution as a limiting case [12]. A univariate symmetric $\alpha$-stable ($S\alpha S$) distribution is most conveniently described by its characteristic function [6]

$$\phi(t) = \exp(-\gamma |t|^\alpha).$$  (1)

Thus, a $S\alpha S$ distribution is completely determined by two parameters: 1) the dispersion $\gamma$ and 2) the characteristic exponent $\alpha$, where $0 < \alpha \leq 2$, and $0 < \gamma$. One of the most important classes of multivariate stable distributions is the class of spherically symmetric (SS) distributions [13]. The real RVs $Y_1, \cdots, Y_n$ are SS $\alpha$-stable, or the real random vector $\mathbf{Y} = [Y_1, \cdots, Y_n]^T \in \mathbb{R}^n$ is SS $\alpha$-stable if the joint characteristic function is of the form

$$\phi(t) = \phi(t_1, \cdots, t_n) = \exp \left( -\gamma \left| \sum_{j=1}^n t_j^2 \right|^{(1/2)\alpha} \right).$$  (2)

Note that the above characteristic function is obtained from the univariate characteristic function $\phi(t)$ in (1) by substituting the $L_2$ norm of $t$ for $t$. More detailed information on stable distributions can be found in [14] and [15].

In this paper, we present a realistic physical mechanism giving rise to SS $\alpha$-stable noise. This is accomplished by considering the nature of noise sources, their distributions in time and space, and propagation conditions. We concentrate on spectrum sharing systems with high likelihood of signals interfering with one another. In radar, noise, which is often referred to as clutter, is an electromagnetic field composed of independent contributions from a large number of scattering centers [16]. In multiple access (MA) radio networks [17],
where one channel is used by many terminals, noise, which is referred to as multiple access interference, is a superposition of many signals from terminals using the same channel. Assuming that the multiuser detection and the power control are not feasible in the system, we analyze frequency hopping spread spectrum (FH SS) radio networks without power control. In many situations, positions of interferers/scatterers are not known, and therefore, they are often assumed to be randomly distributed in the plane or volume [18] according to a Poisson point process [8], [19]. A common feature in radar and multiple access systems without power control is that the noise vector after correlation detection can be written as the sum of contributions from \( N \) sources
\[
\mathbf{Y}_N = \sum_{i=1}^{N} a(r_i)\mathbf{X}_i,
\]
In this equation, the random vector \( \mathbf{X}_i \) with real coordinates corresponds to the signal from the \( i \)th interferer after correlation detection, and the scalar \( a(r_i) \) accounts for signal propagation characteristics. In general, the \( a(r_i) \) depends on the distance \( r_i \) between the detector and the \( i \)th noise source. In our modeling, we assume that the signal strength is attenuated on average as \( 1/r^m \) with distance \( r \). This allows us to predict the interference parameters for a wide range of propagation conditions determined by the attenuation factor \( m \) [20]–[22].

Our approach for \( \alpha \)-stable noise as given in (3) is based on the LePage series decompositions [15]. In Section III, first, we generalize the LePage representation to a multivariate case. Next, we link through the squared/cubed distances the arrangement of interferers/scatterers in the plane/volume to the Poisson process on the line. These two original results allow us to show that the asymptotic distribution for the interference/clutter is \( \alpha \)-stable. Practical constraints in our noise model are investigated in Section IV, and we demonstrate analytically and through simulations that they do not limit our analysis. In Section V, the simulation results are presented to support the accuracy of the proposed model for the MA interference in FH SS radio networks.

Previous approaches to \( \alpha \)-stable noise modeling [7], [8] have been traditionally based on the influence function method [12] and apply only to a limiting noise distribution when the number of interferers \( N \) is infinite; they do not provide any insight into noise distribution when the number of interferers is finite. In contrast, our proof for the limiting \( \alpha \)-stable noise distribution allows us to analyze the convergence of random series in (3). We also introduce randomness into the power propagation law and investigate the effects of the nonuniform distribution of scatterers/interferers.

II. SYSTEM AND INTERFERENCE MODELS

Throughout this section, we will concentrate on interference in multiple access communication systems that do not employ power control. However, a similar scenario applies to clutter resulting from scattering in radar systems and man-made interferences such as automotive ignition noise.

In our system model, a receiver using an omnidirectional antenna is located at the center of a plane where there is a large number of transmitters using the same power and modulation. The distances between the detector and interfering terminals are denoted as \( r_i \), where \( r_1 \leq r_2 \leq \cdots \leq r_N \). We assume initially that the number of interferers \( N \) is infinite (\( N \to \infty \)). The case with a finite number of interferers \( N \) is considered in Section IV-B. A schematic representation of the system is provided in Fig. 1, where \( Tx_i \) denotes the \( i \)th interfering terminal, and \( Rx \) denotes the receiver. The passband interference at the receiver resulting from a superposition of continuous-time waveforms coming from interfering terminals is written as
\[
y(t) = \sum_{i=1}^{\infty} a(r_i)x_i(t)
\]
where \( x_i(t) \) is the signal from the \( i \)th interferer, and \( a(r_i) \) represents the attenuation of signal from this interferer.

