Maximizing the Throughput in Random Wireless Ad Hoc Networks

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Abstract*—This paper focuses on determining the optimum transmission probability that maximizes the per-node throughput in a random wireless ad hoc network. The network self-interference is modeled as a Gaussian random variable which is equal to the sum of the interference from all other transmitting nodes. Given the path loss exponent η, node density λ, and the near field cut-off radius d_0, analytical throughput expressions are derived for three different scenarios - the AWGN channel, block Rayleigh faded channel with no CSI at the transmitter and the same channel assuming perfect CSI at the transmitter.

1 Introduction

Wireless ad hoc networks consist of a set of nodes that communicate among themselves over a wireless channel. There is no centralized control. The nodes cooperate in routing the data packets from the transmitting node to the intended destination node. In doing so, efficient routing protocols must be chosen respecting the power and delay constraints. This paper uses the network self-interference model in [1] to characterize the network throughput. Though [2, 3] focus on optimizing the resource allocation in multiple-access schemes, they do not model the network self-interference. [1] determines the distribution of the interference but does not come up with a closed form expression for the network throughput. [4] characterizes the interference in a random ad hoc network to compare its performance with a regular grid network.

In this paper, the following network model is considered [5]. Nodes are randomly distributed in the plane according to a Poisson point process of density λ. Each node is assumed to transmit at the same frequency f with the same power P_0 using an omnidirectional antenna. Nodes make independent decisions on whether to transmit or listen. In any given time slot, a node transmits with the attempt probability α. This paper does not confine itself to any particular routing scheme. The destination node is always chosen randomly from the nearest neighbors (within radius R) of the transmitting node. It is further assumed that no other node within radius d_0 of the destination node transmits in the same time slot, where d_0 is referred to as the near field cut-off radius. It depends on the path loss exponent η chosen to model the network.

The destination node experiences interference of varying degree from all the transmitting nodes other than its source in the network. For a successful transmission, the following has to hold

$$\frac{P_r}{N_t + I} > \Theta \tag{1}$$

where P_r is the received power, I is the total interference and N_t is the thermal noise at the receiver.

The per-node throughput is the rate at which a node successfully transmits packets. For a network with uniform traffic across all links, the per-node throughput will be the same for all the nodes (boundary effects are ignored). Let A denote the source and B the destination. In terms of probabilities, the per-node throughput can be defined as

$$\zeta = \alpha(1-\alpha)\mathbb{P}\left(\mathrm{SINR} > \Theta\right), \tag{2}$$

where α is the probability that A transmits and $(1-\alpha)$ and the probability that B does not transmit in that same time slot. This manuscript aims at deriving a generalized expression for the throughput in terms of the various network parameters. It further determines the transmit probability that maximizes this throughput by plotting the throughput curve. The analytical results are verified by comparing them with the throughput curves obtained from simulations.

The throughput expressions are derived for three different scenarios - the AWGN channel, the block Rayleigh faded channel with no CSI at the transmitter and the block Rayleigh fading channel with perfect CSI at the transmitter. These expressions help analyze the network throughput for various routing and MAC protocols. The simulation curves illustrate how fading degrades the per-node throughput compared to an AWGN channel and how opportunistic transmission (perfect CSI at transmitter) not only helps overcome this degradation, but also reach a substantially higher throughput than in the non-faded case. The packet loss probabilities for all three cases are also compared.

2 Modeling Interference

The nodes comprising the random network are assumed to be randomly distributed in a square field of side $2L$. Let (X, Y) denote the Cartesian coordinate of a node. Then both X and Y are uniformly distributed in $[-L, L]$, independent of the positions of other nodes in the network. Also, it can be easily shown that the number of nodes in a given area, for all practical purposes, is given by a two-dimensional Poisson distribution [4]. Thus, in an area A in the plane, the probability of finding k nodes is given by

$$\mathbb{P}\left(k \text{ in } A\right) = \frac{e^{-\lambda A}(\lambda A)^k}{k!} \tag{3}$$

where $\lambda = N/(4L^2)$ is the node density. Since each node transmits with probability α, the number of transmitting nodes in this area A follows another Poisson distribution with parameter $\lambda\alpha A$.

