Corrction on $k$-regular trees. A $k$-regular infinite tree is a tree where all vertices have degree $k$. An infinite $k$-regular tree is similar to a $(k - 1)$-branching tree where all but 1 vertex have degree $k$.

Figure 1: 3-regular tree vs 2-branching tree

1 Executive Summary

We have continued with a review on the bond percolation and the introduced the coupling of the bond percolation process, then defined the critical phenomenon and the percolation probability. There is a theorem about the percolation probability is a non decreasing function and we bounded it between 1/3 and 2/3 for the two dimensional case.

2 Bond Percolation

2.1 Preliminaries

We have defined $\mathbb{L} = (\mathbb{Z}^2, E)$ where edges exist between all vertex pairs with distance 1. Let $0 \leq p \leq 1$ and $q = 1 - p$. Denote each edge in $E$ to be open with probability $p$ and closed with probability $q$. We consider the following probability space: As sample space we take

$$\Omega = \prod_{e \in E} \{0, 1\},$$

points of which are represented as $\omega = (\omega(e) : e \in E)$ and called configurations; the value $\omega(e) = 0$ corresponds $e$ being closed, and $\omega(e) = 1$ corresponds to $e$ being open. We take $\mathcal{F}$ to be the $\sigma$-field of subsets of $\Omega$ generated by the finite-dimensional cylinders (since $\Omega$ is a discrete product topology). Finally we take the product measure with density $p$ on $(\Omega, \mathcal{F})$; this is the measure

$$\mathbb{P}_p = \prod_{e \in E} \mu_e,$$
where $\mu_e$ is the Bernoulli measure on $\{0, 1\}$ given by

$$
\mu_e(\omega(e) = 0) = q, \quad \mu_e(\omega(e) = 1) = p.
$$

We drop the subscript $p$ from $P_p$. Similarly, $E$ is the shortcut for the corresponding expectation operator $E_p$.

There is a natural partial order on the set $\Omega$ of configurations, given by $\omega_1 \leq \omega_2$ iff $\omega_1(e) \leq \omega_2(e)$ for all $e \in E$.

There is a one-to-one correspondence between $\Omega$ and the set of subsets of $E$. For $\omega \in \Omega$, we define

$$
K(\omega) \triangleq \{ e \in E : \omega(e) = 1 \};
$$

thus $K(\omega)$ is the set of open edges of the lattice when the configuration is $\omega$. Clearly, $\omega_1 \leq \omega_2$ iff $K(\omega_1) \subseteq K(\omega_2)$.

**Coupling of bond percolation processes.** Suppose that $(X(e) : e \in E)$ is a family of independent RVs indexed by the edge set $E$, where each $X(e)$ is uniform on $[0, 1]$. We may couple all bond percolation processes on $L$ as $p$ ranges over the interval $[0, 1]$ as follows: Define $\eta_p \in \Omega$ by

$$
\eta_p(e) = \begin{cases} 
1 & \text{if } X(e) < p \\
0 & \text{if } X(e) \geq p.
\end{cases}
$$

We say that the edge $e$ is $p$-open if $\eta_p(e) = 1$. The random vector $\eta_p$ has independent components and marginal distributions given by

$$
P(\eta_p(e) = 0) = 1 - p, \quad P(\eta_p(e) = 1) = p.
$$

Clearly $\eta_{p_1} \leq \eta_{p_2}$ whenever $p_1 \leq p_2$. Generally, as $p$ increases, the configuration $\eta_p$ runs through typical configurations of percolation processes with all edge probabilities.

Considering the random subgraph of $L$ containing all vertices but only the open edges, we denote the connected components as *open clusters*. We write $G_p$ for the random graph and $C(x)$ for the set of vertices in the cluster containing $x$.

### 2.2 The Critical Phenomenon

We have defined the *percolation probability*:

Let $C \triangleq C(o)$. The percolation probability $\theta(p)$ is

$$
\theta(p) \triangleq P(|C| = \infty) = 1 - \sum_{k=1}^{\infty} P(|C| = k).
$$

Clearly $\theta$ is a non-decreasing function of $p$ with $\theta(0) = 0$ and $\theta(1) = 1$. It is fundamental to percolation theory that there exists a critical value $p_c$ of $p$ such that

$$
\theta(p) \begin{cases} 
= 0 & \text{if } p < p_c \\
> 0 & \text{if } p > p_c.
\end{cases}
$$

2
$p_c$ is called the *critical probability* defined by

$$p_c = \sup \{ p : \theta(p) = 0 \}.$$ 

We use $p_c(d)$ to denote the critical probability in $d$-dimensional lattices.

