Random-access Poisson networks: stability and delay

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***Abstract*—We consider a Poisson network of sources, each with a destination at a given distance and a buffer of infinite capacity. Assuming independent Bernoulli arrivals, we characterize the stability region when one or two classes of users are present in the network. We then derive a fixed-point equation that determines the success probability of the typical source-destination link and evaluate the mean delay at each buffer.**

***Index Terms*—ALOHA, Poisson point process, queueing, stability.**

I. Introduction

Stochastic geometry and, in particular, point-process theory, has been widely employed in the study of wireless networks [1], [2]. A commonly used model is the Poisson bipolar network model [2], in which the sources are located according to a Poisson point process (PPP) and each source has an associated destination, which is not part of the PPP. This model has been employed in a large number of papers (see [1] for a comprehensive overview) to obtain an analytical handle on the outage probability and the throughput in random-access wireless networks, based on a signal-to-interference-and-noise-ratio (SINR) model for successful reception. By random access, it is understood that the sources follow the ALOHA protocol, i.e., each source transmits with probability p independently of the other sources in the network.

An implicit assumption in this line of work is that the sources always have packets to transmit. This letter departs from this framework and considers the practically important scenario where packets arrive at each source independently from slot to slot with a certain probability, and are accommodated in a buffer. The introduction of buffers to the bipolar model creates the complication that the density of transmitting sources, i.e., sources which are allowed to transmit *and* have packets in their queues, depends on the packet success probability and vice versa. This interaction makes the analysis of the - inherently simple - ALOHA protocol non trivial. A key problem is deriving the stability region, i.e., the set of arrival rates such that the queues are stable, which has been addressed in the past for systems consisting of a finite number of sources and a single destination. E.g., [3] recursively characterized the stability region using Loynes' criterion [4] and stochastic dominance[1]; [5] determined inner and outer bounds to the stability region; and [6] considered ALOHA combined with multi-packet reception.

[1]In the dominant ALOHA system, a subset of the users make "dummy" transmissions when their queues are empty. The queue sizes in the dominant system are never smaller than in the original system, if both systems start from the same initial condition.

This is the first paper to combine queueing theory and the PPP framework in order to study the performance of ALOHA in distributed networks consisting of an infinite population of source-destination pairs. We characterize the stability region for one or two classes of users and derive closed-form expressions for the packet success probability and the mean delay at the source buffers.

II. System model

At the beginning of time, i.e., $t = 0$, the positions of the sources, $\{X_i(0)\}$, $i \in \mathbb{N}$, are determined according to a homogeneous PPP $\Phi(0) \subset \mathbb{R}^2$ of density λ, i.e., $\Phi(0) = \{X_i(0)\}$. Source i has a destination at $Y_i(0)$, such that $|X_i(0)-Y_i(0)| = R$, and a buffer of infinite capacity to accommodate incoming packets. Each source belongs to a traffic class n with probability π_n, $n = 1, \ldots, N$, $\sum_{n=1}^{N} \pi_n = 1$, characterized by a packet arrival probability a_n, and a medium access probability p_n which determines the frequency of channel access for all sources in that class. Note that packet arrival and channel access events are independent across sources and slots. The initial queue lengths $\{Q_i(0)\}$ are also chosen independently according to some probability distribution.

Define the small parameters $\epsilon \ll 1$, $\epsilon'' < \epsilon' \ll 1$ and $\delta \ll 1$. The network operation is depicted in Fig. 1 and described in detail below. For each class n:

