A Simple Approximative Approach to the SIR Analysis in General Heterogeneous Cellular Networks

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***Abstract*—The crushing demands for mobile data traffic drive the current cellular networks to become more heterogeneous, making the signal-to-interference ratio (SIR) distribution more difficult to analyze. In this paper we propose a simple approximative approach to the SIR distribution of heterogeneous cellular networks (HCNs) based on the ASAPPP method which stands for "approximate SIR analysis based on the Poisson point process" and the MISR (mean interference-to-signal ratio)-based gain for each individual tier of the HCN. The results demonstrate that this approach gives a tight approximation and asymptotically a lower bound for the coverage probability.**

I. Introduction

A. Motivation

Heterogeneous cellular networks (HCNs) have been widely regarded as a solution to address the challenge of the explosive mobile date traffic growth [1]. As one of the most important and general metrics, it is important to analyze the signal-to-interference ratio (SIR) distribution in the interference-limited HCNs to further obtain performance metrics such as coverage probability, capacity and throughput. The current theoretic analysis on the SIR distribution mostly focuses on the models based on multi-tier homogeneous independent Poisson point processes (PPPs) [2]. However, for general non-Poisson networks, such analysis is significantly more difficult than that of PPP networks and can be obtained merely by large-scale complicated simulations or at best be expressed using combinations of infinite sums and integrals. Hence it is necessary to explore techniques that provide good approximations of the SIR distribution for general HCN models.

B. Related Work

The homogeneous independent PPP (HIP[1]) model usually gives us highly tractable results for HCNs [2–5] but does not capture the spatial dependence between base stations (BSs). However, for non-Poisson deployments, exact results of the SIR distribution are hard to derive or, even though they could be derived, the resulting expressions are very complex to compute [6–8]. As a result, it is almost impossible to figure out how the network performance is affected by the parameters, such as the density, transmit power, etc. In [9], the authors provide the Padé approximation for the coverage probability of a cellular network model where the BSs form a β-Ginibre point process (β-GPP), but the results show that the Padé approximation becomes very inaccurate as the SIR threshold increases. In addition, since the Maclaurin coefficient computation in the approximation involves multiple-level and infinite integrals, sums and products, the numerical computation of the coverage probability is still complex and time-consuming. Moreover, the Padé approximation can be expected to be much more complex when applied in the heterogeneous scenarios due to the calculation of the Maclaurin coefficients. Fortunately, as shown in [3, 10], the coverage probability $P_{\rm c}(\theta)$ for the single-tier network modeled by different point processes can be tightly approximated by merely scaling the threshold θ to θ/G, i.e., $P_{\rm c}(\theta) \approx P_{\rm c}^{\rm PPP}(\theta/G)$, where $P_{\rm c}^{\rm PPP}(\theta)$ is the coverage probability of PPP networks and G can be quantified based on the mean interference-to-signal ratio (MISR) and is called MISR-based gain. We show that the MISR-based method can be applied to a general HCN, which is modeled by arbitrary (but stationary and independent) point processes.

C. Contributions

The main objective of this paper is to present an approximative approach that yields highly tractable results for the SIR distribution in general HCNs. We extend the ASAPPP-based approximation [11], which stands for "approximate SIR analysis based on the PPP", to general HCNs using the MISR-based gain for each individual tier. Our numerical results demonstrate that the proposed method is an excellent approximation to the SIR distribution in general HCNs.

II. System Model

We first consider a coverage-oriented heterogeneous cellular network (HCN) model comprising two types of nodes, i.e., the macro-BSs (MBSs) and the pico-BSs (PBSs), where the

[1]A model whose tiers are independent Poisson point processes is called HIP model. Its SIR distribution is equivalent to that of the single-tier PPP model when the power path loss law with Rayleigh fading and strongest base station association are adopted [3].

locations of the MBSs are modeled as a stationary point process Φ_{m} with density λ_{m} and the locations of the PBSs are modeled as an another independent stationary point process Φ_{p} with density λ_{p}. We assume that each user is associated with the BS that offers the strongest average received power. Due to the stationarity of both Φ_{m} and Φ_{p}, we consider the typical user located at the origin. Thus, the SIR is expressed as

$$\mathrm{SIR} \triangleq \frac{S}{I} = \frac{\mu_{x_0}\ell(|x_0|)h_0}{\sum\limits_{x\in\Phi_{\mathrm{m}}\bigcup\Phi_{\mathrm{p}}\setminus\{x_0\}} \mu_x \ell(|x|)h_x}, \tag{1}$$

where x_0 denotes the location of the serving BS of the typical user and μ_x denotes the transmit power of node x: if $x \in \Phi_{\mathrm{m}}$, $\mu_x = \mu_{\mathrm{m}}$; otherwise, $\mu_x = \mu_{\mathrm{p}}$. We assume a power path loss law $\ell(r) = r^{-\alpha}$ with a path loss exponent α and Rayleigh fading with unit mean, $\mathbb{E}(h) = 1$. Then, the coverage probability is given by

$$\begin{aligned} P_{\mathrm{c}}(\theta) &= \mathbb{P}(\mathrm{SIR} > \theta) \\ &= \mathbb{P}(\mathrm{SIR} > \theta, x_0 \in \Phi_{\mathrm{m}}) + \mathbb{P}(\mathrm{SIR} > \theta, x_0 \in \Phi_{\mathrm{p}}), \end{aligned} \tag{2}$$

where θ is the SIR threshold. In Section V, the two-tier model will be extended to a K-tier model.

