1 Executive Summary

We proved two important tools (basic) which are fundamental in proving that the bond percolation occurs at $p_c = 1/2$. We started with the FKG inequality which shows that increasing events are positively correlated. We then proved the BK inequality which is in some sense the reverse of the FKG inequality.

2 Basic Techniques

Definition 1 (Increasing event) A RV $X$ is increasing on $(\Omega, \mathcal{F})$ if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$. It is decreasing if $-X$ is increasing. An event $A \in \mathcal{F}$ is increasing whenever its indicator function is an increasing variable, i.e., if $1_A(\omega) \leq 1_A(\omega')$ whenever $\omega \leq \omega'$.

Example:

1. Increasing event: The event $A(x, y)$ that there exists an open path joining $x$ to $y$.

2. Increasing RV: The number $N(x, y)$ of the number of different open paths between $x$ and $y$.

If $A$ is an event on the probability space for bond percolation, then if $A$ is an increasing event, $\mathbb{P}_p(A) \leq \mathbb{P}_{p'}(A)$ whenever $p \leq p'$.

2.1 The FKG Inequality

Named after Fortuin, Kasteleyn, and Ginebre. This was first proved by Harris in 1960. Expresses the fact that increasing events can only be positively correlated.

Theorem 1 (FKG inequality) 1. If $A$ and $B$ are two increasing events, then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

2. If $X$ and $Y$ are increasing random variables such that $\mathbb{E}(X^2) < \infty$ and $\mathbb{E}(Y^2) < \infty$

$$\mathbb{E}(XY) \geq \mathbb{E}(X)\mathbb{E}(Y).$$

The FKG inequality also holds when both $A$ and $B$ are decreasing events. We prove only the first part of theorem in the case where $A$ and $B$ depend on finitely many edges.

Proof: Let $X = 1_A$ and $Y = 1_B$ be the indicators of the increasing events $A$ and $B$, which are increasing RVs. We can then reformulate the FKG inequality as

$$\mathbb{E}(XY) \geq \mathbb{E}(X)\mathbb{E}(Y).$$
This holds for general increasing RVs $X$, $Y$ with finite second moment.

Suppose that $X$ and $Y$ depend only on the state of edges $e_1, e_2, \ldots, e_n$ for some integer $n$. We prove the FKG inequality by induction.

Suppose first that $n = 1$, so that $X$ and $Y$ are only functions of the state $\omega(e_1)$. Pick any two states $\omega_1, \omega_2 \in \{0,1\}$. Since both $X$ and $Y$ are increasing,

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$$

with equality if $\omega_1 = \omega_2$. If $\omega_1 \neq \omega_2$, then either both differences are negative or both are positive. Therefore,

$$0 \leq \sum_{\omega_1=0}^{1} \sum_{\omega_2=0}^{1} (X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2))P(\omega(e_1) = \omega_1)P(\omega(e_1) = \omega_2)$$

$$= \sum_{\omega_1=0}^{1} X(\omega_1)Y(\omega_1)P(\omega(e_1) = \omega_1) + \sum_{\omega_2=0}^{1} X(\omega_2)Y(\omega_2)P(\omega(e_1) = \omega_2)$$

$$- \sum_{\omega_1=0}^{1} \sum_{\omega_2=0}^{1} (X(\omega_1)Y(\omega_2) + X(\omega_2)Y(\omega_1))P(\omega(e_1) = \omega_1)P(\omega(e_1) = \omega_2)$$

$$= 2(E(XY) - E(X)E(Y)).$$

Note that $P(\omega(e_1) = 0) + P(\omega(e_1) = 1) = 1$.

Suppose now that the result is valid for values of $n$ satisfying $n < k$, and that $X$ and $Y$ are increasing functions of the states $\omega(e_1), \omega(e_2), \ldots, \omega(e_k)$ of the edges $e_1, \ldots, e_k$. Then

$$E(XY) = E\left(E(XY \mid \omega(e_1), \ldots, \omega(e_{k-1}))\right)$$

$$\geq E\left(E(X \mid \omega(e_1), \ldots, \omega(e_{k-1}))E(Y \mid \omega(e_1), \ldots, \omega(e_{k-1}))\right),$$

since for given $\omega(e_1), \ldots, \omega(e_{k-1})$, it is the case that $X$ and $Y$ are increasing in the single variable $\omega(e_k)$. Now both $E(X \mid \omega(e_1), \ldots, \omega(e_{k-1}))$ and $E(Y \mid \omega(e_1), \ldots, \omega(e_{k-1}))$ are increasing functions of the states of $k - 1$ edges. Thus

$$E(XY) \geq E\left(E(X \mid \omega(e_1), \ldots, \omega(e_{k-1}))\right)E\left(E(Y \mid \omega(e_1), \ldots, \omega(e_{k-1}))\right)$$

$$= E(X)E(Y).$$

The proof for $X$ and $Y$ depending on infinitely many edges uses the martingale convergence theorem and Cauchy-Schwartz.

