# On the Local Throughput of Large Interference-Limited Wireless Networks

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*Abstract* — **We characterize the interference and the local throughput for large interference-limited networks where the node positions, the channel gains, and the set of simultaneously transmitting nodes are either deterministic or random, governed by a certain probability distribution. In the latter case, we assume a Poisson point process for the node positions, a Rayleigh envelope for the channel gains, and the set of transmitting nodes are drawn from a Bernoulli distribution in each timeslot (slotted ALOHA).**

## I. Introduction

### A. Background

For many applications of large wireless networks, the performance is determined by the inter-node interference. While the scaling behavior of the network throughput or transport capacity is a well-studied area [1–6], relatively few *quantitative* results on the distribution of the interference and the signal-to-interference ratio (SIR) are available. We compare, interpret, and extend the approaches that have been taken in [7–10] to characterize the cumulated interference and outage probabilities. Each of these prior works focused on a particular, rather specific, type of network. We aim at inter- and extrapolating these results for a broad class, specifically one- and two-dimensional networks with random and deterministic node placement (Poisson or deterministic/regular), channel access (slotted ALOHA and TDMA), and channel characteristics (deterministic and Rayleigh fading). So we capture up to three sources of non-determinism: the node positions, the channel gains, and the set of transmitting nodes. On the other hand, since we are concerned with interference-limited networks, we do not consider noise. Consequently, the throughput results are fundamental in the sense that they cannot be exceeded (with a fixed transceiver hardware and modulation/coding scheme) even if infinite power is available.

### B. Models, notation, definitions, and initial remarks

*Network model.* We consider a single link, normally of distance 1, with a (desired) transmitter and receiver in a large network with $n$ other nodes as potential interferers. For infinite networks, $n \to \infty$.

*Transmit power levels.* It is assumed that all nodes transmit at the same power, which is set to 1.[2]

*Channel model.* For the large-scale path loss, we assume the usual power law where the received power $P_r$ decays with $d^\alpha$ for a path loss exponent $\alpha$. We consider deterministic channels where the received power $P_r \propto d^{-\alpha}$ and Rayleigh block fading channels where $P_r \propto S d^{-\alpha}$ with $S$ exponentially distributed with mean 1. If all channels are Rayleigh, this is sometimes referred to as a "Rayleigh/Rayleigh" model. If either only the desired transmitter or the interferers are subject to fading, we speak of *partial fading*.

*Random network.* A network whose nodes are distributed according to a homogeneous Poisson point process of intensity (density) $\lambda = 1$.[3] For random networks, the interference and outage expressions consider all network realizations.

*Normalized node distances $r_i$.* Since absolute distances do not matter, we usually consider *normalized distances* $r_i$, where the distances from the interferers to the receiver are normalized by the link distance. Thus the signal power (deterministic channel) or average signal power (fading channel) at the receiver is 1 (irrespective of $\alpha$).

*Transmit probability $p$.* For slotted ALOHA, every node transmits independently with probability $p$ in each timeslot. Hence for random networks the set of transmitting nodes in each timeslot form a Poisson point process of intensity $p$. Practical values of $p$ are small, *i.e.*, $p \lessapprox 1/3$ due to interference and throughput considerations. This permits certain approximations that would not hold for $p \approx 1$. The mean number of interferers is $np$, and the interference from node $i$ is $I_i = B_i S_i r_i^{-\alpha}$, where $B_i$ is iid Bernoulli with parameter $p$ and $S_i$ is iid exponential with mean 1.

*Success probability $p_s$.* A transmission is successful if the (instantaneous) SIR $\gamma = S_0/I$ exceeds a certain threshold $\Theta$, *i.e.*, $p_s = \mathbb{P}[\gamma > \Theta]$, where $I = \sum_{i=1}^{n} I_i$. This is the reception probability given that the desired transmit-receiver pair transmits and listens, respectively. It is usually assumed that $\Theta > 1$. Note that in the non-fading case, $\gamma = 1/I$, and that the threshold can be related to the rate of transmission $R$ through Shannon's capacity formula: for a rate $R$, the threshold is (at least) $\Theta = 2^{2R} - 1$.

*(Local) throughput $g$.* The local throughput, denoted simply as throughput throughout the paper, is defined to be the success probability multiplied by the probability that the transmit-receiver pair actually transmits and listens (the unconditioned reception probability). For the ALOHA scheme $g := p(1-p)p_s$, whereas for a TDMA line network where nodes transmit in every $m$-th timeslot, $g := p_s/m$.

*Spatial efficiency $\sigma$.* As will be derived, the success probability for slotted ALOHA can often be expressed as $p_s = e^{-p/\sigma}$. The parameter $\sigma$ determines the degree of spatial reuse in a network. Note that in a random network, the success probability equals the probability that a disk of radius

[1]This work was supported by NSF grants ECS03-29766 and CAREER CNS04-47869.