1. At $t+\epsilon$, a source accesses the channel with probability p_n. If it is granted access and $Q_i(t) > 0$ the packet at the head of the queue is transmitted and leaves the queue if the SINR in the slot $(t, t+1)$ is greater than a threshold $\theta > 0$. Otherwise it remains at the head awaiting retransmission.
2. At $t+1-\epsilon'$, a new packet arrives with probability a_n.
3. At $t+1-\epsilon''$, the position of each node is changed according to a high mobility random walk model [2, Ch. 1.3], i.e., a vector $\Delta X_i(t)/\delta$ is added to $X_i(t)$ and $Y_i(t)$. $\Delta X_i(t)$ is a random vector with some smooth probability distribution on $\mathbb{R}^2$, independent of $X_i(t)$.
4. At $t+1$, the queue lengths $\{Q_i(t)\}$ and new node positions $\{X_i(t+1)\}$ are measured. Note that the point process $\Phi(t+1) = \{X_i(t+1)\}$ is also a PPP according to the displacement theorem [2, Ch. 1.3].

Based on these modeling assumptions, each queue length is a Markov chain which obeys [3]

$$Q_i(t+1) = (Q_i(t) - D_i(t))_+ + A_i(t), \quad t = 1, 2, \ldots$$

where $(.)_+ = \max(., 0)$, $A_i(t) = 1$ with probability a_n (and 0 otherwise) and

$$D_i(t) = e_i(t)\mathbf{1}\left(\mathrm{SINR}_i(t) > 0 \mid e_i(t) = 1\right). \tag{1}$$

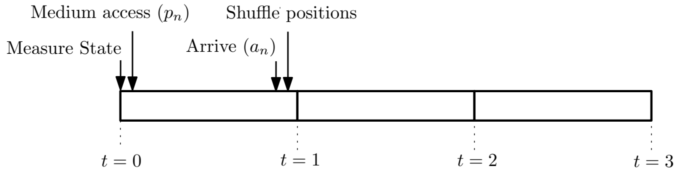

Fig. 1. Network operation. The queue length is measured at $t \in \mathbb{Z}^+$. Medium access follows with probability p_n and, if the queue is not empty, transmission. Right before the end of the slot, a new packet arrives with probability a_n and the positions of the nodes are shuffled.

$e_i(t) = 1$ with probability p_n (and zero otherwise) and

$$\mathrm{SINR}_i(t) = \frac{h_{ii}(t)R^{-b}}{\sum_{j\neq i} h_{ij}(t)e_j(t)\mathbf{1}(Q_j(t) > 0)d_{ij}(t)^{-b} + \gamma^{-1}}$$

where $h_{ij}(t)$, $d_{ij}(t) = |X_j(t)-Y_i(t)|$ are the fading coefficient (constant in $(t, t+1)$) and distance between source j and the destination, respectively, and γ is the signal-to-noise ratio. We assume that $\{h_{ij}(t)\}$ are exponential with unit mean and i.i.d. across space and time, and that the propagation exponent $b > 2$ (otherwise the interference is infinite a.s. [1]).

III. Single-class networks

Let us consider the case where there is only one class of users, i.e., $a_n = a$ and $p_n = p$. The following proposition provides a necessary and sufficient condition for the stability of the buffers in the sense defined in [3], i.e., $\{Q_i(t)\}$ having a limiting distribution for $t \to \infty$.

Proposition 1 *The queues of the PPP network of Section II, with a single user class, are stable if and only if*

$$a < p\mathrm{e}^{-\lambda p c R^2 - \theta R^b \gamma^{-1}} \tag{2}$$

where $c = \Gamma(1 + 2/b)\Gamma(1 - 2/b)\pi\theta^{2/b}$. *The closure of stable arrival rates is thus*

$$a \leq p^*\mathrm{e}^{-\lambda p^* c R^2 - \theta R^b \gamma^{-1}}, \tag{3}$$

where $p^* = \min\left\{\frac{1}{\lambda c R^2}, 1\right\}$.

