Bethe and M-Bethe Permanent Inequalities

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***Abstract*—In [1], it was conjectured that the permanent of a $\mathbf{P}$-lifting $\theta^{\uparrow \mathbf{P}}$ of a matrix θ of degree M is less than or equal to the Mth power of the permanent $\mathrm{perm}(\theta)$, i.e., $\mathrm{perm}(\theta^{\uparrow \mathbf{P}}) \leqslant \mathrm{perm}(\theta)^M$ and, consequently, that the degree-M Bethe permanent $\mathrm{perm}_{M,\mathrm{B}}(\theta)$ of a matrix θ is less than or equal to the permanent $\mathrm{perm}(\theta)$ of θ, i.e., $\mathrm{perm}_{M,\mathrm{B}}(\theta) \leqslant \mathrm{perm}(\theta)$. In this paper, we prove these related conjectures and show some properties of the permanent of block matrices that are lifts of a matrix. As a corollary, we obtain an alternative proof of the inequality $\mathrm{perm}_{\mathrm{B}}(\theta) \leqslant \mathrm{perm}(\theta)$ on the Bethe permanent of the base matrix θ, which, in contrast to the one given in [2], uses only the combinatorial definition of the Bethe-permanent.**

The results have implications in coding theory. Since a $\mathbf{P}$-lifting corresponds to an M-graph cover and thus to a protograph-based LDPC code, the results may help explain the performance of these codes.

I. Introduction

A. Background

The concept of the *Bethe permanent* was introduced in [3], [4] to denote the approximation of a permanent of a non-negative matrix[1] by solving a certain minimization problem of the Bethe free energy with the sum-product algorithm. In his paper [1], Vontobel uses the term *Bethe permanent* to denote this approximation and provides reasons for which the approximation works well by showing that the Bethe free energy is a convex function and that the sum-product algorithm finds its minimum efficiently.[2]

In the recent paper [2], Gurvits shows that the permanent of a matrix is lower bounded by its Bethe permanent, i.e., $\mathrm{perm}_{\mathrm{B}}(\theta) \leqslant \mathrm{perm}(\theta)$, and discusses conjectures on the constant C in the inequality $\mathrm{perm}(\theta) \leqslant C \cdot \mathrm{perm}_{\mathrm{B}}(\theta)$. Related to the results of Gurvits, Vontobel [1] formulates a conjecture that the permanent of an M-lift $\theta^{\uparrow \mathbf{P}}$ of a matrix θ is less than or equal to the Mth power of the permanent $\mathrm{perm}(\theta)$, i.e., $\mathrm{perm}(\theta^{\uparrow \mathbf{P}}) \leqslant \mathrm{perm}(\theta)^M$ and, consequently, that the degree M-Bethe permanent $\mathrm{perm}_{M,\mathrm{B}}(\theta)$ of a matrix θ is less than or equal to the permanent $\mathrm{perm}(\theta)$ of θ, i.e., $\mathrm{perm}_{M,\mathrm{B}}(\theta) \leqslant \mathrm{perm}(\theta)$. A proof of his conjecture would imply an alternative proof of the inequality $\mathrm{perm}_{\mathrm{B}}(\theta) \leqslant \mathrm{perm}(\theta)$ that uses only the combinatorial definition of the Bethe-permanent.[3]

In this paper, we prove this conjecture and explore some properties of the permanent of block matrices that are lifts of a matrix; these matrices are the matrices of interest when studying the degree-M Bethe permanent. Additional examples and explanations of the techniques used can be found in [8].