We assume the use of the conventional detector that first projects the passband signal onto the set of \( \eta \) real and orthogonal basis functions \( \{\varphi_j(t)\}, \ j = 1, \cdots, \eta \}. \) The definition of orthogonality and the projection operation depend on the signaling scheme and the type of demodulation used in the system. In general, after the correlation detection, the interfering signal is represented as an \( \eta \)-dimensional vector given by
\[
\mathbf{Y} = \lim_{N \to \infty} \sum_{i=1}^{N} a(r_i)\mathbf{X}_i
\]
where \( \mathbf{X}_i = [X_{i,1}, \cdots, X_{i,\eta}] \) is a random vector with \( \eta \) coordinates \( X_{i,j}, \ j = 1, \cdots, \eta, \) which are real RV’s. The \( j \)th coordinate of \( \mathbf{X}_i \) is the correlation of \( x_i(t) \) with the function \( \varphi_j(t) \). Because all interfering terminals in the systems considered use the same modulation scheme and transmit at the same power, it is reasonable to assume that the random vectors \( \mathbf{X}_i \) are i.i.d. Moreover, the distribution of \( \mathbf{X}_i \) is independent of \( r_i \).

In this paper, we are concerned with characterizing the distribution of \( \mathbf{Y} \) or its multivariate components \( \mathbf{Y}^{(j)}, \ j = 0, \cdots, M-1 \), where \( \mathbf{Y} = [\mathbf{Y}^{(0)}, \cdots, \mathbf{Y}^{(M-1)}] \). In order to do

\footnote{The projection of \( x_i(t) \) onto \( \varphi_j(t) \) or, equivalently, the correlation of these two, is given as \( X_{i,j} \triangleq \int_{T} \varphi_j(t)x_i(t) \, dt \), where \( T \) is a symbol interval.}
this, we assume that $X_i$, or its multivariate components $X_i^{(j)}$, corresponding to the similar partition of $Y$, are spherically symmetric\(^2\) (SS) RV's. When referring generically to $Y^{(j)}$ and $X_i^{(j)}$, which are the random vectors of interest, we will drop the superscript and simply denote them as $Y$ and $X_i$, respectively. In this notation,

$$Y = \lim_{N \to \infty} \sum_{i=1}^{N} a(r_i) X_i.$$  \hfill (6)

Our task is to find the distribution of $Y$ under the assumption that $X_i$ are SS. Note that the distribution of all subvectors $X_i$ are SS does not imply that $X_i$ is SS. The term SS is applicable to multivariate RV's, but when we specialize our results to the bivariate RV's, we will use the term circularly symmetric (CS) rather than SS. A detailed discussion explaining the SS modeling of $X_i$ is provided at the end of this section. For notational convenience, we denote the dimension of $Y$ and $X_i$ as $n$.

We assume that the signal amplitude loss function with the distance $r$ is given by

$$a(r) = \frac{K}{r^m}$$ \hfill (7)

where the constant $K$ depends on the transmitted power. In free space, where radio frequency (RF) power radiates perfectly in a sphere from the antenna, the received power will decay in proportion to the square of the distance between the transmitter and receiver corresponding to a value of $m = 1$ in (7). In practice, for cellular radio frequencies, $m$ can vary from slightly more than 1 for hallways within buildings to larger than 3 for dense urban environments and office buildings [23]. For the active radar, it is $m = 2$, which corresponds to free-space propagation. For monostatic radars, sea clutter gives a variation of $m$ from $m = 3/2$ at close range to $m = 4$ at the longer ranges [20]. The attenuation of signal strength over distance in (7) is referred to as the deterministic law. Random fluctuations of the signal amplitude for a fixed distance $r$, such as lognormal shadowing and Rayleigh fading, and their effects on noise modeling are discussed in Section IV-A.

Under the assumption that terminals/scatterers transmit/radiate at the same power, combining (6) and (7) results in the noise equation

$$Y = \sum_{i=1}^{\infty} \frac{K}{r_i^m} X_i.$$ \hfill (8)

With respect to the terminal/scatters positions, we assume that interfering terminals/scatters contributing to $Y$ form a Poisson process with the expected number of terminals per unit area/volume given by $\lambda$ [19], i.e., for interferers distributed in the plane, the probability $P_k [k \text{ in } R]$ of $k$ transmitters being in a region $R$ depends only on the area $A$ of the region $R$ and is given by

$$P[k \text{ in } R] = \frac{e^{-\lambda A} (\lambda A)^k}{k!}.$$ \hfill (9)

Note: For scatters distributed in space, the concept of the Poisson point process is derived by replacing the area with volume.

Poisson assumption for the spatial distribution of terminals in wireless networks, although not strictly verified, turns out to be reasonable in the presence of a large population of users [24].

In this paper, we assume that the hopping times of different users are synchronized, i.e., all terminals begin and end symbol transmission at the same time. In [25], we drop this assumption, and we demonstrate that multiple access interference has the same distribution for both synchronous and asynchronous networks, except for one parameter. For active radar systems, the synchronous assumption is realistic because time delays from reflected waveforms at different distances are negligible. The asynchronous model may be applicable to passive sonar.

A Note on SS modeling of $X_i$: To understand the noise modeling in (6), it is useful to consider a system with on--off frequency shift keying (FSK) and noncoherent detection. In this system, $\varphi_1(t) = \cos(2\pi f_0 t)$, and $\varphi_2(t) = \sin(2\pi f_0 t)$. Under the assumption that a tone is transmitted, the projection of $x_i(t)$ onto $\{\varphi_1(t), \varphi_2(t)\}$ results in $X_i = \{\cos(\Theta_i), \sin(\Theta_i)\}$, where $\Theta_i$ is uniformly distributed in $(0, 2\pi)$. When no signal is present, $X_i = [0, 0]$. This means that $X_i$ is CS. We can also view this case as one with $X_i = \{\cos(\Theta_i), \sin(\Theta_i)\}$ and reduced density of terminals to $\lambda = \lambda_{\text{all}} / 2$, where $\lambda_{\text{all}}$ is the density of all terminals.