III. The ASAPPP-based Approach

The approach used in this paper is based on the ASAPPP method [11], which stands for "approximate SIR analysis based on the PPP" and can also be read as "as a PPP", indicating that the network is first treated as if it forms a PPP and then a shift is applied to the SIR distribution. Consequently, the SIR distribution of non-Poisson networks (NPNs) can be accurately approximated by that of a PPP network through scaling the threshold θ with the MISR-based gain G, i.e., $P_{\mathrm{c}}^{\mathrm{NPN}}(\theta) \approx P_{\mathrm{c}}^{\mathrm{PPP}}(\theta/G)$, and the approximation is asymptotically exact as $\theta \to 0$. It is shown in [3] that the MISR-based gain G for a single-tier network Φ is closely related to the corresponding MISR, defined as

$$G \triangleq \frac{\mathrm{MISR}_{\mathrm{PPP}}}{\mathrm{MISR}_{\Phi}}, \tag{3}$$

and the MISR is the mean of interference-to-(average)-signal ratio, defined by

$$\mathrm{MISR} \triangleq \mathbb{E}\left\{\frac{I_\Phi}{\overline{S}}\right\}, \tag{4}$$

where $\overline{S}$ is the desired signal power averaged over the fading and I_Φ represents the sum power of all interferers from the network Φ. The MISR for PPP networks is $\mathrm{MISR}_{\mathrm{PPP}} = 2/(\alpha-2)$, which also holds for the HIP model with an arbitrary number of tiers, densities and transmit powers [3].

The MISR-based gain for HCNs is different for different types of point processes and density and transmit power ratios. We investigate how to extend the ASAPPP method to the general HCNs upon the MISR-based gains of the individual tiers constituting the HCNs. Based on the aforementioned discussions, when a user accesses a BS from a non-Poisson tier, this tier is replaced by a PPP, and the corresponding threshold θ is shifted to θ/G. The interference from the other tier is assumed to be approximated by that from another PPP network with the same density, which is another instance of "as a PPP". Approximating a repulsive point process[2] with a PPP yields an interference power that stochastically dominates the actual interference power. Consequently, the resulting coverage probability is a lower bound to the exact coverage probability, and it turns out to be tight from our numerical results.

IV. Two-Tier Heterogeneous Cellular Networks

A. Main Result

We first focus on two-tier HCNs and assume that the MISR-based gain for the single-tier network Φ_{m} is denoted by G_{m} and the one for Φ_{p} is G_{p}. Let $\delta \triangleq 2/\alpha$, $\varpi = \frac{\lambda_{\mathrm{p}}}{\lambda_{\mathrm{m}}}\left(\frac{\mu_{\mathrm{p}}}{\mu_{\mathrm{m}}}\right)^{\delta}$ and $T(\alpha,\theta) = 1 + \theta^{\delta}\int_{\theta^{-\delta}}^{\infty} \frac{1}{1+t^{\alpha/2}}\mathrm{d}t$, which can be expressed in terms of the Gaussian hypergeometric function ${}_2F_1$ as [12]

$$T(\alpha,\theta) = {}_2F_1(1,-\delta,1-\delta,-\theta). \tag{5}$$

The following theorem gives an accurate approximation and asymptotic bound on the coverage probability.

Theorem 1. *Let*

$$\hat{P}_{\mathrm{c}}(\theta) \triangleq \frac{1}{T\big(\alpha,\frac{\theta}{G_{\mathrm{m}}}\big)+\varpi T(\alpha,\theta)} + \frac{1}{T\big(\alpha,\frac{\theta}{G_{\mathrm{p}}}\big)+\frac{1}{\varpi}T(\alpha,\theta)}. \tag{6}$$

For two-tier HCNs where the typical user is served by the BS with the strongest average received power, the coverage probability $P_{\mathrm{c}}(\theta)$ is approximated by

$$P_{\mathrm{c}}(\theta) \approx \hat{P}_{\mathrm{c}}(\theta). \tag{7}$$

Moreover,

$$P_{\mathrm{c}}(\theta) \gtrsim \hat{P}_{\mathrm{c}}(\theta), \tag{8}$$

where '$\gtrsim$' stands for an asymptotic lower bound, i.e., $\exists t > 0$ s.t. $P_{\mathrm{c}}(\theta) > \hat{P}_{\mathrm{c}}(\theta)\ \forall \theta < t$.

