1 Basic Techniques

Definition Increasing events: Up-sets. \( U \subset \Omega \) is an up-set if \( a, b \subset \Omega, a \in U, a \leq b \Rightarrow b \in U \)

1. FKG (Harris’s Lemma)
   \( A, B \) up-sets: \( P(A \cap B) \geq P(A) \cdot P(B) \)

2. BK
   \( A, B \) up-sets: \( P(A \circ B) \leq P(A) \cdot P(B) \)

3. Russo’s Formula
   “Pivotal” edge (for pair \((A, \omega)\))

Theorem \( A \) up-set depending on the state of finitely many edges of \( \mathbb{L} \), and let \( N(A) \) be the number of pivotal edges for \( A \). Then

\[
\frac{d}{dp} P_p(A) = E_p N(A). \tag{1}
\]

Proof (sketch)
\{\( X(e), e \in E \)\}: collection of i.i.d. RV’s uniformly distributed on \([0, 1]\).
Coupled percolation process
\[
\eta_p(e) = \mathbb{I}\{X(e) < p\} \tag{2}
\]
\[ \Rightarrow P_p(A) = P(\eta_p \in A) \tag{3} \]

Since \( A \) is an up-set, we have, for \( \delta > 0 \),

\[
P_{p+\delta}(A) = P(\eta_{p+\delta} \in A) \tag{4}
\]
\[ = P((\{ \eta_{p+\delta} \in A \} \cap \{ \eta_p \notin A \}) \cup \{ \eta_p \in A \}) \tag{5}
\]
\[ = P(\{ \eta_{p+\delta} \in A \} \cap \{ \eta_p \notin A \}) + P(\eta_p \in A) \tag{6}
\]
\[ = P(\{ \eta_{p+\delta} \in A \} \cap \{ \eta_p \notin A \}) + P_p(A) \tag{7}
\]

If \( \eta_p \notin A \) but \( \eta_{p+\delta} \in A \), there are some edges for which \( \eta_p(e) = 0 \) and \( \eta_{p+\delta}(e) = 1 \). \( A \) only depends on finitely many edges. The probability that there is more than one such edge \( e \) with \( p < X(e) < p + \delta \) is negligible compared to the probability that there is one such edge. If \( e \) is the only such edge, it is pivotal.

\(^{1}\) A \circ B (or \( A \triangledown B \)): \( A, B \) occur on disjoint edges. Also holds for arbitrary events! (Reimer, 2000)

\(^{2}\) If \( P(A) \) depends on infinitely many edges, \( P(A) \) may not be differential. For example, percolation in 2-D, \( \frac{\partial(p) - \theta(p_c)}{p - p_c} = 0 \) if \( p < p_c \); \( \frac{\partial(p) - \theta(p_c)}{p - p_c} = \infty \) if \( p \downarrow p_c \).
i.e., $\eta_p \notin A$ but $\eta_{p'} \in A$, where $\eta_{p'}(e) = 1 - \eta_p(e)$ and $\eta_{p'}(e') = \eta_p(e')$ for all other edges.

The first term in Equation (7) can be expressed as

$$P(\{\eta_{p+\delta} \in A\} \cap \{\eta_p \notin A\}) = \sum_{e \in E} P(\{e \text{ is pivotal for } A\} \cap \{p \leq X(e) < p + \delta\}) + \sigma(\delta)$$

$$= \sum_{e \in E} P(\{e \text{ is pivotal for } A\}) \cdot P(p \leq X(e) < p + \delta) + \sigma(\delta)$$

$$= \delta \cdot \sum_{e \in E} P(\{e \text{ is pivotal for } A\}) + \sigma(\delta)$$

Therefore,

$$P_{p+\delta}(A) = P_p(A) + \delta \cdot \sum_{e \in E} P(\{e \text{ is pivotal for } A\}) + \sigma(\delta).$$

Divided by $\delta$ and let $\delta \to 0$

$$\frac{d}{dp} P_p(A) = \sum_{e \in E} P(\{e \text{ is pivotal for } A\}) = E.N(A).$$

**Integral Form:** for any $0 \leq p_1 < p_2 \leq 1$

$$P_{p_2}(A) = P_{p_1}(A) \cdot \exp\left(\int_{p_1}^{p_2} \frac{1}{p} E_p(N(A) | A) dp\right),$$

which is obtained by integrating

$$\frac{d}{dp} P_p(A) = \frac{1}{p} E(N(A) | A) P_p(A).$$

2 “Square-root trick” (Follows from FKG)

If $A_1, \cdots, A_t$ are up-sets whose union $A$ has “very high” probability, then one of the $A_i$ must have “high” probability. $A_i^c$ are down-sets, so

$$\prod_{i=1}^{t} P(A_i^c) \leq P(A^c).$$

For some $i$ we must have

$$P(A_i^c) \leq (P(A^c))^{1/t}.$$  

Thus,

$$P(A_i) \geq 1 - (1 - P(A_1 \cup \cdots \cup A_t))^{1/t}.$$  

When $t = 2$, it is called “square-root trick”.

2
3 Critical Threshold for Bond Percolation on $\mathbb{L}$

Subcritical Phase: Exponential decrease of the radius of the mean cluster size.  
$S(n)$: diamond of radius $n$ (Manhattan distance).  
$A_n = \{0 \leftrightarrow \partial S(n)\}$: event that there exists an open path connecting the origin to any vertex lying on the surface of $S(n)$.  
Cluster radius of origin’s cluster $C$:  
\[
\text{rad}_C = \max_{x \in C} |x| 
\]
\[
\Rightarrow A_n = \{\text{rad}_C \geq n\}.  
\]

**Theorem** If $p < p_c$, $\exists \Psi(p) > 0$ s.t.  
\[
\mathbb{P}(\text{rad}_C \geq n) = \mathbb{P}(A_n) < \exp(-n \cdot \Psi(p)).  
\]

**Proof** (Sketch) 
Start with Russo’s formula in integral form:  
\[
\mathbb{P}_{p_1}(A_n) = \mathbb{P}_{p_2}(A_n) \cdot \exp\left(- \int_{p_1}^{p_2} \frac{1}{p} \mathbb{E}_p(N(A)|A) \, dp\right)  
\]
\[
\leq \exp\left(- \int_{p_1}^{p_2} \mathbb{E}_p(N(A_n)|A_n) \, dp\right)  
\]

Need to show that the mean number of pivotal edges grows linearly in $n$. Since $p < p_c$, $P(A_n) \to 0$ as $n \to \infty$. If $A_n$ occurs, it must depend on increasing number of pivotal edges. It is plausible that the number of pivotal edges in paths $o \to \partial S(2n)$ is approximately twice the number of such edges in paths to $\partial S(n)$. Thus, $N(A_n)$ gives $A_n = \theta(n)$.

**Theorem** Cluster size distribution. If $0 < p < p_c$, $\exists \lambda(p) > 0$ s.t.  
\[
\mathbb{P}(|C| \geq n) \leq \exp(-n \cdot \lambda(p)),  
\]
and $\exists 0 < S(p) < \infty$ s.t.  
\[
\mathbb{P}(|C| = n) \leq \frac{(1 - p)^2}{p} \cdot n \cdot \exp(-n \cdot S(p)),  
\]
for $n \in \mathbb{N}$.

4 Main Take-Aways
- Basic techniques: Russo’s formula.
- We finally begin to deal with critical threshold for bond percolation. To be continued ...