To describe the self-interference in frequency-hopping (FH) systems with $M$-ary FSK, we recall that at the receiver, after down-converting, the signal is demodulated using the conventional receiver for noncoherent $M$-ary FSK (i.e., a bank of bandpass filters with envelope detectors at their output and a decision rule that chooses the symbol corresponding to the maximum envelope). Let $Y_{1}^{(j)} = \{Y_{11}^{(j)}, Y_{12}^{(j)}\}$ be the output of the correlation filter tuned to the $j$th tone as shown in Fig. 2, where $Y_{11}^{(j)}$ and $Y_{12}^{(j)}$ denote the in-phase and quadrature integrator outputs, respectively. Here, the random vector $X_i$ is a concatenation of $X_i^{(j)} = \{\cos(\Theta_i^{(j)}), \sin(\Theta_i^{(j)})\}$, i.e., $X_i = (1/Mq)[X_i^{(1)}, \ldots, X_i^{(M-1)}]$, where $q$ is the number of frequency hopping slots. The RV's $\Theta_i^{(j)}$ are independent and uniform. Because each terminal transmits only one tone at a time, by grouping terminals that use the $j$th frequency (we denote their distances from the receiver as $r_i^{(j)}$), we can write [25] the self-interference as

$$Y_{1}^{(j)} = \sum_{i=0}^{\infty} a(r_i^{(j)}) X_i^{(j)}.$$ \hfill (10)

From this representation, it is evident that the self-interference at the $j$th branch of the envelope detector $Y_{1}^{(j)}$ has the same distribution as the interference in on--off FSK. Moreover, $Y_{1}^{(j)}$'s are independent from each other. Note that terminals at $r_i^{(j)}$ for $j = 0, \ldots, M - 1$ form independent Poisson field processes with density $\lambda = \lambda_{\text{all}} / Q$, where $Q = Mq$ is the number of orthogonal tones in the systems. In this system, $X_i$ is not a SS random vector, but its components $X_i^{(j)}$, which
are bivariate outputs from branches of a correlation detector, are; more precisely, they are CS.

Even though, in this paper, we concentrate on multidimensional signaling, the model we develop is also applicable to one-dimensional (1-D) signals when $n = 1$. In this case, the requirement that $X_i$ is SS means that the univariate $X_{i,1}$ is symmetric. This condition is always met in any antipodal signaling scheme with coherent demodulation.

In multiple access communication systems, $X_i$ is determined by the signaling scheme employed, but in passive radar, particularly in passive radar, $X_i$ depends on the characteristics of scatterers. If the echo $x_i(t)$ from the $i$th scattering center is a Gaussian process, then usually, $X_i \sim N(\mu_i, \sigma^2)$, where $I$ is the identity matrix, and $X_i$ is a SS RV.

Our objective in this paper is to provide an accurate statistical description of the interference $Y$, which will lead to the design of receivers with improved performance over conventional receivers.

### III. ALPHA-STABLE MODEL FOR INTERFERENCE

In this section, we prove the following.

**Theorem 1:** If the RV’s $X_i$ are i.i.d. and SS and the interferers/scatterers form a Poisson field, then the characteristic function of the interference vector $Y$ in (8) is SS α-stable, i.e.,

$$\phi_Y(t) = \exp(-\gamma ||t||^\alpha)$$

where $\alpha = 2/m$ and $\alpha = 3/m$ for interferers distributed in the plane and volume, respectively. The parameter $\gamma$, which is called dispersion, is given as

$$\gamma = -\lambda \mathcal{P} K \alpha \int_0^\infty \frac{\Phi(x)}{x^\alpha} dx$$

where $\Phi(x) = \Phi_X(||t||)$ is a characteristic function of the SS RV’s $X_i$, and $\mathcal{P}$ denotes differentiation. The constant $\mathcal{P} = \pi$ for interferers in the plane, and $\mathcal{P} = \frac{4}{3}\pi$ for scatterers in the volume.

**Proof:** Our proof of Theorem 1 is based on the multivariate version of the LePage series representation.

**Theorem 2 (The Multivariate LePage Series Representation):** Let $\{\tau_i\}$ denote the “arrival times” of a Poisson process, and let $X_i$ be SS i.i.d. vectors in $\mathbb{R}^n$, independent of the sequence $\{\tau_i\}$, with $E[|X_i|^\alpha] < \infty$, or equivalently, $E[|X_i|^\alpha] < \infty$. Then

$$Y = \sum_{i=1}^\infty \tau_i^{-1/\alpha} X_i^\alpha$$

converges almost surely (a.s.) to a SS α-stable random vector $Y$ with the characteristic function (cf)

$$\phi_Y(t) = \exp(-\gamma ||t||^\alpha).$$

The characteristic exponent $0 < \alpha < 2$, and the dispersion parameter $\gamma$ is given as

$$\gamma = -\lambda \int_0^\infty \frac{\Phi(x)}{x^\alpha} dx.$$

The proof of Theorem 2 is provided in Appendix A.

To link the multivariate version of the LePage series with the noise equation in (8), we need to map a Poisson point process in the plane (volume) onto the homogeneous Poisson process on the line. To achieve this, we use the following two propositions

**Proposition 1:** For a homogeneous Poisson point process in the plane with the rate $\lambda$, assuming that points are at distances $(r_1, r_2, \ldots)$ from the origin, $\Gamma_i$ represents Poisson arrival times on the line with the constant arrival rate $\pi \lambda$.

**Proposition 2:** For a homogeneous Poisson point process in a volume (3-D space) with the rate $\lambda$, $\Gamma_i$ represents “occurrence” times with the arrival rate $\frac{4}{3}\pi \lambda$. These two propositions are proven in Appendix B.