Proof: We first define the nearest-point operator

$$\mathrm{NP}(\Phi) \triangleq \arg\min\{x \in \Phi\colon |x|\} \tag{9}$$

and the reduced point process

$$\Phi^! \triangleq \Phi \setminus \{\mathrm{NP}(\Phi)\}. \tag{10}$$

When the user accesses an MBS, we have $x_0 = \mathrm{NP}(\Phi_{\mathrm{m}})$. Let $\omega = (\mu_{\mathrm{p}}/\mu_{\mathrm{m}})^{1/\alpha}$, $\mathcal{A} = \{|y| \geq \omega|x_0|\}$, and we have

$$\begin{aligned} &\mathbb{P}(\mathrm{SIR} > \theta, x_0 \in \Phi_{\mathrm{m}}) \\ &= \mathbb{E}\left\{\exp\left(-\theta\frac{\sum\limits_{x\in\Phi_{\mathrm{m}}^!}\mu_{\mathrm{m}}\ell(|x|)h_x + \sum\limits_{y\in\Phi_{\mathrm{p}}}\mu_{\mathrm{p}}\ell(|y|)h_y}{\mu_{\mathrm{m}}\ell(|x_0|)}\right)\mathbf{1}_{x_0\in\Phi_{\mathrm{m}}}\right\} \\ &= \mathbb{E}\left\{\prod_{x\in\Phi_{\mathrm{m}}^!}\left(1+\frac{\theta\ell(|x|)}{\ell(|x_0|)}\right)^{-1}\prod_{y\in\Phi_{\mathrm{p}}}\left(1+\frac{\theta\mu_{\mathrm{p}}\ell(|y|)}{\mu_{\mathrm{m}}\ell(|x_0|)}\right)^{-1}\mathbf{1}_{\mathcal{A}}\right\} \end{aligned}$$

[2] A point process whose pair correlation function is at most 1

$$
\begin{aligned}
&\overset{(a)}{\approx}\mathbb{E}\left\{\prod_{x\in\Phi_{\mathrm{m}}^{!\mathrm{PPP}}}\left(1+\frac{\theta\ell(|x|)}{G_{\mathrm{m}}\ell(|x_0|)}\right)^{-1}\prod_{y\in\Phi_{\mathrm{p}}}\left(1+\frac{\theta\mu_{\mathrm{p}}\ell(|y|)}{\mu_{\mathrm{m}}\ell(|x_0|)}\right)^{-1}\mathbf{1}_{\mathcal{A}}\right\}\\
&\overset{(b)}{\gtrsim}\mathbb{E}\left\{\prod_{x\in\Phi_{\mathrm{m}}^{!\mathrm{PPP}}}\left(1+\frac{\theta\ell(|x|)}{G_{\mathrm{m}}\ell(|x_0|)}\right)^{-1}\prod_{y\in\Phi_{\mathrm{p}}^{\mathrm{PPP}}}\left(1+\frac{\theta\mu_{\mathrm{p}}\ell(|y|)}{\mu_{\mathrm{m}}\ell(|x_0|)}\right)^{-1}\mathbf{1}_{\mathcal{A}}\right\}\\
&\overset{(c)}{=}\int_0^{\infty}f_{\mathrm{m}}(r)\exp\left(-\lambda_{\mathrm{m}}\int_r^{\infty}\frac{2\pi t\mathrm{d}t}{1+\frac{G_{\mathrm{m}}t^{\alpha}}{\theta r^{\alpha}}}-\pi\lambda_{\mathrm{p}}\left(r^2\omega^2+\int_{r\omega}^{\infty}\frac{2t\mathrm{d}t}{1+\frac{\mu_{\mathrm{m}}t^{\alpha}}{\theta\mu_{\mathrm{p}}r^{\alpha}}}\right)\right)\mathrm{d}r\\
&=\int_0^{\infty}\exp\left(-rT(\alpha,\theta/G_{\mathrm{m}})-r\overline{\omega}T(\alpha,\theta)\right)\mathrm{d}r\\
&=\frac{1}{T(\alpha,\theta/G_{\mathrm{m}})+\overline{\omega}T(\alpha,\theta)},
\end{aligned}\tag{11}
$$

where $f_{\mathrm{m}}(r)=2\lambda_{\mathrm{m}}\pi re^{-\lambda_{\mathrm{m}}\pi r^2}$ is the distribution of $|\,\mathrm{NP}(\Phi_{\mathrm{m}}^{\mathrm{PPP}})|$. Step (a) uses the ASAPPP approximation of Φ_{m} by shifting θ to θ/G_{m} and replacing Φ_{m} by a PPP. In step (b) the interference from Φ_{p} is upper bounded by that of a PPP, which provides a lower bound for the coverage probability. Since ASAPPP is asymptotically exact and accurate for a large range of θ, the approximation in step (a) is asymptotically exact and step (b) gives an asymptotic lower bound and provides an approximation for the coverage probability. The probability generating functional (PGFL) of the PPP [13] is used in step (c).