[2]Only relative powers matter.

[3]Without loss of generality, since noise is not considered.

$r = 1/\sqrt{\pi\sigma}$ around the receiver is free from interferers[4].

*Effective distances* $\xi_i$. The *effective distance* $\xi_i$ of a node to the receiver is defined as $\xi_i := r_i^\alpha/\Theta$.

*Near and far nodes.* A node with effective distance $\xi \leqslant 1$ is called a *near node*. Otherwise, it is a *far node*. In a random network, the mean number of near nodes can be determined as follows: The boundary radius is $r = \Theta^{1/\alpha}$, so the expected number of near nodes is $\pi\Theta^{2/\alpha}$. With slotted ALOHA, the expected number of *transmitting* near nodes is $\mathbb{E}[N_t] = p\pi\Theta^{2/\alpha}$.

*(Excess) kurtosis* $\kappa$. To characterize the "peakiness" of a distribution, we use the excess kurtosis $\kappa$, henceforth simply denoted as kurtosis, defined as the ratio of the fourth central moment to the squared variance minus 3. Note that the kurtosis of the Gaussian distribution is 0, and that the larger $\kappa$, the "peakier" a distribution. A distribution with $\kappa > 0$ is called leptokurtic.

## II. Related Work

### A. Infinite non-fading random networks with $\alpha = 4$ and slotted ALOHA

This case is studied in [7]. The characteristic function of the interference is determined to be[5]

$$\mathbb{E}[e^{j\omega I}] = \exp\Big(jp\pi\omega\int_0^\infty t^{-2/\alpha}e^{j\omega t}\mathrm{d}t\Big) \tag{1}$$

$$= \exp\big(-\pi p\Gamma(1-2/\alpha)e^{-j\pi/\alpha}\omega^{2/\alpha}\big) \tag{2}$$

and, for $\alpha = 4$,

$$= \exp\big(-\pi\sqrt{\pi/2}(1-j)p\sqrt{\omega}\big)\,. \tag{3}$$

The corresponding density (pdf) is

$$f_I(y) = \frac{\pi}{2}py^{-3/2}e^{-\pi^3p^2/4y}\,, \tag{4}$$

and the distribution function (cdf) is

$$F_I(y) = 1 - \operatorname{erf}\left(\frac{\pi^{3/2}p}{2\sqrt{y}}\right)\,, \tag{5}$$

where erf is the error function, *i.e.*, $\operatorname{erf}(x) = 2/\sqrt{\pi}\int_0^x \exp(-t^2)dt$. It is easily seen that in this case, even though the path loss exponent is relatively large, $\mathbb{E}[I] = \infty$, which is due to the path loss model and the fact that interferers may be arbitrarily close to the transmitter. The kurtosis turns out to be infinite as well. The problem of diverging mean can be avoided if a dead zone around the receiver is assumed. This is the approach taken in [11, 12], where only interferers outside a certain disk of radius $\rho$ around the receiver are considered. The resulting interference distribution is approximated as a Gaussian, which is only accurate if $\rho \gg 1$. In this case, however, the MAC scheme is no longer slotted ALOHA but resembles CSMA.

[4]The probability of not having an interferer within radius $r$ is $e^{-p\pi r^2}$.

[5]Note that their notation is adapted to ours. Also, a small mistake in [7, Eqn. (18)] is corrected here.

### B. Regular fading networks with $\alpha = 2$ and slotted ALOHA

In [9], the authors derive the distribution of the interference power for one- and two-dimensional Rayleigh fading networks with slotted ALOHA and $\alpha = 2$. Both regular and random networks were studied, but only for the regular line network with $r_i = i$, closed-form expressions are given. The interference distribution is given by the convolution of $n$ exponential distributions with means $r_i^{-\alpha}$, under consideration of the slotted ALOHA access scheme. The Laplace transform of the interference [9, Eqn. (5)],

$$\mathcal{L}_I(s) = \prod_{i=1}^n\Big(1 - \frac{p}{1+i^2/s}\Big) \tag{6}$$

$$= \frac{\prod_{i=1}^n(1+(1-p)s/i^2)}{\prod_{i=1}^n(1+s/i^2)}\,,\quad s \geqslant 0\,, \tag{7}$$

can, for $n \to \infty$, be simplified using Euler's product formula

$$\sin(\pi z) \equiv \pi z\prod_{i=1}^\infty(1-z^2/i^2) \tag{8}$$

for $z = j\sqrt{s}$ to [9, Eqn. (8)]

$$\mathcal{L}_I(s) = \frac{\sinh\big(\pi\sqrt{s(1-p)}\big)}{\sqrt{1-p}\sinh\big(\pi\sqrt{s}\big)}\,. \tag{9}$$