A path loss model $g(r)$ is chosen such that it satisfies the following two conditions [1]
$g(r)$ is monotone decreasing,

$$\lim_{r\to 0} g(r) = \infty, \quad \lim_{r\to\infty} g(r) = 0 \tag{4}$$

and

$$\lim_{r\to\infty} r^2 g(r) = 0 \tag{5}$$

(5) is required to ensure finite interference at a node even for an infinite network. It must, however, be kept in mind that this is a far-field approximation model and that it does not hold for transmitters very close to the destination [6].

In this paper, $g(r)$ is modeled for 2 different channel models - the AWGN channel and the Rayleigh faded channel. The mean and the variance of the interference at a receiver node is determined for these 2 channel models in terms of the transmit probability, α, of a node in the network.

2.1 AWGN channel

The channel is assumed to be perfectly Gaussian with no distortion of the transmitted signal. For this channel, the path loss model is taken to be

$$g(r) = \frac{K}{r^\eta}, \qquad \eta > 2 \tag{6}$$

where $K = c^2/(4\pi f)^2$.

Without loss of generality, the receiver node is assumed to be located at the origin. It is shown in [1, 7] that the characteristic function of the total interference at the receiver due to all transmitting nodes beyond d_0 is given by

$$\Phi_I(\omega) = \exp\left(i\lambda\alpha\omega P_0 \int_0^{g(d_0)} \left[g^{-1}(t)\right]^2 e^{i\omega t} dt \right) \tag{7}$$

The first and the second central moments of this characteristic function represent the mean and variance of the interference and are given by

$$\begin{aligned} \mu_1 &= \frac{\lambda\alpha\pi d_0^2}{(1-2/\eta)} \frac{KP_0}{d_0^\eta} \\ \sigma_1^2 &= \frac{\lambda\alpha\pi d_0^2}{(1-1/\eta)} \left(\frac{KP_0}{d_0^\eta}\right)^2. \end{aligned} \tag{8}$$

Observe that the mean and the variance of the total interference power vary linearly with α. Also observe that for $\eta = 2$, the mean interference diverges, thus, justifying the requirement in (5). From the Central Limit theorem, for a large number of interferers, the distribution of the total interference power tends to a Gaussian, i.e.,

$$\mathcal{I}_{\mu_1,\sigma_1^2} \sim \mathcal{N}(\mu_1, \sigma_1^2) \tag{9}$$

This is a valid approximation since the probability of the interference being negative is negligible for the above distribution.

2.2 Rayleigh faded channel

The above analysis is now extended to the Rayleigh faded channel model. Here the channel distorts the signal in addition to adding Gaussian noise at the receiver. The path loss model is taken to be

$$g(r,t) = \frac{t^2}{r^\eta}, \quad 0 \le t < \infty, \quad d_0 \le r < \infty \tag{10}$$

where t is the fading coefficient of the channel. It is Rayleigh distributed with parameter $\sigma = \sqrt{1/2}$ such that $\mathbb{E}\left[t^2\right] = 1$. The interference statistics are derived in a manner similar to what is done in [1]. It is presented here for better understanding.

As discussed earlier, the set of transmitting nodes form a Poisson process with parameter $\lambda' = \lambda\alpha$. Let Y be the total interference power at the destination terminal.

$$Y = \sum g(r_i, t_i) \tag{11}$$

where the summation is over all transmitting nodes in the network. Let Y_a be defined as follows,

$$Y_a = \sum_{r_i \leq a} g(r_i, t_i) \tag{12}$$

i.e., interference power from all transmitters within a disk of radius a from the destination node. Thus, we have $\lim_{a\to\infty} Y_a = Y$. Let ϕ_{Y_a}be the characteristic function of Y_a.

