1 Executive Summary

This lecture gave an overview on two Campbell’s theorems - the mean and variance versions. Using Campbell’s (mean) theorem as a tool, we discussed several examples of its applications such as - mean interference calculation, Monte-Carlo integration and calculation of intensity of a Poisson cluster process.

2 Campbell’s Theorem and its Applications

Let $\Phi$ be a point process (PP) on $S$ and let $f : S \rightarrow \mathbb{R}$ be a measurable function. Then Campbell’s theorem states that the mean of the random sum $T = \sum_{x \in \Phi} f(x)$ is given by $\int_S f(x) \Lambda(dx)$.

1. Interference: Let $\Phi$ be a homogeneous Poisson point process (PPP) with intensity $\lambda$ in the $\mathbb{R}^d$ space. In a path-loss environment where the received power scales as $r^{-\alpha}$, the average interference seen at the origin $E[I] = E[\sum_{x \in \Phi} ||x||^{-\alpha}]$ can be calculated, using the Campbell’s theorem, as

$$E[I] = \lambda c_d d^{-\alpha} \left. \frac{d}{d-\alpha} r^{d-\alpha} \right|_{r=0}^{r=\infty},$$

where $c_d$ is the volume of a unit ball in $\mathbb{R}^d$. Note that $E[I] \rightarrow \infty$, immaterial of the values of $d$ and $\alpha$. Imposing an additional constraint on the minimum distance to the nearest interferer to be $r_0$, the expected interference can be expressed as $E[I] = \lambda c_d \frac{d}{d-\alpha} r_0^{d-\alpha}$ for $\alpha > d$.

2. Monte-Carlo Integration: Suppose we want to compute the integral $I = \int_W f(x)dx$ where $W \subset \mathbb{R}^d$ and $f$ is a nonnegative, integrable, real-valued function. Take any point process $X$ with intensity $\lambda(x) = \begin{cases} c & \text{if } x \in W \\ 0 & \text{else.} \end{cases}$

Then, the integral $I$ can be approximated by evaluating the discrete sum of function $f$ at the random points of $X$ as $\hat{I} = \frac{1}{n} \sum_{x \in X} f(x)$.

3. Suppose $X$ consists of a fixed, finite number of random points in $\mathbb{R}^d$, say $X = \{X_1, \ldots, X_n\}$. Assume $X_i$ has a marginal probability density $f_i(u)$, $u \in \mathbb{R}^d$. Then $X$ has intensity function $\lambda_X(u) = \sum_{i=1}^n f_i(u)$.

4. Poisson Cluster Process: Let $X$ be a homogenous PPP with intensity $\alpha$. Replace each parent point $x \in X$ by a random cluster $Z_x$ which is a finite PP. Suppose $Z_x$ has intensity function $f(u|x)$. Then conditional on $X$, the process $Y = \{Z_x | x \in X\}$ has the intensity function $\lambda_Y|X = \sum_{x \in X} f(u|x)$. Further, the intensity function $\lambda_Y$ of $Y$ is the expectation with respect to $X$,

$$\lambda_Y = \alpha \int_{\mathbb{R}^d} f(u|x)dx.$$

Example: Matérn’s Cluster Process - Here, the cluster $Z_x$ consists of a Poisson($\mu$) random number of points, uniformly distributed in the disc $b(x, r)$ of radius $r$ centered on $x$. This has intensity $f(u|x) = \mu/(\pi r^2)$ if $u \in b(x, r)$ and 0 otherwise. Such a cluster process has an intensity $\lambda_Y = \alpha \mu$. 


3 Variance Version of Campbell’s Theorem

Let \( \Phi \) be a homogeneous PPP on \( \mathbb{R}^d \) with intensity \( \lambda \) and let \( f: \mathbb{R}^d \to \mathbb{R} \) be a measurable function. Then the variance of the random sum \( T = \sum_{x \in \Phi} f(x) \) is given by \( \lambda \int_{\mathbb{R}^d} f^2(x) \, dx \).

Example: Referring back to the Interference Example 1, the variance of the interference (with a minimum distance constraint of \( r_0 \)) can be calculated, using the variance version of Campbell’s theorem, as

\[
\text{Var}[I] = \lambda c d \frac{d - 2 \alpha}{d - 2 \alpha} r_0^{-d - 2 \alpha} \quad \text{for} \quad 2 \alpha > d.
\]

4 Distribution of the Interference from the Nearest Interferer

The probability that the nearest neighbor is closer than \( r \) is given by \( P[\text{dist}(u, X) \leq r] = 1 - e^{-\lambda c d r^d} \). Hence \( P(I_1 \leq x) = P(\text{dist}(u, X) \geq x^{-1/\alpha}) = e^{-\lambda c d r^{-\alpha}} \), where \( \delta = \frac{d}{\alpha} \). The expected value of the interference from the nearest neighbor can be expressed in terms of the gamma function as

\[
\mathbb{E}[I_1] = c_d^\frac{1}{\alpha} \Gamma \left( 1 - \frac{1}{\delta} \right).
\]

Comments:

1. If \( \delta \leq 1 \), then \( \mathbb{E}[I_1] \to \infty \).
2. \( \mathbb{P}(I_1 > x) \sim \lambda c d x^{-\delta} \). For \( \delta \leq 1 \), the tail does not decay fast enough. Thus \( I_1 \) (and consequently \( I \)) has a “heavy-tailed” distribution.

5 Interference at a Given Node

Let \( X \) be a Poisson process in the plane with average number of points per unit area equal to \( \lambda \). A node is assumed to transmit with a probability of \( p \). Hence, the set of transmitting nodes also forms a Poisson process \( X_t \) with intensity \( \lambda t = \lambda p \). Define \( Y = \sum_{i:x_i \in X_t} g(r_i) \) as the total interference power at the origin where \( r_i \) is the distance of \( x_i \) from the origin. Assuming a path loss model \( g(r) = \frac{1}{r^\alpha} \), the density function of \( Y \) for \( \alpha = 4 \) is derived as [1]

\[
f_Y(y) = \frac{\pi}{2} \lambda t y^{-3/2} e^{-\pi y^3/4}.
\]

6 Main Take-Aways

- The mean and variance of the interference in a wireless network with Poisson distributed nodes is infinity for all values of \( d \) (the number of dimensions), and \( \alpha \) (the path loss exponent). Furthermore, its pdf has a heavy-tail and follows a stable distribution. Also note that the interference is not Gaussian-distributed as one would intuitively expect based on the central limit theorem.

- An integral of a nonnegative, integrable, real-valued function can be calculated by taking a discrete sum of the function at random points of a PP (Monte-Carlo Integration).

7 Sources


References