For infinite line networks with $p = 1$ (everybody always transmits), the pdf and cdf are given by[6]

$$f_I(x) = \begin{cases} 2\sum_{i=1}^\infty(-1)^{i+1}i^2e^{-i^2x}\,, & \text{for } x > 0\\ 0 & \text{for } x \leqslant 0\end{cases} \tag{10}$$

and

$$F_I(x) = \begin{cases} 1+2\sum_{i=1}^\infty(-1)^{i}e^{-i^2x}\,, & \text{for } x > 0\\ 0 & \text{for } x \leqslant 0\,,\end{cases} \tag{11}$$

respectively. The mean interference is simply $\sum_{i=1}^\infty i^{-2} = \pi^2/6$ in this case, and the variance is $\pi^4/90$. Since the interferers are independent, the kurtosis is given by the exponential distribution, $\kappa = 6$.

### C. Random fading networks with slotted ALOHA

In [10], the success probability of a transmission over unit distance in a two-dimensional random network with Rayleigh fading and slotted ALOHA and a noise process $N$ is expressed as

$$p_s = \mathbb{P}[S \geqslant \Theta(N+I)] \tag{12}$$

$$= \int_0^\infty e^{-s\Theta}\mathrm{d}\mathbb{P}[N+I \leqslant s] \tag{13}$$

$$= \mathcal{L}_I(\Theta)\cdot\mathcal{L}_N(\Theta)\,. \tag{14}$$

So, remarkably, the success probability for Rayleigh fading can be expressed as the product of the Laplace transforms of the noise $N$ and interference $I$. This elegant equivalence of the Laplace transform evaluated at the SIR threshold and the success probability was also pointed out in [8]. Ignoring the noise term, using the intermediate result

$$p_s = \exp\bigg\{-2\pi p\int_0^\infty\frac{r}{1+r^\alpha/\Theta}\mathrm{d}r\bigg\}\,, \tag{15}$$

[6]Again, a small mistake in [9, Eqn. (12)] is corrected.

(14) evaluates to [10, Eqn. (6)]

$$p_s = e^{-p\Theta^{2/\alpha}C(\alpha)} \tag{16}$$

with $C(\alpha) = (2\pi\Gamma(2/\alpha)\Gamma(1-2/\alpha))/\alpha$. For $\alpha = 3, 4$, for example, $C(3) = 4\pi^2/3\sqrt{3} \approx 7.6$ and $C(4) = \pi^2/2 \approx 4.9$.

So, with the expected number of near interferers $\mathbb{E}[N_t] = p\pi\Theta^{2/\alpha}$ and for $\alpha = 4$, the success probability can be expressed as $p_s = e^{-\mathbb{E}[N_t]\pi/2}$. For $p = 1/(\pi\sqrt{\Theta})$, corresponding to $\mathbb{E}[N_t] = 1$, we obtain $p_s = e^{-\pi/2} \approx 20\%$. To achieve $p_s = 1/2$, the mean number of transmitting near nodes has to be $2\log 2/\pi \approx 0.44$. So, for practical $p_s$, $N_t$ should be kept much smaller than one.

## III. Networks with Random Node Distribution

### A. Non-fading random networks with $\alpha = 4$ and slotted ALOHA

From (5), assuming a transmitter-receiver distance of 1, $\gamma = 1/I$ has the cdf

$$F_\gamma(\Theta) = \mathbb{P}[1/I < \Theta] = \operatorname{erf}\left(\frac{\pi^{3/2}p\sqrt{\Theta}}{2}\right), \tag{17}$$

which is the outage probability for non-fading channels for a transmitter-receiver distance 1. The pdf is

$$f_\gamma(\Theta) = \frac{\pi p e^{-\pi^3 p^2 \Theta/4}}{2\sqrt{\Theta}}, \tag{18}$$

and the expected SIR is $\mathbb{E}[\gamma] = 2/(\pi^3 p^2) \approx 0.064/p^2$. So, even for $\alpha = 4$, the transmit probability $p$ must be rather small for acceptable success probabilities. In fact, if $p$ is chosen such that $\mathbb{E}[\gamma] = \Theta$, *i.e.*, $p = \sqrt{2/(\Theta\pi^3)}$, the outage probability is $\operatorname{erf}(\sqrt{2}/2) \approx 68\%$. For $\Theta = 10$, *e.g.*, a transmit probability of only 8% would still result in more than 2/3 outages. For $\mathbb{E}[\gamma] = 2\Theta$, $p = 1/(\sqrt{\Theta\pi^3})$, the outage probability is still $\operatorname{erf}(1/2) \gtrapprox 1/2$. So the expected SIR must exceed the threshold $\Theta$ by more than a factor of two just to guarantee a success probability of 50%.
The variance is equally simple: $\operatorname{Var}[\gamma] = 8/(\pi^6 p^4) \approx 0.0083/p^4$.