$$\phi_{Y_a}(w) = \mathbb{E}\left[e^{iwY_a}\right] \tag{13}$$

This can be evaluated using conditional expectations as follows,

$$\begin{aligned} \mathbb{E}\left[e^{iwY_a}\right] &= \mathbb{E}\left[\mathbb{E}\left[(e^{iwY_a}/k \text{ in } D_a\right]\right] \\ &= \sum_{k=0}^{\infty} \frac{e^{-\lambda'\pi a^2}(\lambda'\pi a^2)^k}{k!} \mathbb{E}\left[e^{iwY_a}/k \text{ in } D_a\right] \end{aligned} \tag{14}$$

where 'k in D_a' means that there are k interfering nodes in a ring D_a. The nodes are uniformly distributed in this disk so that

$$f_R(r) \sim 2r/(a^2 - d_0^2), \;\; d_0 \leq r \leq a \tag{15}$$

Also, the probability distribution of the fading coefficient is given to be

$$f_T(t) \sim 2te^{-t^2}, \;\; 0 \leq t < \infty \tag{16}$$

The fading is independent from one channel to the other as is the distribution of the node radius. Thus, the conditional expectation can be evaluated to be

$$\mathbb{E}\left[(e^{iwY_a}/k \text{ in } D_a\right] = \left[\int_0^{\infty}\int_{d_0}^{a} \frac{2r}{(a^2 - d_0^2)} 2te^{-t^2} e^{jwg(r,t)} dr dt\right]^k \tag{17}$$

Substituting back in (14)

$$\begin{aligned} \mathbb{E}\left[e^{iwY_a}\right] &= exp\left[\lambda'\pi a^2 \left\{\int_0^{\infty}\int_{d_0}^{a} \frac{2r}{(a^2 - d_0^2)} 2te^{-t^2} e^{jwg(r,t)} dr dt - 1\right\}\right] \\ &= exp\left[\int_0^{\infty} 2te^{-t^2} \lambda'\pi a^2 \left\{\int_{d_0}^{a} \frac{2r}{(a^2 - d_0^2)} e^{jwg(r,t)} dr - 1\right\} dt\right] \end{aligned} \tag{18}$$

Applying the limit $a \to \infty$ as in the non-faded case, the characteristic function of the total interference becomes

$$\Phi(w) = exp\left[\int_0^{\infty} 2te^{-t^2} j\lambda'\pi w \int_0^{g(d_0,t)} \left[g^{-1}(x,t)\right]^2 e^{jwx} dx dt\right] \tag{19}$$

The mean and the variance of the total interference is then calculated to be,

$$\mu_2 = \Phi'(w)|_{w=0} \quad = \quad \frac{\lambda\alpha\pi d_0^{(2-\eta)}}{(1-2/\eta)} \tag{20}$$

$$\sigma_2^2 = \Phi''(w)|_{w=0} - \mu_2^2 \quad = \quad \frac{2\lambda\alpha\pi d_0^{2(1-\eta)}}{(1-1/\eta)} \tag{21}$$

It can be seen that the mean interference is the same as in the non-faded case whereas the variance is doubled. Both statistics are still linear in the transmit probability α. For a large number of interferers, the interference can once again be modeled to be Gaussian such that

$$\mathcal{I}_{\mu_2,\sigma_2^2} \sim \mathcal{N}(\mu_2, \sigma_2^2) \tag{22}$$

3 Throughput Analysis

3.1 AWGN channel

The volatility of the channel is completely ignored and the transmitted signal is assumed to arrive at the destination without any distortion. The channel access scheme is similar to slotted ALOHA where each node transmits packets irrespective of other transmitting nodes. The distance between the source node and the destination node is assumed to be R. Since there is no fading between the transmitter and the receiver, the received power is a constant given by $P_r = KP_0/R^\eta$. The thermal noise at the receiver is taken to be a constant, denoted by σ_n^2. The interference is assumed to be Gaussian distributed as discussed previously. The probability that the SINR is above the required threshold is given by

$$\begin{aligned} \mathbb{P}\,(\mathrm{SINR} > \Theta) &= \mathbb{P}\left(\frac{P_r}{\mathcal{I}_{\mu_1,\sigma_1^2} + \sigma_n^2} > \Theta\right) \\ &= \mathbb{P}\left(\mathcal{I}_{\mu_1,\sigma_1^2} < \frac{P_r}{\Theta} - \sigma_n^2\right) \end{aligned} \tag{23}$$

