1 Executive Summary

The Campbell’s other theorem was introduced to evaluate the Laplace transform of the quantity \( I = \sum_{x \in \phi} f(x) \). This was then generalized to introduce the concept of probability generating functional of a point process. The Laplace transform of the interference was evaluated. Second order moment measure of a point process was introduced.

2 Campbell’s (Other) Theorem.

**Theorem 2.1** Let \( I = \sum_{x \in \phi} f(x) \) for a uniform PPP of intensity \( \lambda \) on \( S \). The sum is absolutely convergent iff

\[
\int_S \min \{1, |f(x)|\} \, dx < \infty.
\]

If it is then

\[
\mathbb{E}[e^{sI}] = \exp \left( \lambda \int_S \left[ e^{sf(x)} - 1 \right] \, dx \right)
\]

**Proof:** The theorem is first proved when \( f(x) \) is a simple function. Let \( A_i \) be measurable disjoint sets on \( S \). Let \( f(x) = \sum_{i=1}^n f_i 1_{A_i}(x) \). We then have

\[
I = \sum_{i=1}^n N_i f_i
\]

where \( N_i = \phi(A_i) \). Observe that \( N_i \sim \text{Poi}(\lambda |A_i|) \). So we have

\[
\mathbb{E}[e^{sI}] = \prod_{i=1}^n \mathbb{E}[e^{sf_i N_i}]
\]

\[
= \prod_{i=1}^n \exp \left( \lambda |A_i| \left( e^{sf_i} - 1 \right) \right)
\]

\[
= \exp \left( \lambda \sum_{i=1}^n |A_i| \left( e^{sf_i} - 1 \right) \right)
\]

\[
= \exp \left( \lambda \int_S \left[ e^{sf(x)} - 1 \right] \, dx \right)
\]

By standard measure theoretic arguments the result follows for general \( f \).

**Definition 2.2** The characteristic functional of a point process is defined as

\[
\mathbb{E} \left[ \exp \left( - \sum_{x \in \phi} f(x) \right) \right]
\]
For a PPP, the characteristic functional is given by

$$\exp\left( -\lambda \int_S \left[ 1 - e^{-f(x)} \right] dx \right)$$

**Theorem 2.3** If the characteristic functional of a point process $\phi$ is given by $\exp\left( -\lambda \int_S \left[ 1 - e^{-f(x)} \right] dx \right)$, then $\phi$ is a PPP of intensity $\lambda$.

**Definition 2.4** For a point process $\phi$ the probability generating functional (PGFL) is defined as

$$G(f) = \mathbb{E}\left( \prod_{x \in \phi} f(x) \right)$$

For a PPP it is equal to $\exp\left( -\lambda \int_S \left[ 1 - f(x) \right] dx \right)$.

**Applications:**

**Empty space function:** A simple application of the PGFL is finding the empty space function. The empty space function of a PP is defined as

$$g(r) = \mathbb{P}(\phi(B(o, R) = \emptyset)) = \mathbb{E}\left( \prod_{x \in \phi} \left[ 1 - 1_{B(o, R)}(x) \right] \right)$$

$$\overset{(a)}{=} G\left( 1 - 1_{B(o, R)}(x) \right)$$

$$= G\left( 1_{B(o, R)^c}(x) \right)$$

For a PPP, we have

$$g(r) = \exp\left( -\lambda \int \left[ 1 - 1_{B(o, R)^c}(x) \right] dx \right)$$

$$= \exp\left( -\lambda \int 1_{B(o, R)}(x) dx \right)$$

$$= \exp(-\lambda \pi R^2)$$

Which matches with the value $\mathbb{P}(\phi(B(o, R) = \emptyset)) = \exp(-\lambda \pi R^2)$.
Thinning of a PPP: Consider a PPP $\Phi$. We thin the process $\Phi$ as follows. Each point is retained with probability $p$ and deleted with probability $1 - p$. The characteristic function of the resultant process is

$$
\mathbb{E} \left( \prod_{x \in \Phi} \exp(-f(x)(x \text{ is retained})) \right)
= \mathbb{E} \left( \prod_{x \in \Phi} [1 + (\exp(-f(x)) - 1)(x \text{ is retained})] \right)
= ^{(a)} \mathbb{E} \left( \prod_{x \in \Phi} [1 + (\exp(-f(x)) - 1)p] \right)
= \exp \left( -\lambda \int \left[ 1 - (1 + (\exp(-f(x)) - 1)p) \right] dx \right)
$$

where $(a)$ follows from the independent thinning. From Theorem 2.3, we have that the resultant process is a PPP of intensity $\lambda p$.