Note that since $p_s$ is given by the error function rather than an exponential, the spatial efficiency is not defined strictly speaking. However, with the fairly sharp approximation $\operatorname{erf}(x) \approx 1 - e^{-3x/\sqrt{\pi}}$, we obtain $\sigma \approx 2/(3\pi\sqrt{\Theta})$ for the spatial efficiency.

### B. Fading random networks with slotted ALOHA

If only the desired link is subject to fading and $\alpha = 4$, we can exploit (3), replacing $j\omega$ by $-\Theta$ to get

$$p_s = \mathcal{L}_I(\Theta) = e^{-p\sqrt{\Theta}\pi^{3/2}}. \tag{19}$$

So, $\sigma = 1/(\sqrt{\Theta}\pi^{3/2})$ for this type of network.

If all the channels are subject to Rayleigh fading, we obtain from (16),

$$\sigma = \frac{\alpha}{2\pi\Theta^{2/\alpha}\Gamma(2/\alpha)\Gamma(1-2/\alpha)}, \tag{20}$$

which interestingly, for practical values of $\alpha$ increases approximately in proportion to $\alpha - 2$ for a fixed $\Theta$, *i.e.*, $\sigma(\alpha) \approx c(\Theta)(\alpha - 2)$, where $c(\Theta)$ is, as expected, a decreasing function of $\Theta$. As a result, if $p_s(\cdot)$ is the success probability as a function of the path loss exponent, we see that $p_s(2(\alpha-2)+2) \approx \sqrt{p_s(\alpha)}$, which quantifies the benefits of a higher path loss exponent. For $\alpha = 4$,

$$p_s = e^{-p\sqrt{\Theta}\pi^2/2} \tag{21}$$

and $\sigma = 2/(\sqrt{\Theta}\pi^2)$. The pdf of the SIR is

$$f_\gamma(\Theta) = \frac{\pi^2 p e^{-\pi^2 p\sqrt{\Theta}/2}}{4\sqrt{\Theta}} \tag{22}$$

with $\mathbb{E}[\gamma] = 8/(\pi^4 p^2) \approx 0.082/p^2$ and $\operatorname{Var}[\gamma] = 320/(\pi^8 p^4) \approx 0.034/p^4$.

Compared with the non-fading case, the mean SIR is increased by a factor $4/\pi$ and the variance by $40/\pi^2 \approx 4.0$. So, fading mainly affects the variance. The kurtosis is $\kappa = 2118/25 \approx 85$, which demonstrates how peaky this distribution is.

## IV. Networks with Deterministic Node Placement

In this section, we assume that $n$ interferers are placed at known relative distances $r_i$ from the intended receiver.

### A. Single-interferer success probability

For a single interferer at effective distance $\xi = r^\alpha/\Theta$, the success probability is

$$p_s = \mathbb{P}[\gamma > \Theta] = 1 - \frac{p}{1+\xi} \tag{23}$$

For small $p$, this is tightly upperbounded by $e^{-p/(1+\xi)}$ since $\log(1+x) \approx x$ for small $x$. For a non-fading interferer but a fading desired link, $I = Br^{-\alpha}$ with $B$ Bernoulli with parameter $p$ and thus

$$p_s = \mathbb{P}[S > \Theta B r^{-\alpha}] = pe^{-1/\xi} + (1-p) \tag{24}$$

$$= 1 - p(1 - e^{-1/\xi}). \tag{25}$$

Note that fading helps the link.

Given that this interferer is actually transmitting, we find $F_\gamma(\Theta) = 1/(1+\xi)$ as the cdf of the SIR in the first case (full fading), and $F_\gamma(\Theta) = 1 - e^{-1/\xi}$ for partial fading. Note that in the full fading case, $\mathbb{E}[\gamma] = \infty$, since the SIR is the ratio of two exponential RVs[7]; in the partial fading case, $\mathbb{E}[\gamma] = r^\alpha$ as for fully deterministic channels.

### B. Networks with slotted ALOHA and fading

In this case, $p_s = \mathbb{P}[S \geqslant \Theta I]$ for $I = \sum_{i=1}^n S_i r_i^{-\alpha}$ and $S_i$ iid exponential with mean 1. As mentioned earlier, since $S$ is exponential, the success probability is simply the Laplace transform of the total interference evaluated at $\Theta$: $p_s = \mathbb{E}[e^{-\Theta I}] = \mathcal{L}_I(\Theta)$. This Laplace transform was determined in (7) for the special case $r_i = i$ (but not interpreted as $p_s$). For general $r_i$ and $\alpha$, we obtain (see also [13]):

$$p_s = \prod_{i=1}^n \Big(1 - \frac{p}{1 + r_i^\alpha/\Theta}\Big) = \prod_{i=1}^n \Big(1 - \frac{p}{1+\xi_i}\Big) \tag{26}$$

where $\xi_i = r_i^\alpha/\Theta$ is the effective distance.