Now, based on Theorem 2 and both Propositions, we are able to give statistics of $Y$ in (8). For interferers distributed...
in the plane, we rewrite $Y$ in (8) as

$$Y = K \sum_{i=1}^{\infty} \frac{1}{(r_i^2)^{m/2}} X_i,$$  \hspace{1cm} (16)

From Proposition 1, $\Gamma_i = r_i^2$ represents Poisson “occurrence” times on the line with the arrival rate $\pi \lambda$, and based on Theorem 2, $Y$ is SS $\alpha$-stable with the characteristic exponent $\alpha = 2/m$ and dispersion $\gamma = -\lambda \pi K^\alpha \int_0^{\infty} \Phi_\alpha(x)/x^\alpha \, dx$. The multiplicative constant $K$ changes the dispersion of $\alpha$-stable RV by $K^\alpha$ [14]. Note that because $\alpha < 2$, this is valid only when $m > 1$, which is almost always the case. However, the free-space propagation is not included in this model.

For interferers distributed in a volume, we rewrite $Y$ in (8) as

$$Y = K \sum_{i=1}^{\infty} \frac{1}{(r_i^2)^{(m/3)}} X_i,$$  \hspace{1cm} (17)

From Proposition 2, $\Gamma_i = r_i^3$ represents Poisson “occurrence” times on the line with the arrival rate $\frac{3}{4} \pi \lambda$, and based on Theorem 2, $Y$ is SS $\alpha$-stable with the characteristic exponent $\alpha = 3/m$ and dispersion $\gamma = -\lambda \pi K^\alpha \int_0^{\infty} \Phi_\alpha(x)/x^\alpha \, dx$. Since $\alpha < 2$, the attenuation exponent $\frac{3}{4} < m$.

A. Parameters of Stable Noise

The characteristic exponent $\alpha$ in (11) controls the heaviness of the pdf tails ($0 < \alpha < 2$) as for univariate $\alpha$-stable RV’s: A small positive value of $\alpha$ indicates severe impulsiveness, whereas a value of $\alpha$ close to 2 indicates a more Gaussian type of behavior [14]. Like a variance for Gaussian RV’s, the dispersion $\gamma$ is the scale parameter from that controls the spread around the origin.

In the case of noncoherent on–off FSK (OOK), $X_i = [\cos(\Theta_i), \sin(\Theta_i)]$, where $\Theta_i$ is uniformly distributed in $[0, 2\pi]$. Therefore, $\Phi_\alpha(x) = J_\alpha(x)$, where $J_\alpha(\cdot)$ is a $\alpha$th-order Bessel function of the first kind [26]. In this case, $X_i$ is bivariate and CS. This model for $X_i$ is assumed in many radar applications. It is also applicable to branch outputs of a noncoherent correlation receiver with orthogonal $M$-ary FSK, as described in Section II. Because $J'_\alpha(x) = -J_1(x)$, [27, formula 6.561.17] can be used to calculate that for $0.5 < \alpha < 2$

$$\gamma = \lambda \pi K^\alpha \frac{\Gamma(1-\alpha/2)}{2^{\alpha/2} \Gamma(1+\alpha/2)}.$$  \hspace{1cm} (18)

In this equation, the admissible range of the path loss exponent is $1 < m < 4$ for interferers distributed in the plane, and $\frac{3}{4} < m < 6$ for scatterers distributed in the volume.

In Fig. 3, the normalized dispersion $\gamma_1 = \gamma/\lambda \pi K^\alpha$ is plotted as a function of $\alpha$ or as a function of $m$ (in parenthesis). For $\alpha$ in the range (0.5, 1.5), the normalized dispersion is almost constant. In this range, as $\alpha$ decreases, the tails of the CS $\alpha$-stable distribution become heavier, and if the dispersion is constant, the area of the tails will increase. This means that the channel will be more impulsive. Therefore, in interference-limited systems, underestimating $m$ may result in performance of some detection schemes that is too optimistic. For $\alpha$ in the range (1.5, 2.0), a considerable increase in dispersion is observed with increasing $\alpha$. In this range, as $\alpha$ decreases, the tails area does not necessarily increase at the same rate as in the previous range.

IV. PRACTICAL CONSIDERATIONS

In Section III, when deriving the formula for the distribution of self-interference, we have made two idealized assumptions: 1) We assumed an infinite network, and 2) we assumed a uniform distribution of points in a Poisson field. In this section, we show that these assumptions do not constrain our analysis. In addition, we discuss the influence of the shadowing effect—random fluctuations of the received power—on noise statistics in our modeling.

A. Shadowing and Fading Effects on Noise Distribution

Thus far, we have assumed that the received signal strength decreases with range raised to some exponent (the deterministic power-law propagation). However, experimental results show that this is only the average behavior of the signal. The received signal at fixed range is not constant due to different terrain characteristics and statistical fluctuations in propagation conditions. Typically, the following random effects should be included in the study.

1) the random link attenuation due to the lognormal shadowing
2) Rayleigh fading.

These are mutually independent and multiplicative phenomena. Furthermore, for most applications, they may be regarded as constant during at least one signaling interval (slow fading).