Proof: We first show sufficiency by proving that (2) guarantees stability in a dominant network, where all sources that have empty queues make dummy transmissions. Select a source in the dominant network, e.g., i. The arrival process $A_i(t)$ is by definition stationary and ergodic. The departure process $D_i(t)$, defined in (1), is also stationary as the point process of transmitting sources, $\Phi'(t) = \{X_j(t) : e_j(t) = 1\}$, is a homogeneous PPP of density λp. Defining $\bar{p}_s = \mathbb{E}[D_i(t)]$, from [1, Eq. (9)] we obtain that $\bar{p}_s = p\mathrm{e}^{-\lambda p c R^2 - \theta R^b \gamma^{-1}}$. We now establish the ergodicity of $D_i(t)$. Due to the mobility model of Section II, the form of the path-loss function $l(r) = r^{-b}$, and the independence of $\{h_{ii}(t)\}$ across t, $\{\mathrm{SINR}_i(t)\}$ are i.i.d. across t (see [2, p. 26]). By the law of large numbers, $D_i(t)$ is ergodic. Since $(A_i(t), D_i(t))$ are jointly stationary and ergodic, by Loynes' theorem, if $a < \bar{p}_s$, then $Q_i(t)$ is stable.

The necessity of (2) is shown as follows: If $a > \bar{p}_s$, then, by Loynes' theorem, it follows that $\lim_{t\to+\infty} Q_i(t) = +\infty$ (a.s.) for all sources in the dominant network. Using the same arguments as in [3, p. 511], if the queues in the actual network initially have a large enough number of packets, both networks are identical. This implies that $Q_i(t) \to +\infty \ \forall i$ is also true for the original network. Finally, (3) follows by maximizing the right-hand side of (2) over p. ■

Note that (2) is a generalization of $a < p(1 - p)^M$, i.e., the stability region of the M-user symmetric ALOHA system with collisions [3], for a network with Poisson distributed sources and an SINR physical-layer model. Having established a condition for stability, we now evaluate the probability of successful transmision.

Proposition 2 *Let $t \to +\infty$. Provided that the network is stable, the packet success probability is*

$$p_s = \exp\left(-\lambda \frac{a}{p_s} c R^2 - \theta R^b \gamma^{-1}\right). \tag{4}$$

Proof: Employing a similar argument as in Proposition 1, $\{\mathrm{SINR}_i(t)\}$ and, by definition, $\{D_i(t)\}$ are i.i.d. across t. By symmetry, the stationary packet success probability is the same across all sources and equal to $p_s = \mathbb{E}[D_i(t)]$. Since the queue of source i is subject to i.i.d. packet arrivals with probability a and i.i.d. packet departures with probability pp_s, the probability that the queue is non-empty is $\rho = a/(pp_s)$. It follows that the point process of transmitting sources $\Phi'(t) = \{X_j(t) : Q_j(t) > 0, e_j(t) = 1\}$ is a PPP with density $\lambda\rho p$. By [1, Eq. (9)], we establish (4). ■

In Proposition 2, we take advantage of the shuffling of the node positions in each slot, which results in independent channel conditions for each source-destination link across time. The interaction between the success probability and the queue sizes is captured by (4). Note that (4) has two solutions iff $a \leq \mathrm{e}^{-\theta R^b \gamma^{-1} - 1}/(\lambda c R^2)$, which is satisfied when (2) holds. The larger of the two solutions is rejected on the basis that it is decreasing in a, i.e., it results in a system with a success probability which increases with increasing incoming traffic. The valid solution is found in closed form

$$p_s(a) = \exp\left(-\theta R^b \gamma^{-1} + W\left(-\lambda a c R^2 \mathrm{e}^{\theta R^b \gamma^{-1}}\right)\right), \tag{5}$$

where W is the principal branch of the Lambert W function [7]. As seen, p_s does not directly depend on p. However, the range of allowable values of a depends on p due to (2).

Eq. (5) may be employed in order to evaluate the mean total time that a packet spends in the typical queue.

