1 Executive Summary
We have reviewed the claim that with thinning we can shape a PPP and proved it. Then the intensity measure
and the mapping theorem was introduced followed by Prof. Renyi’s theorem. The marked point processes
were defined as an extension of a P.P. and a (ball) distance measure was introduced as a specialization of
the capacity function.

2 Shaping an inhomogen PPP

Claim 2.1 Let \( X \) be a stationary \( \text{PPP}(\lambda) \subset \mathbb{R}^d \) and \( g : \mathbb{R}^d \rightarrow [0, 1] \). A new process \( \tilde{X} \) created from \( X \)
by deletion of each points independently from others with probability \( 1 - g(x) \) creates an inhomogen PPP
with intensity function \( \tilde{\lambda}(x) = \lambda \cdot g(x) \).

Intensity (mean) measure: is the integral \( \Lambda(B) = \int_B \lambda(x)dx = \mathbb{E}(\Phi(B)) \) for a PPP \( \Phi \)

Remark: \( \Lambda(\{x\}) = 0, \forall x \in \mathbb{R}^d \). Means that we do not expect to have a point on a specific location. It is
a diffuse measure. It cannot be used for conditioning on points at specific locations.

Theorem 2.1 (Mapping theorem) Let \( \Phi \) be a non homogeneous PPP on \( \mathbb{R}^d \) with intensity function \( \lambda \).
Let \( f : \mathbb{R}^d \rightarrow \mathbb{R}^s \) be measurable and \( \Lambda(f^{-1}(y)) = 0, \forall y \in \mathbb{R}^s \) (f does not shrink a compact set to a point).
Let \( \mu(B) \triangleq \Lambda(f^{-1}(B)) = \int_{f^{-1}(B)} \lambda(x)dx \), where it satisfies \( \mu(B) < \infty, \forall B \), where \( B \) compact.
Then \( f(\Phi) \) is a PPP with intensity measure \( \mu \)

Corollary 2.2 (Linear mapping) Take \( \Phi \) a stationary P.P.P(\( \lambda_\cdot \mathbb{R}^d \)) and let \( A \) be a nonsingular linear mapping,
then \( A(\Phi) = \{A \cdot x : x \in \Phi\} \) is a stationary PPP with intensity \( \lambda \cdot \det(A^{-1}) \)

Theorem 2.3 (Renyi’s theorem) Let \( \Phi \) be a P.P. and \( \lambda : \mathbb{R}^d \rightarrow \mathbb{R} \) it’s intensity function such that
\( \Lambda(B) \int_B \lambda(x)dx < \infty \) \( \forall \) compact \( B \).
If \( P(\Phi \cap B = \emptyset) = e^{-\Lambda(B)}, \forall \) Borel sets \( B \), then \( \Phi \) is a PPP with intensity function \( \lambda(x) \).

proof outline: We have to define a “right” sequence of smaller and smaller partitions that can approximate
the original set \( B \). We should set the little partitions disjoint so we could compute the vacancies of each one
of them. Because of the independence we can use the probability generating function product form, and we
can bound them. After that one can use monotonic convergence to prove the theorem.

Remark: Note that in this theorem \( \Phi \) is a general non stationary point process.

Conditional property: Let \( \Phi \) a non homogeneous, non stationary PPP \( \in \mathbb{R}^d \) with intensity \( \lambda \). Take \( B \) such
that \( 0 < \Lambda(B) < \infty \). Conditioned on \( |\Phi \cap B| = n(\equiv \Phi(B) = n) \),
then the \( n \) points in \( B \) are distributed as \( \frac{\lambda(x)}{\Lambda(B)} \)

Remark: The ALOHA [1] network protocol can be viewed as a classical case of thinning (splitting)
3 Marked point processes

Let’s add a mark \( m \in M \) to each point in the process.

Example:

- \( m \triangleq \text{diameter of trees} \)
- In a WSN \( m \triangleq \text{battery level or transmission power, etc.} \)
- In ALOHA \( m \triangleq \text{the tx or rx state of the node.} \)
- See [2] for further examples.

Marked point process: A marked point process in space \( S \) with marks in space \( M \) is a PP \( Y \) on \( S \times M \) such that \( N_Y(K \times M) < \infty, \forall K \subset S, K \text{ compact}. \)

False example: a PPP on \( \mathbb{R}^3 \) cannot be viewed as a marked PP in \( \mathbb{R}^2 \times \mathbb{R} \).

Explanation: In a general marked PP \( M \) need not be compact. It can be the whole mark space, so if we use the \( 3^{rd} \) dimension as the mark space then in a compact set \( K \in \mathbb{R}^2 \) there will be infinite number of points such that \( N_Y(K \times \mathbb{R}) \not< \infty \)

4 Distances:

Contact distance: \( \text{dist}(u,X), u \in \mathbb{R}^d, X \subset \mathbb{R}^d \) the shortest distance from point \( u \) to the nearest point of \( X \). We usually take the euclidean distance. \( \text{dist}(u,X) \leq r \iff N(\text{ball}(u,r)) > 0 \)

Example: Let \( X \) a homogeneous PPP(\( \lambda \)). \( \mathbb{P}(\text{dist}(u,X) \leq r) = \mathbb{P}(N(\text{ball}(u,r)) > 0) = 1 - e^{-\lambda r^d \kappa_d} \), where \( \kappa_d = |\text{ball}(0,1)|_d \) is the \( d \) dimensional unit ball’s volume, or \( V = \kappa_d \cdot \text{dist}(u,X)^d \): the largest ball that we can fit in before we hit somebody in \( X \). \( \mathbb{P}(V < v) = 1 - e^{-\lambda v} \).

Contact distance function or empty space function: \( F \) is the cdf of \( R = \text{dist}(u,X) \).
\( F(r) = \mathbb{P}(\text{dist}(u,X) \leq r) = \mathbb{P}(N(\text{ball}(u,r)) > 0) = T(\text{ball}(0,r)) = T(\text{ball}(u,r)) \)

Remark: \( F(r) = T(\text{ball}(u,r)) \) is the special case of the capacity function where we use balls for the argument of the capacity function.

5 Main Take-Aways

- Trimming and mapping can be used for shaping a P.P. using the Claim 2.1 or Theorem 2.1
- Marked point processes can be used to introduce a distinction between points in the original process.

6 Sources


References