In this section, first, we show how to include the effect of lognormal shadowing alone in our framework of LePage series, and then, we give results for fading superimposed on shadowing. The lognormal shadowing model applies both to the power and to the amplitude. This is because the multiplication of lognormal RV’s is lognormal [21]. Therefore, the pdf of the signal strength is of the form [21]

$$p[\theta(r)|\theta(r)] = \frac{1}{\sqrt{2\pi} \sigma} \exp \left[ - \frac{1}{2\sigma^2} \ln^2 \left( \frac{\theta(r)}{\theta_0} \right) \right].$$  \hspace{1cm} (19)

where $\theta_0 = K/r^m$ is the median of $\theta(r)$ given as in (7), and $\sigma = 0.5 \sigma_s$. The parameter $\sigma_s$ is the standard deviation of

Fig. 3. Normalized dispersion for noncoherent OOK as a function of $\alpha$ or $(m)$.
the instantaneous power, and it depends on the environment. Values of $\sigma_s$ on the order of 8 to 10 dB are reported in the literature [21]. Therefore, in order to include the lognormal shadowing effect in our model, we have to consider $a(\nu)$ in (6) to be a RV given as $a(\nu) = K/(\nu)^m \exp(\sigma G)$, where $G$ is the standard Gaussian RV with zero mean [21]. The interfering signal is then

$$Y = K \sum_{i=1}^{\infty} \frac{1}{\eta_i^m} \exp(\sigma G_i) X_i.$$  \hfill (20)

We assume here that $G_i$ are i.i.d. The hypothesis of independence between shadowing effects from different users is generally accepted [21]. Therefore, we can apply Theorem 1 in (20) with $X_i$ replaced by $\exp(\sigma G_i)$. Then, in environments with lognormal shadowing, $Y$ is again $\alpha$-stable with $\alpha = 2/m$ and $\alpha = 3/m$ for interferers distributed in the plane and volume, respectively. To calculate the dispersion, we use (28)

$$\gamma = \lambda K^\alpha \mathbb{P}(\gamma^\alpha | \exp(\sigma G_i) X_{i,i} |) = \mathbb{P}(\gamma^\alpha | \exp(\sigma G_i) X_{i,i} |) \exp(\sigma G_i),$$

\hfill (21)

where $\gamma$ is a dispersion of the corresponding system with the deterministic power propagation law. The last equation in (21) follows from the first moment relation for lognormal RV’s. The dispersion of the noise increases with increase in $\alpha$, which depends on environment. When comparing with dispersion in the system without shadowing, dispersion of the noise also increases with an increase in $\alpha$ (decrease in $\sigma_s$).

If $a(\nu)$ is Rayleigh distributed, for a given $\nu$, $a(\nu)$ can be represented [21] as $a(\nu) = (K/(\nu)^m) \sqrt{2/\pi} R$, where the RV $R = \sqrt{G^2 + Q^2}$ Rayleigh distributed with $G, Q \sim N(0, 1)$. Then, we have to substitute $\sqrt{2/\pi} R_i X_i$ for $X_i$ in Theorem 1, and $Y$ is $\alpha$-stable with the same characteristic exponent as in the deterministic power propagation scenario. The dispersion is calculated in the same fashion as in (21). Because $[28] E[R^\alpha] = 2^\alpha/\Gamma(1 + (\alpha/2)]$, the dispersion is

$$\gamma = \gamma_{\text{determin}} \left(\frac{4}{\pi}\right)^{\alpha/2} \Gamma(1 + \frac{\alpha}{2}).$$

\hfill (22)

For the impact of multiplicative Rayleigh fading and lognormal shadowing, we have to substitute $\exp(\sigma G_i) R_i X_i$ for $X_i$ in Theorem 1. In this case, the dispersion is given by

$$\gamma = \gamma_{\text{determin}} \exp\left(\frac{4}{\pi}\right)^{\alpha/2} \Gamma(1 + \frac{\alpha}{2}).$$

\hfill (23)

The dispersion factors $\gamma/\gamma_{\text{determin}}$ for lognormal shadowing, Rayleigh fading, and combined shadowing and fading are calculated based on (21)–(23), respectively. They are shown in Fig. 4 as a function of $\alpha$. The curves are plotted with the shadowing standard deviation $\sigma_s = 10$ dB. We see that in all cases examined, the dispersion factors are increasing functions of $\alpha$. For lognormal shadowing, the dispersion also increases with an increase in shadowing parameter $\sigma_s$.

The RV’s $\exp(\sigma G_i) X_i$ are spherically symmetric (SS) because a product of a univariate RV and a SS RV is SS. In addition, they are independent because $\{G_i\}$ and $\{X_i\}$ are assumed to be independent sequences of mutually independent RV’s.

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2) Experiment 2 (Alpha-Stable Parameter Fitting to Bivariate LePage Series): To investigate the convergence further, we have fitted the bivariate $\alpha$-stable distribution to the 2-D RV's generated with the truncated LePage series; we estimated the parameters of the latter distributions as if they were $\alpha$-stable. The estimation procedure for the parameters of the symmetric $\alpha$-stable RV $Y$ was based on the first- and second-order moments developed by Zolotarev [29] (see also [30]). The truncated LePage series have been generated as described in [15] and [25].

Table I shows the results obtained from the estimation of $\alpha$-stable parameters for the “ideal” $\alpha$-stable generator (column 1) and the truncated LePage series. The generator of “ideal” CS $\alpha$-stable RV’s was implemented as described in [14]. In columns 2–4, we present parameter fitting results for the truncated series with $N = 50$, $N = 20$, and $N = 10$, respectively, where $X_i$ are uniformly distributed on the unit circle ($\Theta_i$ or, in short, $X_i = \exp(j\Theta_i)$). We scaled $X_i$ similarly so that the corresponding LePage series (with infinite number of terms) was standard ($\alpha = 1$).

