1 Executive Summary

We started with an introduction to the notion of percolation over random graphs with a 2-D lattice as an example. We then studied branching processes over random trees and proved conditions under which such a process continues forever with positive probability. Finally, we defined the probability space of a random graph over a 2-D lattice.

2 Introduction to Percolation

Consider the graph formed by a 2-D lattice $\mathbb{L} \triangleq G(V, E)$ with vertex set $V = \mathbb{Z}^2$ and edge set $E = \{(x, y) \in V^2 : ||x - y|| = 1\}$.

We derive a random graph as follows: each edge in $\mathbb{L}$ exists with probability $p$. Such edges are said to be open. This gives rise to the bond percolation model. Alternatively, if we randomly retain vertices instead of edges, we have the site percolation model.

![Figure 1: Bond percolation on $\mathbb{L}$](image)

![Figure 2: Site percolation on $\mathbb{L}$](image)

**Definition 1** A connected component of a random graph is a maximal set of vertices and edges such that for any two vertices $x$, $y$ in the set, there exists an alternating set of distinct vertices and edges that starts with $x$ and ends with $y$.

We often refer to a connected component as simply a component – then, $x$ and $y$ are in the same component if we can “walk” from $x$ to $y$. We denote the set of vertices in the component that contains a vertex $x$ by $C(x)$.

**Definition 2** Let $C = C(0)$. Then, the percolation probability $\theta(p)$ is defined as the probability that the origin is part of an infinite component, i.e.,

$$\theta(p) \triangleq \mathbb{P}(|C| = \infty) = 1 - \sum_{k=1}^{\infty} \mathbb{P}(|C| = k)$$  \hspace{1cm} (1)
It may be shown that $\theta(p)$ is a non-decreasing function of $p$ with $\theta(0) = 0$ and $\theta(1) = 1$. Further, it turns out that

$$\theta(p) = \begin{cases} 0, & \text{for } p < p_c, \\ > 0, & \text{for } p > p_c, \end{cases}$$

for some critical probability $p_c$. For the 2-D lattice $L$, $p_c = 1/2$.

### 3 Random Trees

#### 3.1 Branching Processes

A branching process results in a rooted tree wherein each node at a particular level or generation is connected to a random number of “child” nodes of the next generation. Specifically, a branching process is described as follows:

- The total number of nodes in the $n^{th}$ generation is a random variable (RV) $Z_n$.
- Each member $x_i$ of the $n^{th}$ generation gives birth to a random number of children $X_i$, governed by an offspring distribution, which are members of the $(n+1)^{th}$ generation. Thus, we have

$$Z_{n+1} = X_1 + X_2 + \cdots + X_{Z_n} \quad (3)$$

- We assume $Z_0 = 1$ which gives us the root node $x_0$.
- The $X_i$ are generated i.i.d., with mean $\mu$.

Let the generating function of the “offspring” RV $X$ be given by $G(s) \equiv \mathbb{E}(s^X)$. Let $G_n(s)$ denote the generating function of $Z_n$. Then, from (3), it follows that

$$G_{n+1}(s) = G_n(G(s)) \quad (4)$$

$$= G(G(\cdots G(s) \cdots)) \quad (5)$$

where the composition takes place $n$ times. Further, it can be shown that $\mathbb{E}(Z_n) = \mu^n$, where $\mu \equiv \mathbb{E}(X)$. We then have the following result.

**Theorem 1** The probability $\eta$ that $Z_n = 0$ for some $n$ is equal to the smallest non-negative root of the equation $G(s) = s$. 

![Figure 3: Illustration of a branching process](image-url)
Note that $\eta$ is the probability that the branching process ultimately terminates. This then leads to the following important theorem.

**Theorem 2** If $\mu \leq 1$, the branching process does not grow forever with probability 1, except when $\mathbb{P}(X_1 = 1) = 1$. Conversely, if $\mu > 1$, then the branching process grows forever with positive probability.

### 3.2 Percolation on $k$-regular rooted trees

Consider a rooted tree where each node in a generation has exactly $k$ offsprings ($k$-regular). In this graph, by randomly keeping edges to each offspring with probability $p$, we induce a branching process whose offspring distribution is binomial with parameters $k$ and $p$. Specifically, whenever a node in a particular generation spawns $m < k$ offsprings, we start $k - m$ new branching processes in the succeeding generation.

From Theorem 2, it follows that if the average number of children $kp > 1$, then each time a new branching process is started, there is a positive probability of it going on forever. Consequently, if $kp > 1$, then w.p. 1, an infinite tree is generated starting from some node on the original tree.

### 4 Bond percolation on 2-D lattices: Preliminaries

As mentioned earlier, we consider the graph formed by the 2-D integer lattice $L = (\mathbb{Z}^2, E)$, where the edge set $E$ consists of all vertex pairs with distance 1. Let $0 \leq p \leq 1$ and let $q = 1 - p$. We derive a random graph from the base graph $L$ by declaring each edge in $L$ to be open w.p. $p$ and closed w.p. $q$.

#### 4.1 Probability space

- **Sample Space:** Let $\omega : E \to \{0, 1\}$ such that $\omega(e) = 1$ if $e$ is open and 0 if $e$ is closed. Then, the sample space $\Omega$ is the space of all 0-1 valued functions $\omega$, i.e., $\Omega = \prod_{e \in E} \{0, 1\}$ (where $\prod$ denotes the product space). Each sample point $\omega \in \Omega$ is also called a configuration.

- **Measurable events:** The measurable events belong to the $\sigma$-field $\mathcal{F}$ of subsets of $\Omega$ generated by the finite-dimensional cylinders, i.e., each event in $\mathcal{F}$ corresponds to specifying $\omega$ on a finite set of edges of $E$.

- **Probability Measure:** We take the product measure $\mathbb{P}_p$ on $(\Omega, \mathcal{F})$:

\[ \mathbb{P}_p = \prod_{e \in E} \mu_e \]  \hspace{1cm} (6)

where $\mu_e$ is the Bernoulli measure on $\{0, 1\}$:

\[ \mu_e(\omega(e) = 0) = q; \quad \mu_e(\omega(e) = 1) = p \]  \hspace{1cm} (7)

We denote the expectation with respect to $\mathbb{P}_p$ as $\mathbb{E}_p$.

There is a natural **partial order** on the set $\Omega$ of configurations: for $\omega_1, \omega_2 \in \Omega$, we say $\omega_1 \leq \omega_2$ iff $\omega_1(e) \leq \omega_2(e), \forall e \in E$.

### 5 Main Take-Aways

- For bond percolation over the 2-D lattice, the percolation probability $\theta(p)$ exhibits a **threshold** behavior as a function of $p$.

- Similarly, a branching process continues forever with positive probability iff the average number of children per node is **strictly larger** than 1.