Also, $\mathcal{I}_{\mu_1,\sigma_1^2} \geq 0$ since the interference power cannot be negative. Thus, the probability of packet success is given by

$$\begin{aligned} \mathbb{P}\,(\text{packet success}) &= \mathbb{P}\left(0 \leq \mathcal{I}_{\mu_1,\sigma_1^2} < \frac{P_r}{\Theta} - \sigma_n^2\right) \\ &= \mathbb{P}\left(-\frac{\mu_1}{\sigma_1} \leq \mathcal{I}_{0,1} < \frac{\frac{P_r}{\Theta} - \sigma_n^2 - \mu_1}{\sigma_1}\right) \\ &= Q\left(\frac{\mu_1 + \sigma_n^2 - P_r/\Theta}{\sigma_1}\right) - Q\left(\frac{\mu_1}{\sigma_1}\right), \end{aligned} \tag{24}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-x^2/2}dx$. The per-node throughput is then given by

$$\zeta = \alpha(1-\alpha)\left[Q\left(\frac{\mu_1 + \sigma_n^2 - P_r/\Theta}{\sigma_1}\right) - Q\left(\frac{\mu_1}{\sigma_1}\right)\right]. \tag{25}$$

3.2 Block Rayleigh Faded Channel without CSI

The Rayleigh fading characteristics of the underlying physical channel are now taken into account. The channel access scheme, however, remains the same as in the previous case. The fading coefficient A is distributed as

$$p_A(a) = \frac{a}{\sigma_a^2} e^{-a^2/(2\sigma_a^2)} \tag{26}$$

where $\sigma_A^2 = 1/2$ such that $\mathbb{E}\left[A^2\right] = 1$. This paper considers a block faded channel where the fading is independent from one block to the next. This implies that the received power, $A^2 P_r$, is exponentially distributed. The mean probability that the SINR at the receiver exceeds the threshold is given by

$$\begin{aligned} \mathbb{E}\left[\mathbb{P}\left(\mathrm{SINR} > \Theta\right)\right] &= \mathbb{E}\left[\mathbb{P}\left(\frac{A^2 P_r}{\mathcal{I}_{\mu_2,\sigma_2^2} + \sigma_n^2} > \Theta\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\frac{\Theta\left(\mathcal{I}_{\mu_2,\sigma_2^2} + \sigma_n^2\right)}{P_r}\right)\right], \end{aligned} \tag{27}$$

where the expectation is taken over the Gaussian distribution. Simplifying further, and keeping in mind that $\mathcal{I}_{\mu_2,\sigma_2^2} \geq 0$, the probability of packet success becomes

$$\begin{aligned} \mathbb{P}\left(\text{packet success}\right) &= \exp\left(-\frac{\Theta\sigma_n^2}{P_r}\right) \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_0^{\infty} \exp\left(-\frac{\Theta y}{P_r}\right) \exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right) dy \\ &= \exp\left(-\frac{\Theta\left(\mu_2 + \sigma_n^2\right)}{P_r}\right) \frac{1}{\sqrt{2\pi}} \int_{-\frac{\mu_2}{\sigma_2}}^{\infty} \exp\left(-\frac{\Theta\sigma_2 t}{P_r}\right) \exp\left(-t^2/2\right) dt \\ &= \exp\left(-\frac{\Theta\left(\mu_2 + \sigma_n^2\right)}{P_r}\right) \exp\left(\frac{\Theta^2\sigma_2^2}{2P_r^2}\right) Q\left(\frac{\Theta\sigma_2}{P_r} - \frac{\mu_2}{\sigma_2}\right) \end{aligned} \tag{28}$$

The per-node throughput is then obtained by substituting back in (2).