B. Related work

The literature on permanents and on adjacent areas (of counting perfect matchings, counting 0-1 matrices with specified row and column sums, etc.) is vast. Apart from the previously mentioned papers, the most relevant papers to our work are the one by Chertkov & Yedidia [4] that studies the so-called fractional free energy functionals and resulting lower and upper bounds on the permanent of a non-negative matrix, the papers [9] (on counting perfect matchings in random graph covers), [10] (on counting matchings in graphs with the help of the sum-product algorithm)[4], and [3], [11], [12] (on max-product/min-sum algorithms based approaches to the maximum weight perfect matching problem). Relevant is also the work on approximating the permanent of a non-negative matrix using Markov-chain-Monte-Carlo-based methods [13], or fully polynomial-time randomized approximation schemes [14] or Bethe-approximation based methods or sum-product-algorithm (SPA) based method [3], [15].[5]

C. Notation and definitions

A non-negative matrix is here a matrix with non-negative real entries. Rows and columns of matrices and entries of vectors will be indexed starting at 1. For a positive integer M, we will use the common notation $[M] \triangleq \{1, \ldots, M\}$. We will also use the common notation h_{ij} or $\mathbf{H}_{ij}$ to denote the (i, j)th entry of a matrix $\mathbf{H}$ when there is no ambiguity in the indices and $h_{i,j}$ or $\mathbf{H}_{i,j}$, respectively, when one of the two indices is not a simple digit, e.g., $h_{i,m-1}$, $\mathbf{H}_{i,m-1}$, respectively. $|\alpha|$ denotes the cardinality (number of elements) of the set α. For positive integers m, M, the set of all permutations on the set $[m]$ is denoted by $\mathcal{S}_m$, while the set of all $M \times M$ permutation matrices is denoted by $\mathcal{P}_M$. In addition, $\mathcal{M}_m(\mathcal{P}_M)$ will be the

[1] A non-negative matrix contains only non-negative real entries.

[2] Although its definition looks simpler than that of the determinant, the permanent does not have the properties of the determinant that enable efficient computation [5]. In terms of complexity classes, the computation of the permanent is in the complexity class ♯P [6], where ♯P is the set of the counting problems associated with the decision problems in the class NP. Even the computation of the permanent of 0-1 matrices restricted to have only three ones per row is ♯P-complete [7].

[3] The formal definition of the Bethe and M-Bethe permanents is given in Definition 1.

[4] Computing the permanent is related to counting perfect matchings.

[5] See [1] for a more detailed account of these and other related papers.

set of all $m \times m$ block matrices with entries in $\mathcal{P}_M$, i.e., the entries are permutation matrices of size $M \times M$:

$$\mathcal{M}_m(\mathcal{P}_M) \triangleq \{\mathbf{P} = (P_{ij}) \mid P_{ij} \in \mathcal{P}_M, \forall i, j \in [m]\}.$$

Finally, the permanent of an $m \times m$-matrix with real entries is defined to be

$$\operatorname{perm}(\theta) \triangleq \sum_{\sigma \in \mathcal{S}_m} \prod_{i \in [m]} \theta_{i\sigma(i)} \ .$$

Note that in contrast, the determinant of θ is

$$\det(\theta) \triangleq \sum_{\sigma \in \mathcal{S}_m} \operatorname{sgn}(\sigma) \prod_{i \in [m]} \theta_{i\sigma(i)} \ ,$$

where $\operatorname{sgn}(\sigma)$ is the signature operator.

Definition 1. *Let m, M be two positive integers and θ be a non-negative $m \times m$ matrix.*

- *For a matrix $\mathbf{P} \in \mathcal{M}_m(\mathcal{P}_M)$, the $\mathbf{P}$-lifting $\theta^{\uparrow \mathbf{P}}$ of θ of degree M is defined as*

$$\theta^{\uparrow \mathbf{P}} \triangleq \begin{bmatrix} \theta_{11}P_{11} & \dots & \theta_{1m}P_{1m} \\ \vdots & & \vdots \\ \theta_{m1}P_{m1} & \dots & \theta_{mm}P_{mm} \end{bmatrix},$$

  i.e., as an $m \times m$ block matrix with its (i, j)-th entry equal to the matrix $\theta_{ij}P_{ij}$, where P_{ij} is an $M \times M$ permutation matrix in $\mathcal{P}_M$. (It results in an $mM \times mM$ matrix.)
- *The degree-M Bethe permanent of θ is defined as*