Proposition 3 *Let $\rho(a) = a/(pp_s(a))$. The mean packet delay in a stable network with a single class of users is*

$$D = \frac{\rho(a)}{a} \frac{1 - a}{1 - \rho(a)}. \tag{6}$$

Proof: The steady-state distribution of $Q_i(t)$ is

$$\mathbb{P}(Q_i(t) = k) = \frac{\rho - \rho^2}{1 - a}\left(\frac{\rho - a}{1 - a}\right)^{k-1}, \quad k \geq 1.$$

The result follows by Little's formula, $D = \mathbb{E}[Q_i]/a$. ■

We plot D vs. a in Fig. 2 for $\lambda c R^2 = 0.5, 1, 2$, $p = p^*$ and $\mathrm{SNR} \to \infty$. We also plot D with $\bar{p}_s = p^*\mathrm{e}^{-\lambda p^* c R^2}$ in place of

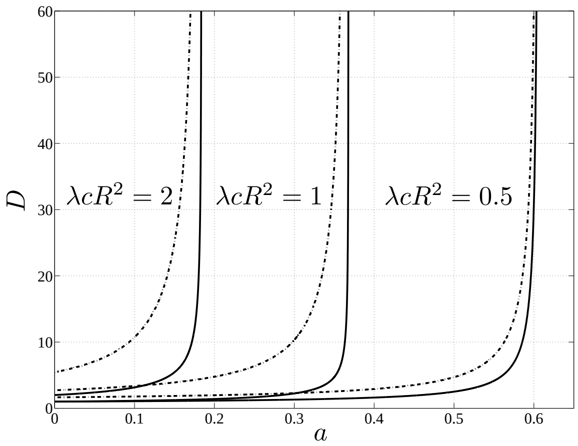

Fig. 2. Delay vs. a for $\lambda cR^2 = \{0.5, 1, 2\}$, $p = p^*$, and SNR $\to \infty$. The delay in the dominant network is also shown (dash-dot) for comparison.

p_s, i.e., the delay at the typical queue in the dominant system. The discrepancy between the pairs of curves decreases with decreasing λcR^2.

IV. Two-class networks

We now consider a network with $N = 2$, i.e., two classes of users. For ease of exposition, let the SNR $\gamma \to \infty$ and define $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{p} = (p_1, p_2)$.

Proposition 4 *The set of stable arrival rates for the PPP network of Section II, with two classes of users, is* $\mathcal{S}_{\boldsymbol{p}} = \bigcup_{(n,m)\in\mathcal{I}} \mathcal{S}_{n,m,\boldsymbol{p}}$, *where* $\mathcal{I} = \{(1,2),(2,1)\}$ *and*

$$\mathcal{S}_{n,m,\boldsymbol{p}} = \Big\{\boldsymbol{a} : a_m < p_m e^{-\lambda c(\pi_1 p_1 + \pi_2 p_2)R^2},$$
$$a_n < p_n \exp\Big(-\lambda c \pi_n p_n R^2 + W(-\lambda c \pi_m a_m R^2 e^{\lambda c \pi_n p_n R^2})\Big)\Big\}$$

Moreover, if $\mathcal{S} = \bigcup_{\boldsymbol{p}} \mathcal{S}_{\boldsymbol{p}}$, *then* $C(\mathcal{S}) = \mathcal{T}$, *where*

$$\mathcal{T} \triangleq \Big\{\boldsymbol{a} : a_1 = p_1 e^{-\lambda cR^2(\pi_1 p_1 + \pi_2 p_2)},$$
$$a_2 = p_2 e^{-\lambda cR^2(\pi_1 p_1 + \pi_2 p_2)}, \boldsymbol{p} \in [0,1]\times[0,1]\Big\}. \quad (7)$$

and $C(\mathcal{S})$ *denotes the closure of* $\mathcal{S}$.