3 Interference Distribution.

The interference at the origin is given by $I = \sum_{x \in \Phi} h_x \|x\|^{-\alpha}$ where $h_x$ are i.i.d random variables. The Laplace transform of $I$ is given by

$$
\mathcal{L}(s) = \mathbb{E} \left[ \exp(-sI) \right]
= \mathbb{E} \left[ \prod_{x \in \Phi} \exp(-s h_x \|x\|^{-\alpha}) \right]
= ^{(a)} \mathbb{E} \left[ \prod_{x \in \Phi} \mathbb{E}_{h_x} \exp(-s h_x \|x\|^{-\alpha}) \right]
= ^{(b)} \exp \left( -\lambda \int_{\mathbb{R}^d} 1 - \mathbb{E}_{h} \exp(-s h \|x\|^{-\alpha}) dx \right)
= \exp \left( -\lambda \mathbb{E}_{h} \int_{\mathbb{R}^d} 1 - \exp(-s h \|x\|^{-\alpha}) dx \right)
$$

where $(a)$ follows from the i.i.d property of $h_x$ and $(b)$ follows from the PGFL. Just considering the inner integral

$$
T = \int_{\mathbb{R}^d} 1 - \exp(-s h \|x\|^{-\alpha}) dx
= cd \int_{\mathbb{R}} \left[ 1 - \exp(-s r^{-\alpha}) \right] r^{d-1} dx
= cd \int_{\mathbb{R}} 1 - \exp \left( -\frac{s h}{x} \right) dx
$$

(2)
For a R.V. $X$, the $r$-th moment is given by $E[X^r] = \int rx^{r-1}(1 - F(x))dx$. We observe that (2), is the $\delta$-th moment of a RV with CDF $\exp \left( -\frac{sh}{x} \right)$. Observe that this is the CDF of a RV $X^{-1}$ where $X$ is exponential with mean $1/sh$. (2) is given by

$$c_d(sh)^{\delta}\Gamma(1 - \delta)$$

So (1) is equal to

$$L(s) = \exp \left( -\lambda c_d s^{\delta} \mathbb{E}[h^{\delta}] \Gamma(1 - \delta) \right)$$

When $h$ is exponential with mean 1, we have $\mathbb{E}[h^{\delta}] = \Gamma(1 + \delta)$. We also have $\Gamma(1 + \delta)\Gamma(1 - \delta) = \pi \delta / \sin(\pi \delta)$. So Interference is a stable RV with exponent $\delta$.

4 Success Probability

- PPP of intensity $\lambda$
- Received power $s \sim \exp(1)$
- Path loss model $hr^{-\alpha}$, $h \sim \exp(1)$

The success probability is given by

$$\mathbb{P}(s > \theta I) = \mathbb{E}\exp(-\theta I) = L_I(\theta)$$

5 Second Moment Measure

**Definition 5.1** Let $\phi$ be a PP on $S$. Then $\phi \times \phi$ is a PP on $S \times S$ consisting of all the ordered pairs $(x, y), x, y \in \phi$. The intensity measure $\mu^{(2)}$ of $\phi \times \phi$ is a measure on $S \times S$ defined as

$$\mu^{(2)}(A \times B) = \mathbb{E}(\phi(A)\phi(B))$$

If $A = B$,

$$\mu^{(2)}(A^2) = \mathbb{E}(\phi(A)^2)$$

So we have

$$\text{Cov}(\phi(A), \phi(B)) = \mu^{(2)}(A \times B) - \Lambda(A)\Lambda(B)$$