$$\operatorname{perm}_{\mathrm{B},M}(\theta) \triangleq \left(\left\langle \operatorname{perm}(\theta^{\uparrow \mathbf{P}}) \right\rangle_{\mathbf{P} \in \mathcal{M}_m(\mathcal{P}_M)} \right)^{1/M},$$

  where the angular brackets represent the arithmetic average of $\operatorname{perm}(\theta^{\uparrow \mathbf{P}})$ over all $\mathbf{P} \in \mathcal{M}_m(\mathcal{P}_M)$.
- *The Bethe permanent of θ is then defined as*

$$\operatorname{perm}_{\mathrm{B}}(\theta) \triangleq \limsup_{M \to \infty} \operatorname{perm}_{\mathrm{B},M}(\theta).$$

Since the permanent operator is invariant to the elementary operations of interchanging rows or columns, when taking the permanent, we can assume, without loss of generality, that matrices $\mathbf{P} \in \mathcal{M}_m(\mathcal{P}_M)$ have $P_{1j} = P_{i1} = I_M$, for all $i, j \in [m]$, where I_M is the identity matrix of size $M \times M$. We call such matrices *reduced*.

Definition 2. *A matrix $\mathbf{P} = (P_{ij}) \in \mathcal{M}_m(\mathcal{P}_M)$ is* reduced *if $P_{1j} = P_{i1} = I_M$, for all $i, j \in [m]$.*

Remark 1. Note that a $\mathbf{P}$-lifting of a matrix θ corresponds to an M-graph cover of the protograph (base graph) described by θ. Therefore we can consider $\theta^{\uparrow \mathbf{P}}$ to represent a protograph-based LDPC code and θ to be its protomatrix (also called its base matrix or its mother matrix) [16]. □

II. The Permanent of a Matrix Lift

In [1], it was conjectured that for any non-negative square matrix θ and for any $\mathbf{P} \in \mathcal{M}_m(\mathcal{P}_M)$, we have the inequality

$$\operatorname{perm}(\theta^{\uparrow \mathbf{P}}) \leqslant \operatorname{perm}(\theta)^M.$$

In this section we prove this conjecture and several related results on the structure of the $\operatorname{perm}(\theta^{\uparrow \mathbf{P}})$ of the lift $\theta^{\uparrow \mathbf{P}}$ of the matrix θ, for any non-negative matrix θ.

A. Rewriting the permanent products of lifts of matrices

In this subsection, we present an algorithm that lets us rewrite the permanent-products of a $\mathbf{P}$-lifting of θ into a form useful for proving the conjecture.

Let $\theta = (\theta_{ij})$ be a non-negative matrix of size $m \times m$ and let $\mathbf{P} = (P_{ij}) \in \mathcal{M}_m(\mathcal{P}_M)$. Let $\tau \in \mathcal{S}_{mM}$ be a permutation on the set $[mM]$ and let

$$A_\tau \triangleq \prod_{i \in [mM]} (\theta^{\uparrow \mathbf{P}})_{i\tau(i)}$$

be a non-zero permanent-product of $\theta^{\uparrow \mathbf{P}}$, which is a non-zero term of

$$\operatorname{perm}(\theta^{\uparrow \mathbf{P}}) = \sum_{\tau \in \mathcal{S}_{mM}} \prod_{i \in [mM]} (\theta^{\uparrow \mathbf{P}})_{i\tau(i)}.$$

We first observe that, since A_τ is assumed to be non-zero, for each $i \in [mM]$, there exists $j, l \in [m]$ such that $(\theta^{\uparrow \mathbf{P}})_{i\tau(i)} = \theta_{jl}$. Indeed, let $i \in [mM]$, then $i \in \mathcal{J}$ and $\tau(i) \in \mathcal{L}$, for some $j, l \in [m]$, where $\mathcal{I} \triangleq \{(j-1)M + 1, \dots, jM\}$ and $\mathcal{L} \triangleq \{(l-1)M + 1, \dots, lM\}$. Therefore $(\theta^{\uparrow \mathbf{P}})_{i\tau(i)}$ is a non-zero entry in the matrix-entry $\theta_{jl}P_{jl}$ of $\theta^{\uparrow \mathbf{P}}$. Since all its nonzero entries of $\theta_{jl}P_{jl}$ are equal to θ_{jl}, we obtain that $(\theta^{\uparrow \mathbf{P}})_{i\tau(i)} = \theta_{jl}$. Therefore, the product $A_\tau \triangleq \prod_{i \in [mM]} (\theta^{\uparrow \mathbf{P}})_{i\tau(i)}$ can be rewritten as a product of entries θ_{jl} of the matrix θ, $j, l \in [m]$. Let