In columns 5 and 6, we present parameter fitting results for the truncated LePage series, with $\gamma = 0.5$ and $\gamma = 1$, where $\gamma$ is uniformly distributed on the unit circle. The dispersion is modified by the factor $\exp(\sigma_2^2 / 2)$ [see (21)]. We calculated this factor in the last column. Table I gives the average ($\delta_\alpha, \delta_\gamma$) and the standard deviation ($\delta_\alpha, \delta_\gamma$) values from Monte Carlo simulations. In the simulations, the sample size was 5000, and the experiments were repeated independently 1000 times. We observe that for approximately $\alpha < 1.5$, we have a good fit of the ideal $\alpha$-stable distribution to the truncated LePage series with $N$ as small as 10. For a constant $\alpha$, as the number of terms $N$ in the truncated LePage series decreases, the distribution of $Y_N$ becomes more heavy tailed (with smaller fitted $\gamma$) and more “peaked” (concentrated more around the zero-location parameter—with smaller $\gamma$). In addition, we see that the shadowing (the distribution of $X_i$) does not affect the convergence much. Moreover, the factor obtained in (21) is in a good agreement with simulation results. For $\alpha$ close to 2, the convergence is much slower.

3) A Note on Modeling the Truncated LePage Series: For $\alpha$-stable RV’s, with $\alpha < 2$, only moments of order less than $\alpha$ exist [6]. Therefore, theoretically, there is a contradiction in observing that the truncated LePage series can be represented by a RV with infinite variance. A finite sum of RV’s with finite second moments should result in the distribution with a finite variance. It has to be emphasized that the LePage series is only an asymptotic representation. We use stable distributions to model the truncated LePage series because both exhibit similar statistical behavior; they are heavy tailed. Analytical calculations and data-fitting experiments in this section show good convergence properties of the truncated LePage series. Therefore, we conjecture that for the modeling purposes, an $\alpha$-stable distribution is a proper choice when we try to represent the truncated LePage series with a distribution of a simpler form.

C. Nonhomogeneous Poisson Processes

Until now, we have been concentrating on Poisson point processes with a constant rate. In particular, we map the processes in the plane (volume) through the squared (cubed) distances into homogeneous processes on the line. This is because the LePage series representation applies only to Poisson processes with a constant rate. However, as any nonhomogeneous process can be made homogeneous by a monotone transformation [19], it is of interest to determine what rate functions for the density of interferers/scatterers can be handled in our framework. Because, in (13), the form factors have to be $\Gamma_x^{(1/\alpha)}$, where $\Gamma_x$ are occurrence times of a homogeneous Poisson process, and because from (8), $\Gamma_x^{(1/\alpha)} = 1/\rho \sigma^\alpha$, then the mapped process must be of the form $\Gamma_x = r_x^{\alpha}$, where $\alpha = \beta/\rho$. In the rest of this section, we limit our attention to interferers distributed in the plane. The mapping should be from the original Cartesian coordinates $(x, y)$ into the “polar” coordinates $(w, \phi)$, with $w = r^{\alpha}$ (see Appendix B). This allows us to work with nonhomogeneous point processes with the rate function $\lambda(r) = \lambda_0 r^{\alpha-2}$ and $\beta < 2\alpha$. Note that since $x = w^{1/\beta} \cos(\phi)$ and $y = w^{1/\beta} \sin(\phi)$,
then the induced measure in the new plane \((w, \phi)\) is 
\[ \mu^* = \int \lambda(r) \, dx \, dy = \int \int \left(1/\beta\right) w^{\beta-2} w^{-\left(\beta-2/\beta\right)} \, dw \, d\phi. \]

Ignoring \(\phi\), the homogeneous Poisson process \(r^\beta\) has the rate \(\lambda = \lambda_0(2\pi/\beta)\). Then, we could proceed as in Section II-B and arrive at the stable model with \(\alpha = \beta/m\) and \(\gamma = -\lambda_0(2\pi/\beta)K\alpha \int_0^\infty [\Psi(\alpha)/\alpha^\alpha] \, dx.\)

Similarly, if we assume that the interferers are Poisson distributed only in a sector of the plane with an angle \(\phi_s\) and that their density is \(\lambda\), then we can map such a process to a homogeneous Poisson point process in the whole plane with the rate \(\lambda^* = \lambda\int_0^{\phi_s} \phi/2\pi = \lambda\phi_s/2\pi\). This scenario is applicable to directional antennas as opposed to the omnidirectional ones discussed so far.

Although the results presented above are specific to Poisson processes in the plane, Poisson processes in the higher dimensional spaces (volume in particular) can be handled in a similar fashion. More information on modeling the noise in nonhomogeneous Poisson fields of interferers can be found in [25].

The Poisson assumption for interferer positions is consistent with that followed in many papers on wireless radio networks [23], [31]; however, its choice is often motivated by analytical convenience [24]. With this model, we may apply the noise modeling to a network with dynamically changing topology or to obtain average performance for a collection of random networks.

V. SIMULATION RESULTS

To demonstrate the practicality of the proposed noise model, we simulated a small, single-cell, code division multiple access (CDMA) wireless network [17] and verified that stable distributions provide good description of MA interference statistics. The modulation employed is binary noncoherent FSK, and the network is synchronous, without power control. Each of users in the network is assigned a hopping sequence being a Reed–Solomon (RS) codeword that minimizes the number of simultaneous transmissions between code sequences (users) at the same frequency—“hits.” The description of the frequency-hopping pattern design using RS codes is beyond the scope of this paper and can be found in [32]. Briefly, if \(q\) is a prime number, we have a set of \(q^2 - q\) sequences of length \(q - 1\) such that any two sequences hit at most \(t - 1\) times in each period of \(q - 1\) symbols. The parameter \(q\) determines the number of frequency slots (gain), whereas \(q^2 - q\) determines the number of users in the system. In our simulations, we have experimented with different values of \(q\) and \(t\). In this section, we disclose the results for \(q = 11\) and \(t = 2\). In the case considered, we have 110 users, and nine \((q - 2)\) out of 110 terminals hopping to the same frequency as our receiver (provided that all users are active in the network). As to the user positions, we assume that, initially, they are Poisson distributed and they are moving around an area of \(100 \times 100\) m\(^2\). Their velocity is random between 2 and 4 km/hr. In the simulations, we ensure that the terminals do not get closer than 5 m to the receiver. The sample positions of terminals moving over the period of 5 s, with the receiver in the center of the square, are shown in Fig. 6. The transmission rate in the system is 1 kb/s. We assume deterministic power propagation law with the attenuation factor \(m = 3, 2, 1.5,\) and