Since we are mostly interested in the behavior for small $p$ (and $\xi_i \gg 1$ for most $i$, *i.e.*, most interferers are far for non-negligible success probabilities), we approximate $\log p_s$ as a

[7]The pdf of $Z := X_1/X_2$ with $X_i$ exponential with mean $\mu_i$ is $f_Z(z) = \rho/(\rho z + 1)^2$ with $\rho = \mu_2/\mu_1$.

sum of terms $\log(1-p/(1+\xi)) \approx -p/(1+\xi)$, such that $p_s$ can be expressed as $p_s \lessapprox e^{-p/\sigma}$ for

$$\sigma = \frac{1}{\sum_{i=1}^{n} \frac{1}{1+\xi_i}}\,. \tag{27}$$

So, the approximation shows that $p_s$ has the same exponential form as for random networks, and the spatial efficiency is given by the "parallel connection" (or $1/n$ times the harmonic mean) of $1+\xi_i$.

### C. Regular networks with fading and slotted ALOHA

In regular networks, it is assumed that a relationship of the type $r_i^\alpha = i^\beta$ holds. For equidistant line networks, *e.g.*, $\beta = \alpha$.

*Special case:* $\beta = 2$. For $\beta = 2$ and $n \to \infty$[8],

$$\sigma = \frac{2}{\pi\sqrt{\Theta}\coth(\pi\sqrt{\Theta})-1} \approx \frac{2}{\pi\sqrt{\Theta}-1}\,, \tag{28}$$

resulting in

$$p_s \approx e^{-p(\pi\sqrt{\Theta}-1)/2}\,. \tag{29}$$

If the network extends to both sides of the link, $\sigma$ is cut in half. We expect to obtain the same approximation starting from (9). Indeed, with $\exp(-\pi\sqrt{\Theta}) \ll 1$, which is valid for $\Theta > 1$, we have

$$p_s \approx \frac{\exp(\pi\sqrt{\Theta}\sqrt{1-p})}{\sqrt{1-p}\exp(\pi\sqrt{\Theta})}, \tag{30}$$

and employing the linear approximations $\sqrt{1-p} \approx 1-p/2$ and $\log\sqrt{1-p} \approx -p/2$ yields

$$\log p_s \approx -p\pi\sqrt{\Theta}/2 + p/2 = -p\Big(\frac{\pi\sqrt{\Theta}}{2} - \frac{1}{2}\Big)\,, \tag{31}$$

which is the same as (29).

Note that the case $\beta = 2$ ($r_i^\alpha = i^2$) does not only include regular line networks with $\alpha = 2$, but also two-dimensional networks with $\alpha = 4$ where $r_i = \sqrt{i}$, which is an important case since the (ordered) distances in a regular lattice or random network increase approximately with $\sqrt{i}$.

The approximation is compared with the precise expression (9) in Fig. 1. Note that for practical values of $p$, *i.e.*, $p \lessapprox 1/3$, the match is very good over a large range of $\Theta$.

*General* $\beta$. For general $\beta$, the spatial efficiency $\sigma$ can be tightly upperbounded by interpreting (27) as a Riemann sum (or, equivalently, employing Euler's summation formula):

$$\sum_{i=1}^{n} \frac{1}{1+i^\beta/\Theta} \approx \int_{1/2}^{n+1/2} \frac{1}{1+x^\beta/\Theta}\mathrm{d}x \tag{32}$$

For general $\beta$ and $n$, this integral can be expressed by the Lerch transcendent $\Phi$-function. For certain values, it has a closed-form solution. For $\beta = 1$,

$$\sigma \approx \frac{1}{\Theta\ln\left(1+\frac{n}{\Theta+1/2}\right)} \tag{33}$$

which goes to zero as $n \to \infty$. Other values for which closed-form expressions of the integral (32) exist are $2\beta \in \mathbb{N} \leqslant 10$. For $\beta = 4$, the following approximation can be found:

$$p_s \approx e^{-p\left(\frac{\sqrt{2}}{4}\pi\Theta^{1/4}-1/2\right)}\,. \tag{34}$$

[8]For finite $n$, the sum can be expressed by the Digamma function.

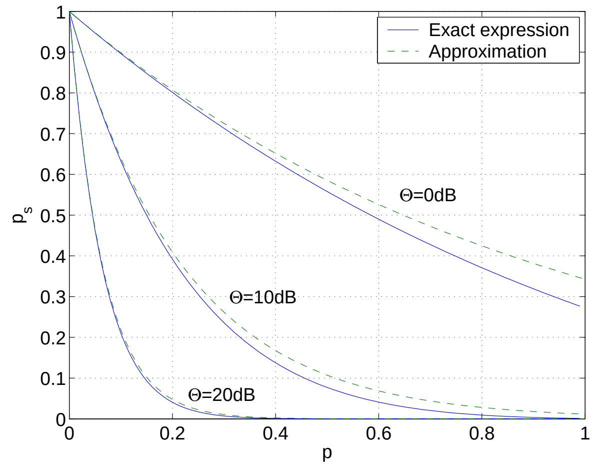


Fig. 1: Success probability $p_s$ for slotted ALOHA for an infinite line network and $\alpha = 2$. Solid lines: exact expression (9). Dashed lines: approximation (29).