What is $p_c(1)$? Clearly $p_c(1) = 1$.

Generally it is apparent that $\theta_d(p)$ is non-decreasing in $d$ since if the origin belongs to an infinite open cluster in $\mathbb{L}^d$ it also belongs to an infinite open cluster in the “augmented” lattice $\mathbb{L}^{d+1}$. So

$$p_c(d + 1) \leq p_c(d), \quad d \geq 1.$$

It is known that $\theta_d(p)$ is continuous except possibly at $p = p_c$. For $3 \leq d \leq 19$, the possibility of a discontinuity at $p_c(d)$ has not been ruled out.

**Theorem 1** If $d \geq 2$, then $0 < p_c(d) < 1$.

**Theorem 2** The probability $\psi(p)$ that there exists an infinite open cluster in the graph is

$$\psi(p) = \begin{cases} 
0 & \text{if } \theta(p) = 0 \\
1 & \text{if } \theta(p) > 0.
\end{cases}$$

This will be proved by an application of the zero-one law. Note that is does not tell how many infinite open clusters there are (but whenever it exists it will almost surely be unique).

**Theorem 3** When $0 < p_1 < p_2 < 1$, we have that $\theta(p_1) \leq \theta(p_2)$.

**Proof:** Let

$$p_1 = p_2 \frac{p_1}{p_2},$$

where $p_1/p_2 < 1$. Take a realization of $G_{p_2}$ and delete each edge independently with probability $1 - p_1/p_2$. The resulting graph is a realization of $G_{p_1}$ that contains less edges than $G_{p_2}$. So if there is an infinite cluster in $G_{p_1}$, then there must also be one in $G_{p_2}$. \hfill \Box

Back to the two-dimensional case:

**Theorem 4** There exists a $1/3 \leq p_c \leq 2/3$ such that $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$.

**Beginning of proof:**

First, we show that for $p < 1/3$, $\theta(p) = 0$. Let $\sigma(n)$ denote the number of self-avoiding (consisting of distinct vertices and edges) paths of length $n$ in $\mathbb{L}$ starting at the origin. $\sigma(n)$ is unknown but a simple bound is

$$\sigma(n) \leq 4 \cdot 3^{n-1}$$

since at each step there are 3 choices of directions, except for the first step where there are 4. Note that this is a loose upper bound because we only avoid returning to itself at the first step.
Now let the RV $N(n)$ be the number of open path of length $n$ in the random grid. Any such path is open with probability $p^n$, so 
$$\mathbb{E}N(n) = p^n \sigma(n).$$

Now, if the origin belongs to an infinite open cluster, then for each $n$ there must exist at least one open path of length $n$ starting at $o$, so that 
$$\theta(p) \leq \mathbb{P}(N(n) \geq 1) \leq \mathbb{E}N(n) = p^n \sigma(n) \quad \forall n.$$ 
Hence 
$$\theta(p) \leq p^n 4 \cdot 3^{n-1} \quad \forall n$$

So by choosing $p < 1/3$ and letting $n \to \infty$ we have proven the first part.

The second part is to show that for $p > 2/3$, $\theta(p) > 0$. This part is based on a counting argument known as Peierls' argument [Sir Rudolf Ernst Peierls; 1936 paper on ferromagnetism in Ising models]. We need the concept of a dual lattice. The dual lattice to $L$ is the lattice where a vertex is put in the center of each cell in the original lattice, i.e., the translated lattice by $(1/2,1/2)$.

We can also construct a dual to the random lattice by declaring edges to be closed if they cross a closed edge of the original lattice, and open if they cross an open edge of the original lattice:

![Dual Lattice and Realization](image)

Figure 2: dual lattice and realization of a dual lattice

A circuit is a closed path (start and end vertex are the same; all vertices have degree 2). Note that any finite connected component in the random grid is surrounded by a closed circuit in the dual random grid. So the event $|C| < \infty$ is the equivalent to the event that $o$ lies inside a closed circuit in the dual random grid.

Now consider first some deterministic quantities. Let $C$ denote the set of all circuits in the dual lattice that contain the origin $o$, and let $C_k \subset C$ be the subset of circuits that surround a box of size $k$ centered at $o$. Let $\rho(n)$ be the number of circuits of length $n$ of the dual lattice that surround the origin. This deterministic quantity satisfies 
$$\rho(n) \leq n \sigma(n-1),$$

since any circuit of length $n$ surrounding the origin contains a path of length $n - 1$ starting at some point $x = (k + 1/2,1/2)$ for some $0 \leq k < n$. 

4