![Fig. 6. Positions of moving terminals with the initial state given by a realization of the Poisson field.](image)
TABLE II

<table>
<thead>
<tr>
<th>Network with duty cycle 100%</th>
<th>Network with duty cycle 50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>m = 3 m = 2 m = 1.3 m = 1.2</td>
<td>m = 2 m = 3 m = 1.5 m = 1.2</td>
</tr>
<tr>
<td>α</td>
<td>0.666 0.635 0.666 0.666 0.666 0.666 0.666 0.666</td>
</tr>
<tr>
<td>α</td>
<td>0.6795 1.24 1.24 1.26 0.6804 0.6804 0.6804 0.6804</td>
</tr>
<tr>
<td>α</td>
<td>0.26 0.31 0.25 0.25 0.26 0.26 0.26 0.26</td>
</tr>
<tr>
<td>γ</td>
<td>7.7 8.3 8.7 8.7 7.7 7.7 7.7 7.7</td>
</tr>
<tr>
<td>δ</td>
<td>2.63 3.1 3.1 3.1 2.636 2.636 2.636 2.636</td>
</tr>
</tbody>
</table>

With respect to the second point, recall that in our noise modeling (Section II), we assume that on different hops, we have different (random) subsets of users hopping to the frequency to which our receiver is tuned. This scenario corresponds to SS MA networks where we regard the code sequences as statistically independent processes [17]. Experiment 2 in Section IV-B already confirmed that for this type of multiuser systems, the alpha-stable distributions provide a good description of MA noise. In CDMA networks, we have only pseudo-random subsets of terminals using the same frequency, and for a small number of users and small gains, there can be instances where the initial positions of terminals influence the MA interference to the point where stable model is inappropriate. Generally, CDMA FH patterns of long periods (supporting large number of users) behave like SS MA signals [17], and this was a premise on which we projected the MA interference model in SS MA networks to the MA interference in CDMA FH networks.

To demonstrate further that the stable model describes efficiently the MA interference, in Fig. 7, we plot the histogram of the envelope of MA interference in the network with duty cycle of 50%, for $m = 1.5$ ($\alpha = 1.333$). There, we also show the pdf of the envelope based on two fitted models: 1) Rayleigh and 2) the envelope of the bivariate alpha-stable random vector. The parameter of the Rayleigh model was obtained by estimating the mean of the series (the realization length was 100,000 samples). The calculation of the envelope pdf for the CS $\alpha$-stable RV was carried out using Fourier–Bessel expansion [25]. In Fig. 7, we show the center part of the pdf and the tail region using linear and logarithmic scales, respectively. It is evident that the bivariate Gaussian RV does not capture the heavy-tail character of MA noise, and the stable model provides much better fit to the histogram. The use of the Gaussian model for the MA interference in probability of error calculations will result in performance prediction of the networks that is too optimistic. Even though we do not have the perfect fit of the stable model to simulated MA interference, the advantages of defining the noise model in terms of just two parameters linked to physical scenarios are far more important.

There are many aspects of CDMA radio networks that have been simplified in our simulations and that may affect the noise parameters estimated. In general, these networks are more “random”:

1) They are asynchronous.
2) The user activity factors are more complex.
3) There is fading and multipath propagations.

4) There are many more effects which have not been considered here.

By experimenting with the simulated network parameters, we observed that the more random the network, the more accurate fit the stable model was providing. Specifically, longer periods of hoping patterns ($q$) result in more independent Poisson-like fields of interferers and follow closer the assumptions in Section II. For a given $q$, the number of hits controlled by the $t$ parameter is of lesser importance to the overall fit of stable distributions to MA interference. This is in agreement with the relatively fast convergence of the LePage series. In our simulations, we did not observe a better fit of stable distributions to MA interference at low values of $\alpha$ as expected from the convergence analysis in Fig. 4. This is related to the exclusion of close to the receiver interferers, which makes the MA interference more Gaussian-like and affects stable distributions more for low values of $\alpha$.

In summary, in this section, we verified the applicability of stable distributions to MA interference modeling in FH CDMA networks without power control. We pointed out some limitations in assumptions made when building the analytical model in Section II. However, as always, certain simplifications have to be made when transforming complicated intrinsic processes in the radio networks into a nearly equivalent MA interference model, which is credible, analytically tractable, and computationally efficient.
VI. SUMMARY

In this paper, we have characterized interference for multiple access communication systems without power control in which interferers are assumed to be Poisson-distributed in the plane. The same development applies for radar backscatter. Assuming inverse power attenuation of signal strength with distance, interference in the system is shown to be an SS \( \alpha \)-stable noise. This model has the advantage of specifying system noise with two parameters: the characteristic exponent \( \alpha \) and the dispersion \( \gamma \). By using the LePage series representation of an \( \alpha \)-stable RV, through simulations and numerical calculations, we have shown that a theoretical assumption about an infinite number of interferers does not limit our analysis. Moreover, we have demonstrated that the \( \alpha \)-stable model applies when interferers form nonhomogeneous Poisson fields. In addition, we have derived the formulas that allow us to predict noise statistics in complex interference environments with lognormal shadowing and Rayleigh fading, and we have verified the validity of our modeling by simulating small wireless networks.

The hypothesis of \( \alpha \)-stable noise is partially confirmed by the impulsive character of clutter and multiple access interference. In the end, however, it must be resolved against experimental data.