The integral approximation also permits the analysis of networks with a finite number of nodes and networks whose closest interferer has an arbitrary distance from the transmitter.

The main difference to the expressions for random networks is that $\sigma$ is not linear in $\Theta^{-2/\alpha}$ but affine in $\Theta^{-1/\beta}$. This difference stems from the fact that the nearest interferer is not closer than the desired transmitter, which benefits $p_s$.

*Density functions.* When interpreting $1-p_s$ as the cdf of the SIR $\gamma$, we face the problem that for very small values of $\Theta$, it may be negative. For $\beta = 2$, this is the case for $\Theta < 1/\pi^2$. So the support of the cdf has to be restricted to $\Theta \in [1/\pi^2, \infty)$. The pdf is

$$f_\gamma(\Theta) = \frac{p\pi}{4\sqrt{\Theta}} e^{-p\left(\frac{\pi\sqrt{\Theta}}{2}-\frac{1}{2}\right)}\,, \quad \Theta \in [1/\pi^2, \infty) \tag{35}$$

and the expected interference is the rational expression

$$\mathbb{E}[\gamma] = \frac{p^2+4p+8}{p^2\pi^2}\,. \tag{36}$$

The expected SIR is 10dB at $p \approx 0.3$. The variance is

$$\mathrm{Var}[\gamma] = 16\frac{p^2+8p+20}{p^4\pi^4}\,. \tag{37}$$

Mean and variance are displayed in Fig. 2. Note the good match between this approximation and the numerical evaluation based on (9)[9] for $p \lessapprox 1/3$. The variance is at least an order of magnitude larger than the mean for practical $p$, which is due to the long tail of the density.

Note that for $r_i^\alpha = i^2$ and networks where the desired link is not subject to fading but the interferers are (*e.g.*, if power control is used to compensate for the fading), we can obtain $F_\gamma(\Theta) = \mathbb{P}[1/I < \Theta]$ from (11):

$$F_\gamma(\Theta) = 2\sum_{i=1}^{\infty} (-1)^{i+1} e^{-i^2/\Theta}\,, \quad \Theta > 0. \tag{38}$$

[9]$\mathbb{E}[\gamma] = \int_0^\infty s\,\mathrm{d}(1-\mathcal{L}_I(s))$. Analogously, $\mathrm{Var}[\gamma] = \int_0^\infty s^2\,\mathrm{d}(1-\mathcal{L}_I(s)) - \mathbb{E}^2[\gamma]$.

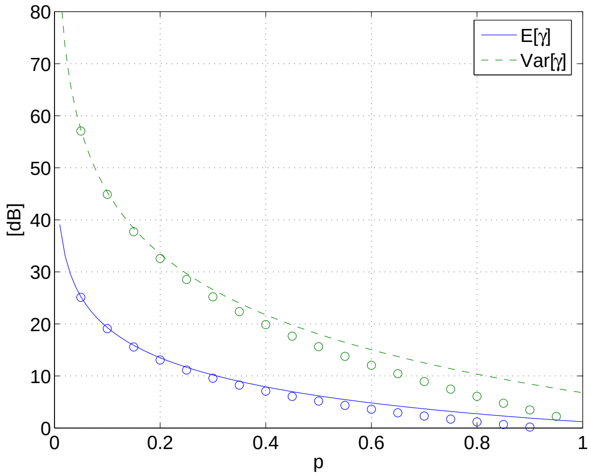


Fig. 2: Expected SIR $\mathbb{E}[\gamma]$ and variance $\mathrm{Var}[\gamma]$ (in dB) as a function of the transmit probability $p$. The circles denote the values obtained by numerical evaluation based on (9).

### D. Partially fading regular networks

If only the desired link is subject to fading, the success probability is given by

$$p_s = e^{-p\Theta \sum_{i=1}^{n} r_i^{-\alpha}}\,, \tag{39}$$

thus $\sigma = (\sum_{i=1}^{n} 1/\xi_i)^{-1}$. Compared with (27), $1+\xi$ is replaced by $\xi$. So, $\sigma$ is the "parallel connection" of all the $\xi_i$, and it is trivially upperbounded by $\min_i\{\xi_i\}$.