When the user accesses a PBS, let $\mathcal{B}=\{|x|\geq|x_0|/\omega\}$, and we have

$$
\begin{aligned}
&\mathbb{P}(\mathrm{SIR}>\theta,x_0\in\Phi_{\mathrm{p}})\\
&\gtrsim\mathbb{E}\left\{\prod_{x\in\Phi_{\mathrm{m}}^{\mathrm{PPP}}}\left(1+\frac{\theta\mu_{\mathrm{m}}\ell(|x|)}{\mu_{\mathrm{p}}\ell(|x_0|)}\right)^{-1}\mathbf{1}_{\mathcal{B}}\prod_{y\in\Phi_{\mathrm{p}}^{!\mathrm{PPP}}}\left(1+\frac{\theta\ell(|y|)}{G_{\mathrm{p}}\ell(|x_0|)}\right)^{-1}\right\}\\
&=\frac{1}{T(\alpha,\theta/G_{\mathrm{p}})+\frac{1}{\overline{\omega}}T(\alpha,\theta)}.
\end{aligned}\tag{12}
$$

By substituting (11) and (12) into (2), we obtain the result. ■

When the ASAPPP-based method is applied to two-tier HIP networks, (7) reduces to $P_{\mathrm{c}}(\theta)\approx1/T(\alpha,\theta)$, which is the exact result for the coverage probability of two-tier HIP networks and consistent with Eq. (16) in [4]. In the following, we divide this highly general model into two types, where for the first one, one tier is a non-Poisson (NP) network and the other is a PPP network; while for the second one, both tiers are NP networks.

B. NP/PPP Deployment

In this subsection, we consider two kinds of NP point processes, namely, the β-GPP and the lattice model.

1) Special Case: β-GPP/PPP: The locations of the MBSs Φ_{m} are modeled by a β-GPP, and the locations of the PBSs Φ_{p} are modeled by a PPP. Through simulations, we find that the MISR-based gain of the β-GPP is quite exactly $G\approx1+\beta/2$, irrespective of α, as can be seen in Figure 1. Therefore, the coverage probability of the user served by a β-GPP network is approximately the same as that of a user served by a Poisson

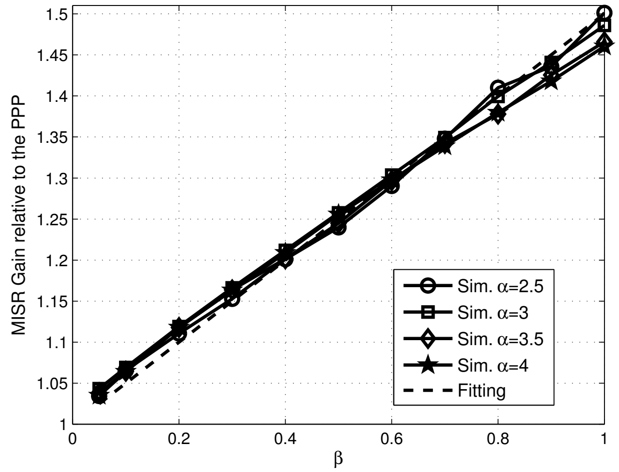

Fig. 1. The MISR-based gain of β-GPP for different α.

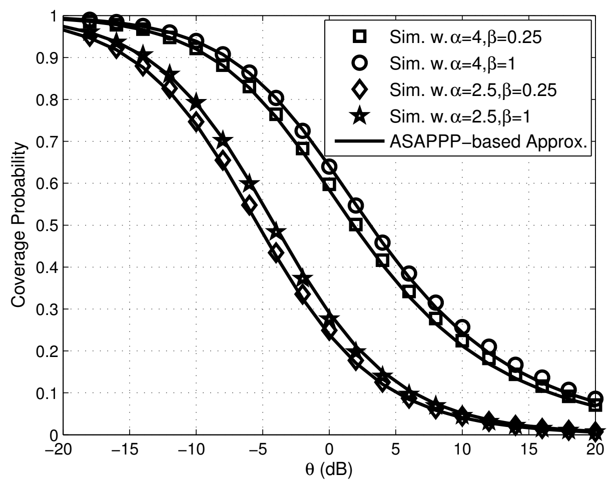

Fig. 2. The ASAPPP-based approximation of β-GPP network for different α and β.

network and scaling the SIR threshold θ to θ/G, which is verified in Figure 2.