$$\alpha_{jl}^\tau \triangleq \{i \in \mathcal{J} \mid \tau(i) \in \mathcal{L}\}, \tag{1}$$

$$r_{jl}^\tau \triangleq |\alpha_{jl}^\tau|. \tag{2}$$

Then, $(\theta^{\uparrow \mathbf{P}})_{i\tau(i)} = \theta_{jl}$, for all $i \in \alpha_{jl}$ and for all $j, l \in [m]$, therefore

$$\prod_{i \in \mathcal{J}} (\theta^{\uparrow \mathbf{P}})_{i\tau(i)} = \theta_{j1}^{r_{j1}} \theta_{j2}^{r_{j2}} \cdots \theta_{jm}^{r_{jm}} = \prod_{l=1}^m \theta_{jl}^{r_{jl}}, \quad \forall j \in [m].$$

Since each row and each column of $\theta^{\uparrow \mathbf{P}}$ must contribute to the product exactly once, the matrix $\alpha_\tau \triangleq (\alpha_{jl}^\tau)_{j,l}$ with the set α_{jl}^τ as its entry (j, l) satisfies

$$\alpha_{jl}^\tau \bigcap \alpha_{jl'}^\tau = \emptyset, \forall j, l, l' \in [m], l \neq l', \quad \bigcup_{l=1}^m \alpha_{jl}^\tau = \mathcal{J}, \tag{3}$$

from which it follows that $0 \leqslant r_{jl}^\tau \leqslant M, \quad \forall j, l \in [m]$ and

$$\sum_{l=1}^m r_{jl}^\tau = M, \quad \forall j \in [m],$$

$$\sum_{j=1}^m r_{jl}^\tau = M, \quad \forall l \in [m]. \tag{4}$$

Therefore, the matrix $R_\tau \triangleq (r_{ij}^\tau)_{i,j \in [m]}$ corresponding to A_τ has positive entries and all row and column sums equal M. It will henceforth be referred to as the *exponent matrix*.

For each $\sigma \in \mathcal{S}_m$, let $P_\sigma \in \mathcal{P}_m$ be the $m \times m$ permutation matrix corresponding to σ and let $t_{\tau\sigma} \triangleq \min\{r_{1\sigma(1)}^\tau, r_{2\sigma(2)}^\tau, \dots, r_{m\sigma(m)}^\tau\} \geqslant 0$. Then $R_\tau - t_{\tau\sigma}P_\sigma$ is a positive matrix with the sums of all entries on each row and each column equal to $M - t_{\tau\sigma}$ and with all its entries equal to the ones on the same positions of R_τ except for the entries corresponding to the permutation σ, which decreased by the same amount $t_{\tau\sigma}$. We can index the set $\{\sigma \in \mathcal{S}_m\} \triangleq \{\sigma_k \in \mathcal{S}_m, k \in [m!]\}$ and compute sequentially

$$R_{\tau,1} \triangleq R_\tau$$

$$R_{\tau,k+1} \triangleq R_{\tau,k} - t_{\tau\sigma_k} P_{\sigma_k} = R_\tau - \sum_{s=1}^{k} t_{\tau\sigma_s} P_{\sigma_s}, k \geqslant 2,$$