## V. Throughput

### A. Networks with slotted ALOHA

For networks with slotted ALOHA, the goal is to maximize the throughput $g(p) = p(1-p)p_s$. For fading networks,

$$g = p(1-p)\prod_{i=1}^{n}\Big(1 - \frac{p}{1 + r_i^\alpha/\Theta}\Big)\,. \tag{40}$$

This is a polynomial of order $n+2$. The roots are 0, 1, and $1 + r_i^\alpha/\Theta$, so the polynomial is positive for $0 < p < 1$ as expected.

With the approximation $g = p(1-p)e^{-p/\sigma}$, maximizing $\log(g)$ yields the quadratic equation $p_{\text{opt}}^2 - p_{\text{opt}}(1+2\sigma) + \sigma = 0$. Hence $p_{\text{opt}}$ is given by

$$p_{\text{opt}} = \sigma + \frac{1}{2}\big(1 - \sqrt{1 + 4\sigma^2}\big)\,. \tag{41}$$

Note that $0 < p_{\text{opt}} < 1/2$, and that there is a simple upper bound $p_{\text{opt}} \lessapprox \sqrt{\sigma}/2$ for $\sigma < 4/5$. In this case, $p_s \approx e^{-1/(2\sqrt{\sigma})}$, or, with $1/\sigma \propto \Theta^{2/\alpha}$, we obtain

$$p_s \approx e^{-c\Theta^{1/\alpha}}\,, \tag{42}$$

where $c$ depends on the network topology.

The *transmit efficiency* $\eta$ is defined as $\eta := g/p = (1-p)e^{-p/\sigma}$ which is monotonically increasing from $\lim_{\sigma\to 0}\eta = e^{-1} \approx 37\%$ to $\lim_{\sigma\to\infty}\eta = 1/2$. The upper bound is achieved if the interference goes to zero, in which case $p = 1/2$ and $g = 1/4$. The lower bound is more interesting. It shows that even for large interference ($\alpha/\Theta \ll 1$), a transmit efficiency of $e^{-1}$ can be achieved. Of course, for $\sigma \to 0$, we have $p \to 0$ and $g \to 0$, but the ratio $\eta$ converges to $e^{-1}$.

This is precisely the transmit efficiency of conventional slotted ALOHA. Indeed, for $\alpha/\Theta \to 0$, the interference prohibits that some positive fraction of the nodes transmits simultaneously. For such a network with $n$ nodes, $p = 1/n$ maximizes the throughput, and $\eta = \lim_{n\to\infty}(1-1/n)^{n-1} = e^{-1}$. So, even if no spatial reuse is possible, an efficiency of 37% is achievable.

### B. Regular line networks with TDMA and $\alpha = 2$

If in a TDMA scheme, only every $m$-th node transmits, the relative distances of the interferers are increased by a factor of $m$. So, the *effective* distances change from $r^\alpha/\Theta$ to $(mr)^\alpha/\Theta = r^\alpha/(\Theta m^{-\alpha})$, so having every $m$-th node transmit is equivalent to reducing the threshold $\Theta$ by a factor $m^\alpha$ and setting $p = 1$.

Using L'Hôpital's rule for $p = 1$ in (9) and replacing $\Theta$ by $\Theta m^{-\alpha}$ gives the success probability if every $m$-th node transmits:

$$p_s = \frac{y}{\sinh y}\,, \quad \text{where } y := \frac{\pi\sqrt{\Theta}}{m}\,. \tag{43}$$

The throughput $g = p_s/m$ can be expressed as

$$g = \frac{y^2}{\pi\sqrt{\Theta}\sinh y} = \frac{1}{m^2}\cdot\frac{\pi\sqrt{\Theta}}{\sinh\Big(\frac{\pi\sqrt{\Theta}}{m}\Big)}\,. \tag{44}$$

For large $m$, the sinh in the denominator is $\pi\sqrt{\Theta}/m$, so the throughput is $g \lessapprox 1/m$ as $m$ grows large, as expected, since the interference becomes negligible. Tighter bounds can be derived from $x + x^3/6 < \sinh(x) < e^x/2$:

$$\frac{1}{m^2}\cdot\frac{2\pi\sqrt{\Theta}}{\exp\Big(\frac{\pi\sqrt{\Theta}}{m}\Big)} < g < \frac{6m}{6m^2 + \pi^2\Theta}\,. \tag{45}$$

The upper bound is tighter for $m > 2\sqrt{\Theta}$. Unfortunately, maximizing these bounds does not necessarily maximize the throughput. So, to find $m_{\text{opt}} = \arg\max_{m\in\mathbb{N}} g(m)$, we need to determine the optimum value of $y$, which is the solution of

$$c\,y^* = \tanh y^*\,, \quad \text{where } c := \frac{1}{2}\,. \tag{46}$$

The optimum *real* value of $m$, denoted as $\tilde{m}_{\text{opt}}$, is then given by $\tilde{m}_{\text{opt}} = \pi\sqrt{\Theta}/y^*$. There is no closed-form expression for $y^*$, but solving (46) numerically and choosing $m_{\text{opt}} = \lfloor\tilde{m}_{\text{opt}}\rceil$ yields the optimum integer $m$ for virtually all[10] $\alpha$ and $\Theta$.