Figure 3 and 4 show the coverage probability of the heterogeneous networks with different α and β when $\lambda_{\mathrm{m}}=\lambda_{\mathrm{p}}=10^{-5}$, $\mu_{\mathrm{m}}=\mu_{\mathrm{p}}=1$. It is apparent that the approximation is excellent over a wide range of θ in different cases, which validates the effectiveness of the proposed ASAPPP-based method. The tiny gap between each simulation and its corresponding approximation can be attributed to the approximation of the interference from the NP tier by that of a PPP, which yields the lower bound to the coverage probability.

2) Special Case: Square lattice/PPP: The locations of the MBSs Φ_{m} are modeled by a square lattice point process[3] and the locations of the PBSs Φ_{p} are modeled by a PPP. From [3], the MISR of the square lattice is quite exactly half of

[3]This is a randomly translated square lattice and thus a stationary point process.

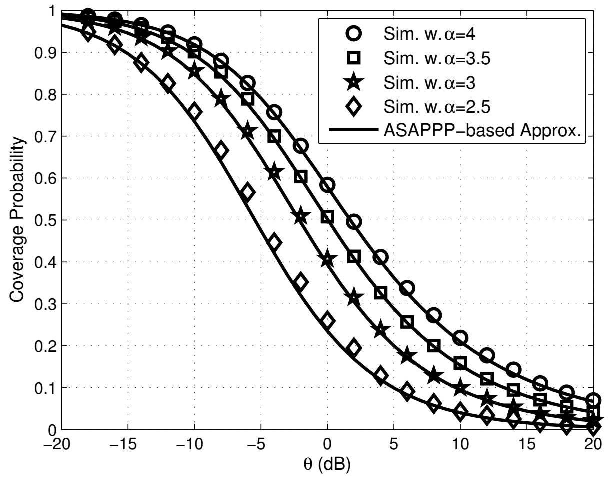

Fig. 3. The ASAPPP-based approximation of β-GPP/PPP networks for different α with $\beta = 1$.

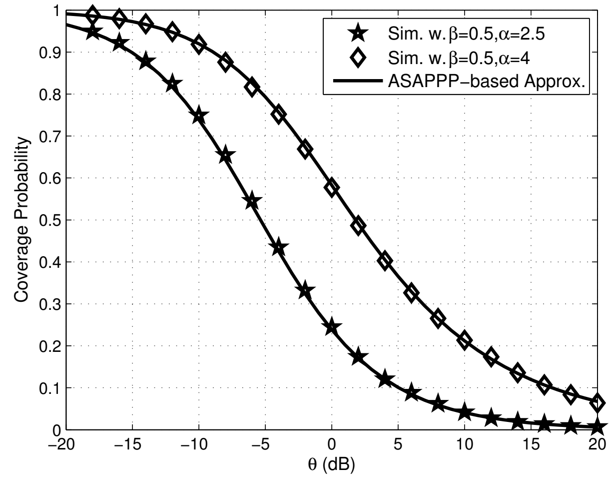

Fig. 4. The ASAPPP-based approximation of β-GPP/PPP networks for $\beta = 0.5$.

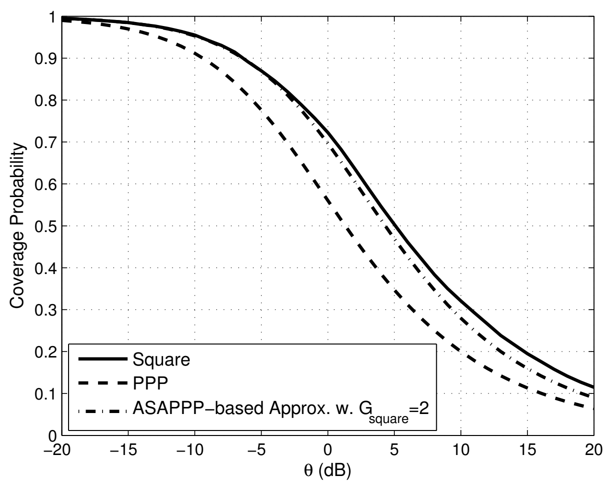

Fig. 5. The ASAPPP-based approximation of the square lattice network with $\alpha = 4$.

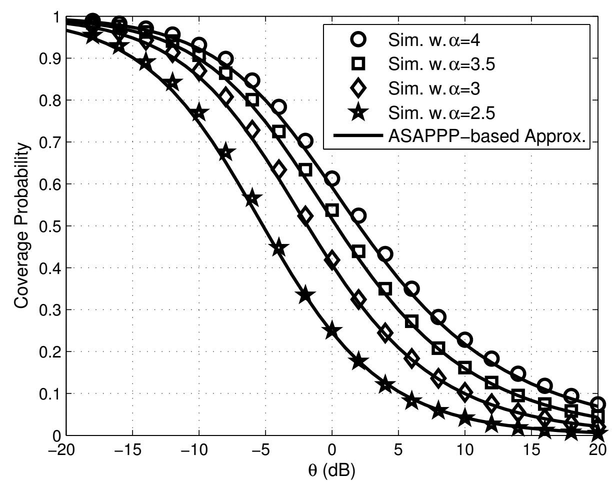

Fig. 6. The ASAPPP-based approximation of the square lattice/PPP networks for different α.