where the sums of all entries on each row and each column of $R_{\tau,k+1}$ are all equal to $M - \sum_{s=1}^{k} t_{\tau\sigma_s}$. The algorithm runs until all non-zero entries get changed into zero entries, see Example 1 for an illustration of this process. Consequently, the matrix $R - \sum_{\sigma\in\mathcal{S}_m} t_{\tau\sigma} P_\sigma = 0$. This yields $R = \sum_{\sigma\in\mathcal{S}_m} t_{\tau\sigma} P_\sigma$, leading to $A_\tau = \prod_{i\in[mM]} (\theta^{\uparrow\mathbf{P}})_{i\tau(i)} = \prod_{j=1}^{m}\prod_{l=1}^{m} (\theta_{jl})^{r_{jl}}$

$$= \prod_{\sigma\in\mathcal{S}_m} \left(\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)}\right)^{t_{\tau\sigma}}$$

and $\sum_{\sigma\in\mathcal{S}_m} t_{\tau\sigma} = M$.

This algorithm always terminates, which follows from the Birkhoff-von Neumann theorem on the decomposition of doubly stochastic matrices[6] into a convex combination of permutation matrices[7]. Hence the doubly stochastic matrix $\frac{1}{M}R_\tau$ can be written as a convex sum of permutation matrices.

We will refer to this algorithm of rewriting any permanent-product in $\operatorname{perm}(\theta^{\uparrow\mathbf{P}})$ as a product of powers of permanent-products in θ as the *decomposition algorithm*, and the decomposition is called the *standard decomposition*.

Example 1. *Let $M = 7$ and $\theta \triangleq \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Suppose that $A_\tau \triangleq a^3b^2c^2e^3f^4g^4h^2i$ is a product in $\operatorname{perm}(\theta^{\uparrow\mathbf{P}})$. Then, this product corresponds to the following exponent matrix R_τ and the corresponding $\theta^{R_\tau} \triangleq (\theta_{ij}^{r^\tau_{ij}})$:*

$$R_\tau \triangleq \begin{bmatrix} 3 & 2 & 2 \\ 0 & 3 & 4 \\ 4 & 2 & 1 \end{bmatrix}, \quad \theta^{R_\tau} = \begin{bmatrix} a^3 & b^2 & c^2 \\ d^0 & e^3 & f^4 \\ g^4 & h^2 & i^1 \end{bmatrix}$$

Following the algorithm we obtain

$$R_\tau = \begin{bmatrix} \mathbf{3} & 2 & 2 \\ 0 & \mathbf{3} & 4 \\ 4 & 2 & \boxed{\mathbf{1}} \end{bmatrix} \to (aei) \to \begin{bmatrix} 2 & \boxed{\mathbf{2}} & 2 \\ 0 & 2 & \mathbf{4} \\ \mathbf{4} & 2 & 0 \end{bmatrix} \to (bfg)^2$$

$$\to \begin{bmatrix} 2 & 0 & \boxed{\mathbf{2}} \\ 0 & \mathbf{2} & 2 \\ \mathbf{2} & 2 & 0 \end{bmatrix} \to (ceg)^2 \to \begin{bmatrix} \boxed{\mathbf{2}} & 0 & 0 \\ 0 & 0 & \mathbf{2} \\ 0 & \mathbf{2} & 0 \end{bmatrix} \to (afh)^2.$$

So $a^3b^2c^2e^3f^4g^4h^2i = (aei)(bfg)^2(ceg)^2(afh)^2$. It can be easily seen that this factorization is unique (which is not always the case though).

B. Grouping entries in the permanent product

The rewriting algorithm presented in Section II-A provides a way to rewrite the product $\prod_{j=1}^{m}\prod_{l=1}^{m} (\theta_{jl})^{r_{jl}}$ as a product $\prod_{\sigma\in\mathcal{S}_m} \left(\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)}\right)^{t_{\tau\sigma}}$ but does not tell us exactly how to combine the entries $(\theta^{\uparrow\mathbf{P}})_{i\tau(i)}$ to obtain this rewriting. Is there a way to algorithmically combine the indices of the sets α^τ_{jl} to form the products $\left(\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)}\right)^{t_{\tau\sigma}}$ for all $\sigma \in \mathcal{S}_m$? The answer is yes, as we explain in the next example of a concrete $\mathbf{P}$-lifting of θ from Example 1 with $\mathbf{P}$ reduced.