To find a closed-form approximation for $m_{\text{opt}}$, we note that $y^*$ is the value where $y/2$ intersects $\tanh y$. Since tanh is upperbounded by 1, $y_u = 2$ is an upper bound for $y^* = 1.9150$. So, $m_{\text{opt}} > 1/2\pi\sqrt{\Theta}$. A "natural" choice is to round this lower bound up to the next integer, *i.e.*,

$$m_{\text{opt}} = \lceil\frac{1}{2}\pi\sqrt{\Theta}\rceil\,. \tag{47}$$

Note that the same approximation can be derived if the sinh in (44) is approximated by $e^x/2$. The resulting approximation on the achievable throughput

$$g_{\max} \approx \frac{8}{\pi e^2\sqrt{\Theta}} \approx \frac{0.34}{\sqrt{\Theta}} \tag{48}$$

[10] There are very small intervals where the resulting throughput is up to about 1% smaller than the theoretical optimum.

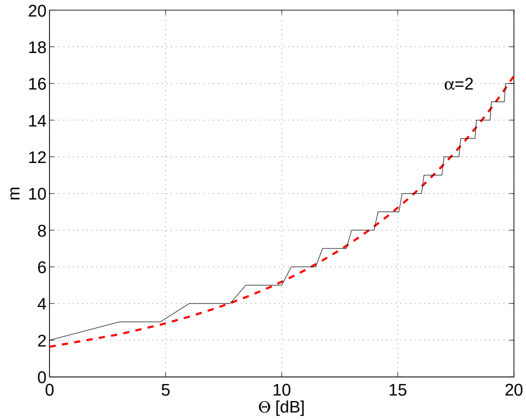


Fig. 3: TDMA parameter $m$ as a function of $\Theta$ [dB] for $\alpha = 2$. The dashed lines show the (real) theoretical optimum $\tilde{m}_{\text{opt}}(\Theta)$, the solid line is from (47).

is sharp. In a line network with slotted ALOHA, for comparison, the approximation $g \approx 0.2/\sqrt{\Theta}$ can be derived from (29) and (41). So, TDMA has an about 70% higher throughput.

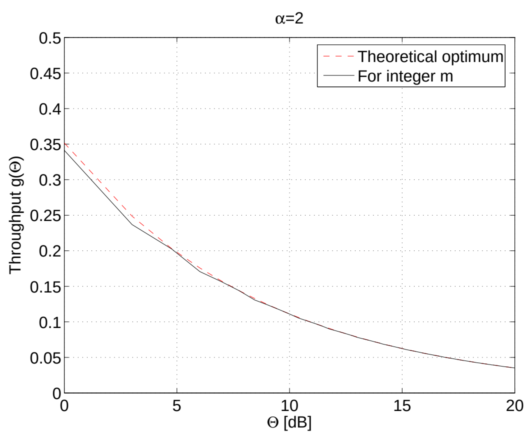


Fig. 4: Throughput $g$ as a function of $\Theta$ [dB] for $\alpha = 2$. The dashed lines show the (real) theoretical optimum $\tilde{m}_{\text{opt}}(\Theta)$, the solid lines the achievable throughput for $m$ given by (47).

If there are interferers on the left and right side of the receiver, $p_s$ in (43) needs to be squared, resulting only in a change of the coefficient $c$ in (46) to $c := 2/3$.

## VI. Conclusion

We have characterized the interference and SIR for large wireless networks with different degrees of non-determinism, and we analytically derived throughput expressions for random and deterministic networks as a function of the node positions, the transmit probability, the path loss exponent, and the SIR threshold. For slotted ALOHA, the throughput can be expressed as $g = p(1-p)e^{-p/\sigma}$, where the *spatial efficiency* $\sigma$ is approximately proportional to $1/\Theta^{2/\alpha}$ for two-dimensional networks. The maximum achievable throughput is determined for slotted ALOHA and deterministic $m$-phase TDMA. Not surprisingly, the TDMA scheme has a substantially better throughput performance at comparable energy consumption.

The success probability $p_s = e^{-p/\sigma}$ as a function of $\Theta$ can be interpreted as the complementary cumulative distribution of the SIR, which permits a complete characterization of the SIR and/or interference. This is important since the mean interference or SIR is not a meaningful measure. Indeed, it they are often infinite; the SIR even in the simple case of Rayleigh fading with only one single interferer.

Many extensions are possible, such as the inclusion of power control, opportunistic access schemes, and relating $\Theta$ to the rate of transmissions.

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