that of the PPP, irrespective of the path loss exponent, i.e., $G_{\rm square} \approx 2$. Figure 5 gives the ASAPPP approximation for the single-tier square lattice networks and the approximation is tight for coverage probabilities above 70% and becomes less but acceptably accurate as θ increases. Figure 6 shows the coverage probability with different α when $\lambda_{\rm m} = \lambda_{\rm p} = 10^{-5}$, $\mu_{\rm m} = \mu_{\rm p} = 1$, which further corroborates the effectiveness of the ASAPPP-based method. We can see the gap between the simulation and its corresponding approximation is bigger than in the β-GPP/PPP case. It can be explained as follows: the less accurate ASAPPP-based approximation for the square lattice deployment leads to a less accurate approximation in this kind of HCNs.

C. NP/NP Deployment

In this subsection, we again consider two types of HCNs: one is composed of two independent β-GPP deployments, and the other consists of a square lattice and a β-GPP deployment.

1) Special Case: Two independent β-GPPs: The locations of the MBSs $\Phi_{\rm m}$ and the PBSs $\Phi_{\rm p}$ are modeled by two independent β-GPPs. Figure 7 shows the coverage probability with different α when $\lambda_{\rm m} = \lambda_{\rm p} = 10^{-5}$, $\mu_{\rm m} = \mu_{\rm p} = 1$, $\beta = 1$, which again demonstrates the accuracy of the ASAPPP-based approximation. We also see from (6) and (7) that the coverage performance for the two-tier independent GPP networks is the worst with $\overline{\omega} = 1$ (while better than that of PPP networks) because in this case the independence between the two tiers reduces the regularity property of a single GPP the most. Conversely, as $\overline{\omega}$ tends to zero or infinity, these HCNs tend to single-tier GPP networks, since of the two tiers dominates.

2) Special Case: Square lattice/β-GPP: The locations of the MBSs $\Phi_{\rm m}$ are modeled by a square lattice point process and the locations of the PBSs $\Phi_{\rm p}$ are modeled by a β-GPP. Figure 8 gives the coverage probability for different α when

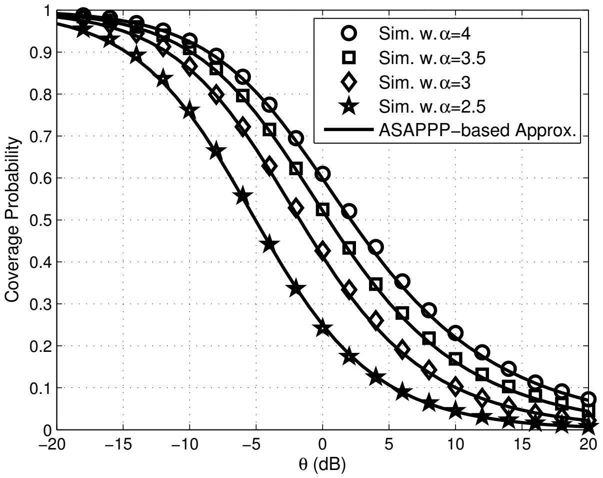

Fig. 7. The ASAPPP-based approximation of the two-tier GPP networks for different α.

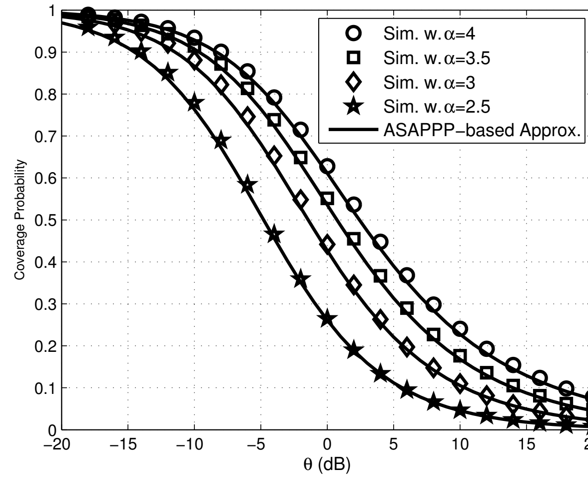

Fig. 8. The ASAPPP-based approximation of square lattice/GPP networks for different α.

$\lambda_{\rm m}=\lambda_{\rm p}=10^{-5}$, $\mu_{\rm m}=\mu_{\rm p}=1$ and $\beta=1$. We can see that similar to the case of square lattice/PPP, the ASAPPP-based approximations are tight when θ tends to zero and become less accurate as θ increases. The reason is the same, i.e., the less accurate ASAPPP-based approximation for the square lattice deployment leads to the less accurate approximation in the HCNs.