Before presenting it, let us introduce a new matrix $\overline{\alpha}_\tau \triangleq (\overline{\alpha}^\tau_{jl})$ obtained from α_τ by substituting each index $(j-1)M+k$ in an entry set by k, $k \in [M]$. Then, the properties (3) of the matrix α_τ carry over to the following properties of the matrix $\overline{\alpha}_\tau$:

$$\overline{\alpha}^\tau_{jl} \bigcap \overline{\alpha}^\tau_{jl'} = \emptyset, \forall j, l, l' \in [m], l \neq l', \quad \bigcup_{l=1}^{m} \overline{\alpha}^\tau_{jl} = [M]. \quad (5)$$

The following example uses the matrix $\overline{\alpha}$ and provides a unique method of combining the indices $\overline{\alpha}^\tau_{jl}$ to obtain the desired rewriting of the product A_τ. This method follows the steps of the algorithm illustrated in Example 1 for modifying the matrix R_τ.

Example 2. *Let θ be the 3×3 matrix in Example 1, $\mathbf{P} = (P_{ij}) \in \mathcal{P}_3^3$, $\theta^{\uparrow\mathbf{P}}$ and $A_\tau = a^2bdf^2h^2i$ as follows:*

$$\mathbf{P} \triangleq \begin{bmatrix} I_3 & I_3 & I_3 \\ I_3 & Q & Q^2 \\ I_3 & I_3 & Q^2 \end{bmatrix}, Q \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, Q^2 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\theta^{\uparrow\mathbf{P}} \triangleq \begin{bmatrix} \boxed{a_1} & 0 & 0 & b_1 & 0 & 0 & c_1 & 0 & 0 \\ 0 & a_2 & 0 & 0 & \textcircled{b_2} & 0 & 0 & c_2 & 0 \\ 0 & 0 & \colorbox{pink}{a_3} & 0 & 0 & b_3 & 0 & 0 & c_3 \\ d_1 & 0 & 0 & 0 & 0 & e_1 & 0 & \boxed{f_1} & 0 \\ 0 & \textcircled{d_2} & 0 & e_2 & 0 & 0 & 0 & 0 & f_2 \\ 0 & 0 & d_3 & 0 & e_3 & 0 & \colorbox{pink}{f_3} & 0 & 0 \\ g_1 & 0 & 0 & \boxed{h_1} & 0 & 0 & 0 & i_1 & 0 \\ 0 & g_2 & 0 & 0 & h_2 & 0 & 0 & 0 & \textcircled{i_2} \\ 0 & 0 & g_3 & 0 & 0 & \colorbox{pink}{h_3} & i_3 & 0 & 0 \end{bmatrix} \quad (6)$$

where I_3 denotes the identity matrix of size 3 and the entries boxed in $\theta^{\uparrow\mathbf{P}}$ correspond to the permutation τ that gives the product $A_\tau = a^2bdf^2h^2i$. Here we wrote the matrix $\theta^{\uparrow\mathbf{P}}$ with its entries indexed by their row, e.g., $a_1 = a_2 = a_3 = a$ and a_i is on the ith row of the first block P_{11}.

[6]A matrix is doubly stochastic if it has positive entries and both its rows and columns sum to 1.

[7]See http://staff.science.uva.nl/∼walton/Notes/Hall_Birkhoff.pdf for a short presentation of the Birkhoff-von Neumann theorem and the decomposition algorithm.