**Example.** Let $\Pi_1, \Pi_2, \ldots, \Pi_k$ be families of path in $\mathbb{L}$, and let $A_i$ be the event that there exists some path in $\Pi_i$ that is open. The $A_i$ are increasing events, and therefore

$$P\left(\bigcap_{i=1}^{k} A_i\right) \geq P(A_1)P\left(\bigcap_{i=2}^{k} A_i\right),$$

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since the intersection of increasing events is increasing. We iterate this to obtain

\[ \mathbb{P} \left( \bigcap_{i=1}^{k} A_i \right) \geq \prod_{i=1}^{k} \mathbb{P}(A_i) , \]

*Example:* Let \( G = (V,E) \) be an infinite connected graph with countably many edges and consider a bond percolation process on \( G \). For any vertex \( x \), we write \( \theta_x(p) \) for the probability that \( x \) lies in an infinite open cluster and

\[ p_c(x) = \sup\{p: \theta_x(p) = 0\} \]

for the associated critical probability. We have by the FKG inequality that

\[ \theta_x(p) \geq \mathbb{P}(\{x \leftrightarrow y\} \cap \{y \leftrightarrow \infty\}) \geq \mathbb{P}(x \leftrightarrow y) \theta_y(p) \]

so that \( p_c(x) \geq p_c(y) \). This inequality also holds with \( x \) and \( y \) interchanged. So we obtain the following theorem:

**Theorem 2** Let \( G \) be an infinite connected graph with countably many edges. The value of the critical probability \( p_c(x) \) is independent on the choice of \( x \).

*Another example.* Take a box \( B(n) \subset G \) of size \((2n-1)^2\). Let \( A \) be the event that there is an open path connecting a vertex of the top face to a vertex of the bottom face. This is a top-bottom (TB) crossing. Let \( B \) the event that there is an open path from a vertex of the left face to a vertex of the right face. This is a left-right (LR) crossing. The probability that there are both top-bottom and left-right crossings is at least the product that there is a TB crossing and a LR crossing.

*An equivalent continuous analogue*

**Theorem 3 (Chebyshev’s association inequality)** Let \( f \) and \( g \) be nondecreasing real-valued functions defined on the real line. If \( X \) is a real valued random variable, then

\[ \mathbb{E}(f(X)g(X)) \geq \mathbb{E}(f(X))\mathbb{E}(g(X)) . \]

If \( f \) is non increasing and \( g \) is nondecreasing then

\[ \mathbb{E}(f(X)g(X)) \leq \mathbb{E}(f(X))\mathbb{E}(g(X)) . \]

**Proof:** Let the random variable \( Y \) be distributed as \( X \) and independent of it. If \( f \) and \( g \) are nondecreasing, \(((f(x) - f(y))(g(x) - g(y)) \geq 0 \) so that

\[ \mathbb{E}(((f(x) - f(y))(g(x) - g(y))) \geq 0 . \]

Expanding we get the first inequality. The proof of the second is similar. \( \square \)
2.2 The BK Inequality

Named after van den Berg and Kesten, proved in 1985. Can be viewed as the reverse of the FKG inequality, but it applies to the event $A \circ B$ that two increasing events occur on disjoint sets of edges, rather than the larger event $A \cap B$ that $A$ and $B$ occur on any set of edges. $A \circ B$ is the set of configurations $\omega \in \Omega$ for which there are disjoint sets of open edges such that the first set guarantees the occurrence of $A$ while the second set guarantees the occurrence of $B$.

Canonical example. Let $G$ be a finite subgraph of $\mathbb{L}$, and let $A_G(x,y)$ be the event that there exists an open path joining $x \leftrightarrow y$ in $G$. Then $A_G(x,y) \circ A_G(u,v)$ is the event that there exist two edge-disjoint open paths in $G$, one joining $x$ and $y$, the other joining $u$ and $v$. Suppose now that we are given that $A_G(u,v)$ occurs, and we ask for the (conditional) probability of $A_G(u,v) \circ A_G(x,y)$. The conditioning on $A_G(u,v)$ amounts to knowing some information about the occurrence of open edges, but we are not allowed to use all such open edges in finding an open path from $x$ to $y$ disjoint from one of the open paths from $u$ to $v$. Thus it is plausible that

$$
\mathbb{P}(A_G(u,v) \circ A_G(x,y) \mid A_G(u,v)) \leq \mathbb{P}(A_G(x,y)).
$$

This is essentially the assertion of the BK inequality.