V. K-TIER HETEROGENEOUS CELLULAR NETWORKS

Based on the ASAPPP method for two-tier HCNs, we now extend the above analysis to general K-tier heterogeneous networks where $\Phi_k, k=1,2,\ldots,K$ are the locations of the BSs in the k-th tier and G_k is the corresponding MISR-based gain. Let μ_k and λ_k be the transmit power and node density of the k-th tier, respectively.

Theorem 2. *Let*

$$\hat{P}_{\rm c}(\theta)\triangleq\sum_{k\in[K]}\frac{1}{T(\alpha,\theta/G_k)+\sum\limits_{i\in[K]^!}\frac{\lambda_i}{\lambda_k}(\frac{\mu_i}{\mu_k})^{\delta}T(\alpha,\theta)}. \tag{13}$$

For K-tier HCNs where the typical user is served by the BS with the strongest average received power, the coverage probability $P_{\rm c}(\theta)$ is approximated by

$$P_{\rm c}(\theta)\approx\hat{P}_{\rm c}(\theta). \tag{14}$$

Moreover,

$$P_{\rm c}(\theta)\gtrsim\hat{P}_{\rm c}(\theta). \tag{15}$$

Proof: When a user is served by a BS in the k-th tier, we have $x_0=\mathrm{NP}(\Phi_k)$. Let $\mathcal{A}_{i,k}=\{\mu_i\ell(|x|)\le\mu_k\ell(|x_0|)\}$ and we have

$$\begin{aligned}
&\mathbb{P}(\mathrm{SIR}>\theta,x_0\in\Phi_k)\\
&\gtrsim\mathbb{E}\left\{\bigcap_{i\in[K]^!}\prod_{x\in\Phi_i^{\rm PPP}}\left(1+\frac{\theta\mu_i\ell(|x|)}{\mu_k\ell(|x_0|)}\right)^{-1}\mathbf{1}_{\mathcal{A}_{i,k}}\prod_{y\in\Phi_k^{!\rm PPP}}\left(1+\frac{\theta\ell(|y|)}{G_k\ell(|x_0|)}\right)^{-1}\right\}\\
&=\int_0^\infty 2\pi\lambda_k r\exp\left(-\pi\lambda_k r^2-2\pi\lambda_k\int_r^\infty\frac{t{\rm d}t}{1+\frac{G_k t^\alpha}{\theta r^\alpha}}\right.\\
&\qquad\left.-\sum_{i\in[K]^!}\pi\lambda_i\left(r^2\left(\frac{\mu_i}{\mu_k}\right)^{\delta}+\int_{r(\frac{\mu_i}{\mu_k})^{\frac{1}{\alpha}}}^\infty\frac{2t{\rm d}t}{1+\frac{\mu_k t^\alpha}{\theta\mu_i r^\alpha}}\right)\right){\rm d}r\\
&=\int_0^\infty 2\pi\lambda_k r\exp\left(\pi\lambda_k r^2T(\alpha,\theta/G_k)\right.\\
&\qquad\left.-\sum_{i\in[K]^!}\pi\lambda_i r^2\left(\frac{\mu_i}{\mu_k}\right)^{\delta}T(\alpha,\theta)\right){\rm d}r\\
&=\int_0^\infty\exp\Big(-rT(\alpha,\theta/G_k)-\sum_{i\in[K]^!}\rho_{i,k}rT(\alpha,\theta)\Big){\rm d}r\\
&=\frac{1}{T(\alpha,\theta/G_k)+\sum\limits_{i\in[K]^!}\rho_{i,k}T(\alpha,\theta)},
\end{aligned} \tag{16}$$

where $\rho_{i,k}=\frac{\lambda_i}{\lambda_k}(\frac{\mu_i}{\mu_k})^{\delta}$, $[K]=\{1,2,\ldots,K\}$ and $[K]^!=[K]\setminus\{k\}$. The result follows by summing over $[K]$. ∎

As for the two-tier networks, when the ASAPPP-based method is applied to K-tier HIP networks, (14) reduces to the exact result for K-tier HIP networks, i.e., $P_{\rm c}(\theta)=1/T(\alpha,\theta)$. We give an example of K-tier heterogeneous networks with $K=3$ based on the β-GPP. Figure 9 shows the coverage probability for three-tier HCNs with $\beta_1=1,\beta_2=0.5$ and $\beta_3=0$, i.e., GPP, 0.5-GPP and PPP for different α. It is shown that ASAPPP can also approximate the simulation results in the three-tier HCNs case, which demonstrates the effectiveness of the proposed method for K-tier heterogeneous networks.