3.3 Block Rayleigh Faded Channel with Perfect CSI at the Transmitter

In this case, a transmitting node is fully aware of the fading coefficient in the channel leading to its intended receiver. The channel access scheme is opportunistic instead of the blind ALOHA strategy. The transmitting node estimates the SNR at the receiver based on knowledge of the thermal noise and transmits only if the SNR is higher than a certain threshold τ. The transmit probability of the node depends on this threshold as

$$\begin{aligned} \alpha &= \mathbb{P}\left(\mathrm{SNR} > \tau\right) \\ &= \mathbb{P}\left(\frac{A^2 P_r}{\sigma_n^2} > \tau\right) \\ &= \exp\left(-\frac{\tau\sigma_n^2}{P_r}\right) \end{aligned} \tag{29}$$

Once this is determined, the packet success probability is given by

$$\mathbb{E}\left[\mathbb{P}\left(\mathrm{SINR} > \Theta \left| \mathrm{SNR} > \tau \right.\right)\right] = \mathbb{E}\left[\mathbb{P}\left(\frac{A^2 P_r}{\mathcal{I}_{\mu_2,\sigma_2^2} + \sigma_n^2} > \Theta \left| \frac{A^2 P_r}{\sigma_n^2} > \tau \right.\right)\right] \tag{30}$$

Using the memoryless property of the exponential distribution and simplifying as before, the probability of packet success is

$$\mathbb{P}\left(\text{packet success}\right) = \min\left\{\exp\left(\frac{\Theta^2\sigma_2^2}{2P_r^2} - \frac{\Theta\left(\mu_2 + \sigma_n^2\right) - \tau\sigma_n^2}{P_r}\right) Q\left(\frac{\Theta\sigma_2}{P_r} - \frac{\mu_2}{\sigma_2}\right),\ 1\right\} \tag{31}$$

For very small α, the SNR threshold, τ, takes on very large values due to which the above expression exceeds 1. Since a probability of success greater than 1 does not make sense, the '$\min$' operator is used. As before, the per-node throughput is calculated from (2).

For all three cases discussed above, the optimum transmit probability, α^*, that maximizes the throughput is obtained by solving

$$\left.\frac{d\zeta}{d\alpha}\right|_{\alpha^*} = 0 \quad \text{and} \quad \left.\frac{d^2\zeta}{d\alpha^2}\right|_{\alpha^*} < 0 \tag{32}$$

Closed form analytical expressions for α^* are unwieldy and difficult to obtain without approximations. More practical solutions can be obtained from numerical methods or by simply plotting the throughput expressions as a function of α.

4 Simulation Results

A Poisson field of 1600 network nodes is considered. The field is chosen to be a square of side 40m, so that the node density is $\lambda = 1/\mathrm{m}^2$. The path loss exponent η is varied from 3.5 to 4 in steps of 0.1. The attempt probability of the nodes is varied from 0 to 1. R is chosen to be 1m and d_0 is taken to be 3m. The threshold SINR, Θ, is 10dB. The transmit power is 10 nW and the thermal noise power is taken to be 5 fW.

The throughput curves for the $\eta = 4$ case are shown in Figure 1. In all three cases, the simulation curves agree to a great extent with the analytical curves. Also shown in the figure is the variation of the maximum throughput and the optimum transmit probability with variations in η.

Channel model	$\eta = 3.5$			$\eta = 4$		
	$\zeta_{\max}$	α^*	$P_{loss}(\%)$	$\zeta_{\max}$	α^*	$P_{loss}(\%)$
AWGN	0.076	0.100	23.8	0.145	0.2	27.4
Rayleigh w/o CSI	0.043	0.125	65.5	0.075	0.225	66.8
Rayleigh with CSI	0.160	0.200	20.0	0.210	0.300	30.0

Table 1: Comparison of $\zeta_{\max}$, α^* and P_{loss} for different channel models