APPENDIX A

THE LEPAGE TYPE SERIES REPRESENTATIONS

The following result from [14, Th. 1.4.2] for univariate RV’s forms the basis in the proof of Theorem 2:

Theorem 3 (The LePage Series Representation): Let \( \{ \tau_i \} \) be as given in Theorem 2, and let \( \{ R_i \} \) be symmetric univariate i.i.d. RV’s, independent of the sequence \( \{ \tau_i \} \) with \( E[R_i] \alpha < \infty \). Then, the sum

\[
Y = \sum_{i=1}^{\infty} \tau_i^{-\alpha} R_i
\]

(26)

converges almost surely (a.s.) to a symmetric \( \alpha \)-stable \( (S\alpha S) \) RV \( Y \) whose characteristic function is

\[
\phi_Y(t) = \exp \left( -\gamma |t|^\alpha \right).
\]

(27)

The characteristic exponent \( 0 < \alpha < 2 \), and the dispersion \( \gamma \) is

\[
\gamma = \lambda C^{-1}_\alpha \left( E \left[ |R_i| \right] \right)^\alpha
\]

(28)

where \( C_\alpha = (1 - \alpha)/[\Gamma(2 - \alpha) \cos(\pi\alpha/2)] \).

Now, to prove Theorem 2, we proceed in three steps.

Step 1) First, we show that \( Y \) is SS.

The RV’s \( \tau_i^{-\alpha} X_i \) are spherically symmetric (SS) because a product of a univariate RV, and a SS RV is SS. The random vector \( Y \) is then a sum of SS RV’s and, therefore, is SS.

Step 2) Next, we show that each coordinate of \( Y \) is univariate \( S\alpha S \).

In the \( j \)th coordinate (\( j = 1, \cdots, n \)), the sum in (13) has the form

\[
Y_j = \sum_{i=1}^{\infty} \tau_i^{-\alpha} X_{i,j}
\]

(29)

where \( X_{i,j} \) being a \( j \)th coordinate of SS \( X_i \) is a symmetric univariate distribution. Based on Theorem 2, RVs \( Y_j \) are symmetric \( \alpha \)-stable \( (S\alpha S) \) with the dispersion \( \gamma \) as in (28). Now, since each coordinate \( Y_j \) of \( Y \) is \( S\alpha S \) with the same characteristic exponent \( \alpha \) and dispersion \( \gamma \), and because \( Y \) is SS, \( Y \) is SS \( \alpha \)-stable with the characteristic function as in (14).

Step 3) We express now the dispersion \( \gamma \) in terms of the characteristic function of \( X_{i,j} \).

We start with the integral formula ([27], 3.823)

\[
\int_{0}^{\infty} \frac{1 - \cos(zt)}{t^{\alpha+1}} \, dt = \left| z \right|^\alpha \frac{\Gamma(1 - \alpha) \cos \left( \frac{\pi}{2} \alpha \right)}{\alpha}
\]

(30)

for any real constant \( z \), and \( 0 < \alpha < 2 \). Replacing the constant \( z \) with \( RV \ X_{i,j} \) and taking expectation of both sides, after some algebra, we obtain

\[
\gamma = -\lambda \int_{0}^{\infty} \frac{d\alpha(x)}{x^\alpha} \, dx.
\]

(31)

Similar integral manipulation as in the proof of Step 3 has been used in [6].

To confirm the equivalence of (28) and (31), we evaluate them for \( X_i \sim N_m(0, \sigma^2 I) \) with \( \lambda = 1 \). In both cases, using \( \int_{0}^{\infty} x^{\alpha-1} \exp(-\mu x) \, dx = (1/\mu^\alpha) \Gamma(\alpha) \) ([27], 3.81.4) and after some manipulations, we obtain

\[
\gamma = \Gamma \left( 1 - \frac{\alpha}{2} \right) \left( \frac{\sigma^2}{2} \right)^{\alpha/2}
\]

(32)

APPENDIX B

MAPPING OF POISSON POINT PROCESSES

Before we prove Proposition 1 and 2, we recall that for Poisson point processes in \( \mathbb{R}^d \), in addition to the rate (density) of points, we often specify mean measure \( \mu \) (expected number of points) defined as

\[
\mu(A) = \int_A \lambda(x) \, dx
\]

(33)

where \( d\lambda \) stands for \( dx_1 \cdots dx_d \). In the special case, when \( \lambda \) is constant, \( \mu(A) = \lambda |A| \).

Now, both propositions arise as a consequence of the following mapping theorem, which is often overlooked in the engineering literature [19].

Theorem 4: Let \( \Pi \) be a Poisson process with mean measure \( \mu \) in the space \( \mathcal{S} \), and let \( f: \mathcal{S} \rightarrow \mathcal{T} \) be a “smooth” function. Then, \( f(\Pi) \) is a Poisson process on \( \mathcal{T} \) having as its mean measure the induced measure \( \mu^* \)

\[
\mu^*(B) = \mu[f^{-1}(B)]
\]

(34)

where \( f^{-1}(B) \) is the inverse image of set \( B \) in \( \mathcal{T} \).
Proof of Proposition 1: When we map the Poisson point process in the Cartesian coordinate system \((x, y)\) into the “polar” coordinates \((\rho, \phi) = (w, \bar{\phi})\), we again obtain a Poisson point process in the strip \((w, \bar{\phi}) z w \geq 0, 0 \leq \phi < 2\pi\) with the induced measure \(\mu = \int \int \lambda(\rho, \phi) d\rho d\phi = \int \int \lambda_2(\sigma(\phi), \rho) d\rho d\phi\).

The proof of Proposition 2 parallels that of Proposition 1.

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REFERENCES


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