**Definition 2 (Disjoint occurrence)** Let $A$ and $B$ be two increasing events which depend on the states $\omega(e_1), \ldots, \omega(e_n)$ of $n$ distinct edges $e_1, \ldots, e_n$ of $\mathbb{L}$. Each such configuration is specified uniquely by the subset $K(\omega) = \{e_i : \omega(e_i) = 1\}$ of open edges among these $n$ edges. Then $A \circ B$ is the set of $\omega$ for which there exists a subset $H \subset K(\omega)$ such that any $\omega'$ determined by $K(\omega') = H$ is in $A$ and any $\omega''$ determined by $K(\omega'') = K(\omega) \setminus H$ is in $B$.

In short, $A$ and $B$ occur disjointly if they occur on disjoint edge sets of $\mathbb{L}$.

**Theorem 4 (BK inequality)** If $A$ and $B$ are two increasing events,

$$
\mathbb{P}(A \circ B) \leq \mathbb{P}(A)\mathbb{P}(B).
$$

The theorem also holds when both $A$ and $B$ are decreasing events.

Proof: (Sketch.) Consider the case where $A$ and $B$ are the existence of two open paths between different pairs of vertices. Let $G$ be a finite subgraph of $\mathbb{L}$. Let

$A = \{u \leftrightarrow v\}; \quad B = \{x \leftrightarrow y\}, \quad u, v, x, y \in G.$

Then $A \circ B$ is the event that there exist two edge-disjoint open paths from $u$ to $v$ and $x$ to $y$. Let $e$ be an edge in $G$. Replace $e$ by two parallel edges $e'$ and $e''$ with the same end-vertices, each of which being open with the same probability $p$ independently of each other and all other edges. The splitting of $e$ can only make the search for two disjoint path easier. After splitting, the probability of finding two disjoint open paths can therefore only increase or remain equal. We continue this splitting process, replacing every edge $f \in G$ by two parallel edges $f'$ and $f''$. At each stage we look for two open paths, the first one avoiding all edges marked $'$ and the second avoiding all edges marked $''$. At each stage the probability of finding two disjoint paths can only increase or remain equal. When all edges of $G$ have been split in two, we end up
with two independent copies of $G$. In the first we look for an open path $u \leftrightarrow v$, and in the second we look for $x \leftrightarrow y$. Since such paths occur independently in each copy of $G$, the probability that they both occur is $P(A)P(B)$. □

Continuing the example above, what is the probability that there is a TB open path and a disjoint LR open path? The probability of this event is not more than the product of the probabilities of that there is a TB open path and that there is a LR open path. So these events are negatively correlated.

2.3 Russo’s Formula

The third relation estimates the rate of change of the probability of occurrence of an increasing event $A$ as $p$ increases. We first need to introduce the notion of a pivotal edge. If $A$ is increasing, an edge $e$ is pivotal iff $A$ occurs when $e$ is open and does not occur when $e$ is closed. A pivotal edge is thus a critical edge for $A$.

**Definition 3 (Pivotal edge)** Let $A \in \mathcal{F}$ and $\omega \in \Omega$. The edge $e$ is pivotal for the pair $(A, \omega)$ if the occurrence of $A$ critically depends on $e$, i.e., if $1_A(\omega) \neq 1_A(\omega')$ where $\omega'$ is the configuration such that $\omega'(e) = 1 - \omega(e)$ and $\omega'(f) = \omega(f)$ for all $f \in E \setminus \{e\}$.

The event “$e$ is pivotal for $A$” is the set of all configurations $\omega$ for which $e$ is pivotal for $(A, \omega)$. Observe that this event is independent of the state of $e$ itself but only depends on the state of the other edges.

For example, let $A$ be the event that there is a LR open crossing of the box $B(n) \subset \mathbb{L}$. Any edge $e$ of $B(n)$ is pivotal for $A$ if, when it is removed from the graph, there is no more LR open crossings of $B(n)$ but one endvertex of $e$ is joined to the left side of $B(n)$ while the other endvertex is joined to the right side of $B(n)$.

**Theorem 5 (Russo’s formula)** Let $A$ be an increasing event which depends on the state of finitely many edges of $\mathbb{L}$, and let $N(A)$ denote the number of edges that are pivotal for $A$. Then

$$\frac{d}{dp}P(A) = E(N(A)).$$

3 Main Take-Aways

1. FKG-inequality: If $A$ and $B$ are two increasing events, then $P(A \cap B) \geq P(A)P(B)$.

2. BK inequality: If $A$ and $B$ are two increasing events, $P(A \circ B) \leq P(A)P(B)$. 