VI. CONCLUSIONS

In this paper, we provided a simple approximative approach to the SIR analysis in general HCNs based on the MISR-based gain for each individual tier. We first established the ASAPPP-based approximation for general two-tier HCNs and then extended it to K-tier HCNs. The results indicate that

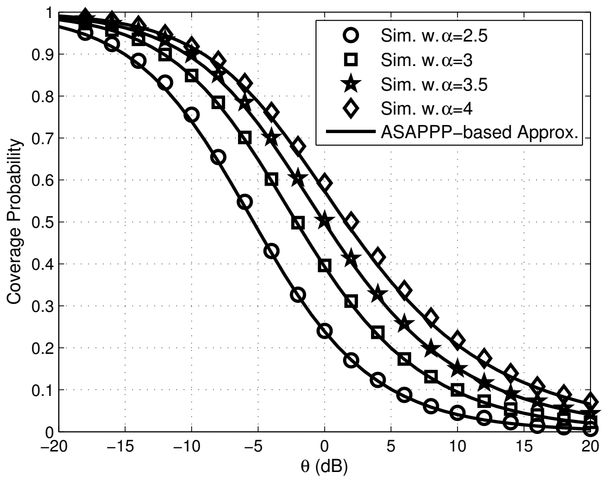

Fig. 9. The coverage probability of three-tier networks for different α with $\lambda_1 = 10^{-5}$, $\mu_1 = 1$, $\lambda_2 = 2\lambda_1$, $\mu_2 = \mu_1/5$, $\lambda_3 = 5\lambda_1$ and $\mu_3 = \mu_1/25$.

the approximations are tight lower bounds to the coverage probability over a wide range of SIR thresholds, thus providing useful approximations for practical network models where an exact calculation of the SIR distribution is unfeasible.

ACKNOWLEDGMENT

The work of H. Wei, N. Deng and W. Zhou has been supported by National programs for High Technology Research and Development (2014AA01A707) and the work of M. Haenggi has been supported by the U.S. NSF (grant CCF 1216407).

REFERENCES

[1] A. Ghosh, N. Mangalvedhe, R. Ratasuk, B. Mondal, M. Cudak, E. Visotsky, T. Thomas, J. Andrews, P. Xia, H. Jo, H. Dhillon, and T. Novlan, "Heterogeneous cellular networks: From theory to practice," *Communications Magazine, IEEE*, vol. 50, no. 6, pp. 54–64, June 2012.

[2] H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, "Modeling and analysis of K-tier downlink heterogeneous cellular networks," *Selected Areas in Communications, IEEE Journal on*, vol. 30, no. 3, pp. 550–560, 2012.

[3] M. Haenggi, "The mean interference-to-signal ratio and its key role in cellular and amorphous networks," *Wireless Communications Letters, IEEE*, vol. 3, no. 6, pp. 597–600, Dec 2014.

[4] G. Nigam, P. Minero, and M. Haenggi, "Coordinated multipoint joint transmission in heterogeneous networks," *Communications, IEEE Transactions on*, vol. 62, no. 11, pp. 4134–4146, Nov 2014.

[5] S. Mukherjee, "Distribution of downlink SINR in heterogeneous cellular networks," *IEEE Journal on Selected Areas in Communications*, vol. 30, no. 3, pp. 575–585, 2012.

[6] N. Deng, W. Zhou, and M. Haenggi, "Heterogeneous cellular network models with dependence," *IEEE Journal on Selected Areas in Communications*, 2015, Accepted. Available at http://www3.nd.edu/~mhaenggi/pubs/jsac15b.pdf.

[7] I. Nakata and N. Miyoshi, "Spatial stochastic models for analysis of heterogeneous cellular networks with repulsively deployed base stations," *Performance Evaluation*, vol. 78, pp. 7–17, Aug 2014. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0166531614000546

[8] V. Suryaprakash, J. Moller, and G. Fettweis, "On the modeling and analysis of heterogeneous radio access networks using a Poisson cluster process," *Wireless Communications, IEEE Transactions on*, vol. 14, no. 2, pp. 1035–1047, Feb 2015.

[9] H. Nagamatsu, N. Miyoshi, and T. Shirai, "Padé approximation for coverage probability in cellular networks," in *Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt), 2014 12th International Symposium on*, May 2014, pp. 693–700.

[10] A. Guo and M. Haenggi, "Asymptotic deployment gain: A simple approach to characterize the SINR distribution in general cellular networks," *Communications, IEEE Transactions on*, vol. 63, no. 3, pp. 962–976, March 2015.

[11] M. Haenggi, "ASAPPP: A simple approximative analysis framework for heterogeneous cellular networks," Dec 2014, Keynote presentation at the 2014 Workshop on Heterogeneous and Small Cell Networks (HetSNets'14) Available at http://www3.nd.edu/~mhaenggi/talks/hetsnets14.pdf.

[12] X. Zhang and M. Haenggi, "A stochastic geometry analysis of inter-cell interference coordination and intra-cell diversity," *Wireless Communications, IEEE Transactions on*, vol. 13, no. 12, pp. 6655–6669, Dec 2014.

[13] M. Haenggi, *Stochastic geometry for wireless networks*. Cambridge University Press, 2012.