Proof: $\mathcal{S}_{\boldsymbol{p}}$ follows from [3, Th. 1], Proposition 1 and 5. We create two dominant systems. In each one, one of the two classes of users makes dummy transmissions. The stability region is found by the union of stable arrival rates over the two systems. $C(\mathcal{S}) = \mathcal{T}$ is proved by showing that the Jacobian determinant $\mathcal{J} = |\partial \boldsymbol{a}/\partial \boldsymbol{p}|$ is the same at the boundaries of $\mathcal{S}$ and $\mathcal{T}$. ∎

In Fig. 3, we plot $\mathcal{S}$ for different values of λcR^2. The following may be verified:

- $\lambda cR^2 \le 1$: $\mathcal{S} = \mathcal{S}_{(1,1)}$, which is the generalization of (3) for two user classes. Moreover, if $\lambda cR^2 \max\{\pi_1, \pi_2\} \le 0.5$, then $\max \mathcal{T} = 2e^{-\lambda cR^2}$ at $\boldsymbol{p} = (1,1)$.
- $\lambda cR^2 \min\{\pi_1, \pi_2\} \ge 1$: the boundary of $\mathcal{T}$ is linear, i.e., time sharing between the two classes achieves all rate pairs.

Note that the success probability and the delay for each class may be evaluated similarly to Propositions 2 and 3.

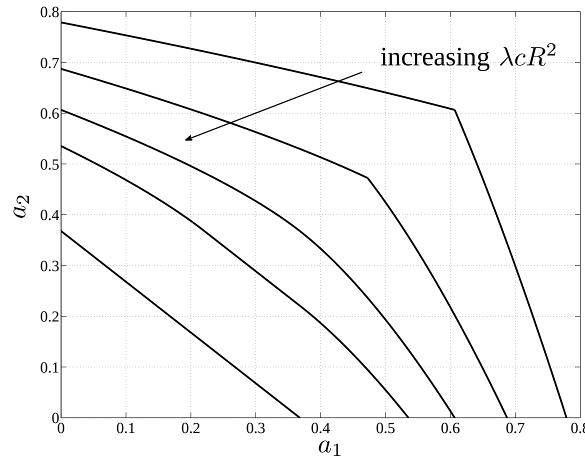

Fig. 3. The boundary of $\mathcal{S}$ plotted for $\lambda cR^2 = 0.5, 0.75, 1, 1.25, 2$ and $\pi_1 = \pi_2 = 0.5$. The shape of $\mathcal{S}$ is determined by the physical-layer parameter λcR^2 and the probabilities π_1, π_2.

V. Conclusions

This paper studied the stability and delay performance of slotted ALOHA in a Poisson network. Based on the fact that the transmitters in each slot form a PPP, we determined the set of stable arrival rates for one or two classes of users and provided a closed-form expression for the mean delay. These results bridge the gap between work on PPPs that has exclusively considered backlogged nodes and work on ALOHA that has relied on crude physical-layer models and not considered the effect of the node spatial distribution.

Acknowledgments

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References

[1] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse, and M. Franceschetti, "Stochastic geometry and random graphs for the analysis and design of wireless networks," *IEEE J. Sel. Areas Commun.*, vol. 27, no. 7, pp. 1029–1046, Sep. 2009.

[2] F. Baccelli and B. Błaszczyszyn, *Stochastic Geometry and Wireless Networks*, ser. Foundations and Trends in Networking. NOW, 2009.

[3] W. Szpankowski, "Stability conditions for some distributed systems: buffered random access systems," *Adv. in Applied Prob.*, vol. 26, pp. 498–515, 1994.

[4] R. Loynes, "The stability of a queue with non-independent inter-arrival and service times," *Proc. Camb. Phil. Soc.*, vol. 58, pp. 497–520, 1962.

[5] W. Luo and A. Ephremides, "Stability of N interacting queues in random-access systems," *IEEE Trans. Inf. Theory*, vol. 45, pp. 1579–1587, 1999.

[6] V. Naware, G. Mergen, and L. Tong, "Stability and delay of finite-user slotted ALOHA with multipacket reception," *IEEE Trans. Inf. Theory*, vol. 51, pp. 2636–2656, 2005.

[7] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert function," *Adv. Computational Maths.*, vol. 5, pp. 329–359, 1996.