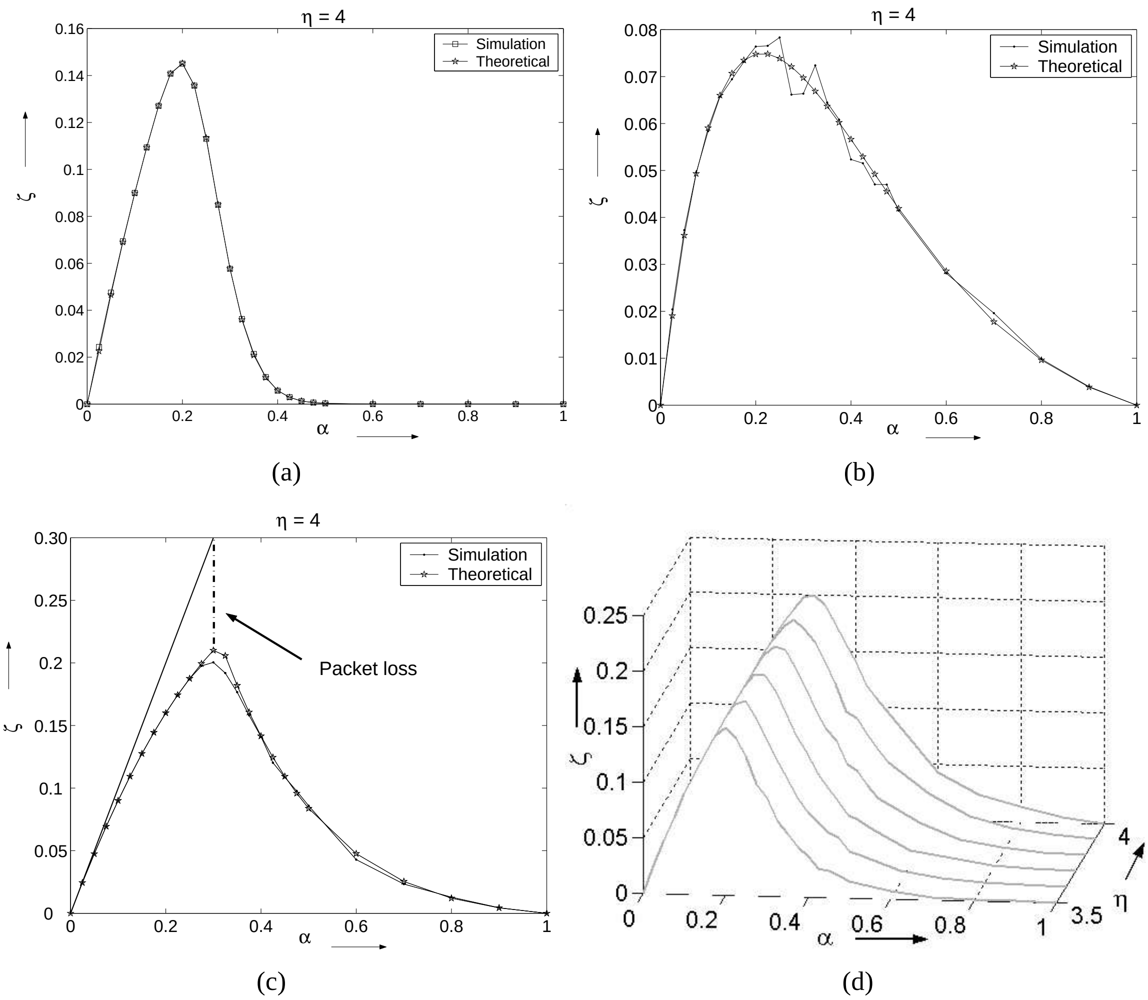

Figure 1: (a)AWGN Channel (b) Rayleigh fading with no CSI (c) Rayleigh fading with perfect CSI at Tx (d) Variation of throughput with α and η for the case with perfect CSI

Another parameter that characterizes the network performance is the packet loss probability P_{loss}. It is defined as

$$P_{loss} = \frac{\alpha^* - \zeta_{\max}}{\alpha^*} \tag{33}$$

The throughput curves also indicate a significant decrease in the packet loss probability for the perfect CSI case. The results are summarized in Table 1.

5 Conculsion

A performance evaluation of random wireless ad hoc networks is presented. First and second order statistics of the interference are used to derive closed form throughput expressions for three different cases. The exact transmission probability that maximizes the per-node throughput in each case is also determined. It is further observed that fading deteriorates the throughput. Opportunistic transmission helps recover from this loss and, in fact, increases the network throughput and the optimum transmission probability compared to the AWGN channel. The packet loss probability for the fading channel is also comparable to the AWGN case when there is CSI. The throughput is sensitive to two other parameters - η and d_0. Increasing η seems to affect the interference power more than the signal power leading to an increased network throughput. Increasing d_0 ensures that all interfering nodes are farther away; this also increases the throughput. (It cannot, however, be increased arbitrarily keeping in mind the fact that it affects the activity in the rest of the network.)

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