The matrices α_τ, $\overline{\alpha}_\tau$ and R_τ are

$$\overline{\alpha}_\tau = \begin{bmatrix} \boxed{\{1,3\}} & \textcircled{\{2\}} & \emptyset \\ \textcircled{\{2\}} & \emptyset & \boxed{\{1,3\}} \\ \emptyset & \boxed{\{1,3\}} & \textcircled{\{2\}} \end{bmatrix}, \quad R_\tau = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

Note that $\overline{\alpha}_\tau$ corresponds to the indices of the boxed entries in $\theta^{\uparrow \mathbf{P}}$. In the matrix $\overline{\alpha}_\tau$, we use circles and boxes to show how to group the boxed entries of $\theta^{\uparrow \mathbf{P}}$: we combine entries in $\theta^{\uparrow \mathbf{P}}$ in rows indexed by the circled entries in $\overline{\alpha}_\tau$, and we combine entries in $\theta^{\uparrow \mathbf{P}}$ in rows indexed by the boxed entries in $\overline{\alpha}_\tau$, thus obtaining a unique rewriting of the product A_τ as $A_\tau = (afh)^2(bdi)$, which is in correspondence to the rewriting steps of the matrix R_τ. In terms of the indexed entries of $\theta^{\uparrow \mathbf{P}}$, the above grouping corresponds to $A_\tau = (a_1 f_1 h_1)(a_3 f_3 h_3)(b_2 d_2 i_2)$ which is exemplified through circles, boxes and shaded boxes in the version of $\theta^{\uparrow \mathbf{P}}$ with indexed entries in (6).

Is a decomposition like the one drawn in $\overline{\alpha}_\tau$ of Example 2 always possible? The answer is yes due to the following simple fact. Each row and column of $\theta^{\uparrow \mathbf{P}}$ participates with exactly one element to a permanent-product. In the matrix $\theta^{\uparrow \mathbf{P}}$ of (6), once we choose d on the second column, or, equivalently, d_2, none of the entries a_2 or g_2 on that column can be part of the permanent-product anymore and, therefore, the second row of matrix P_{11} (where a_2 is positioned) and the second row of the matrix P_{31} (where g_2 is positioned) must contribute each with exactly one entry other than the entries a_2 and g_2 that are not allowed. These are the boxed entries b_2 and i_2. We group these entries with d_2 uniquely and continue the same way to group each of the a entries with the entries f and h that are on the two rows associated with the other two entries on the columns of the entries a to obtain $(a_1 f_1 h_1)$ and $(a_3 f_3 h_3)$.

In terms of the entries of the matrix $\overline{\alpha}_\tau$, this corresponds to the grouping we showed in Example 2 because the matrix $\mathbf{P}$ is reduced, so the first matrices P_{l1} in each row and P_{1l} in each column are equal to the the identity matrix, for all $l \in [m]$. Therefore, for each of the first M columns, the nonzero entries on the jth column are all positioned on the jth row of the matrices P_{l1}, for all $l \in [m]$. Of course, this is not valid for a column that is not among the first M. Indeed, the boxed i of $\theta^{\uparrow \mathbf{P}}$ in (6) is on row 2 of matrix P_{33} and has the nonzero entries on rows 3 of matrix P_{13} and 2 of matrix P_{23}. However, it still holds that the rows corresponding to these non-zero entries must contribute to the product with exactly one entry that cannot be on the column of i. In this case, d_2 on position $(2,2)$ in P_{21} and a_3 on position $(3,3)$ of P_{11} are these entries. We can group these together as well. In fact any such grouping of three where two of them are on the rows corresponding to the non-chosen entries of the column of the third of the group is a good association; the permanent-product A_τ is then a product of some of these three-products with the property that the entries in the products are taken only once and they cover all the entries in the permanent-product A_τ (i.e., they form a partition). Such a partition is surely given by the three-sets of the boxed entries in the first M columns, because each of these sets must be disjoint and they are exactly M, the number of boxed entries from the first M columns, so the union of all entries in these products is equal to all entries in the product A_τ. In fact, any three-sets associated to the boxed entries in a set $(j-1)M+1, \ldots, jM$ of columns corresponds to a partition of the entries in A_τ. For simplicity, however, we choose the partition corresponding to the first M columns, or, equivalently, to the matrix $\overline{\alpha}_\tau$. We call this decomposition *same-index decomposition*.

Therefore, the *same-index decomposition of a permanent-product* in $\theta^{\uparrow \mathbf{P}}$ is the writing of the permanent-product as a product of M sub-products of m entries in θ each indexed by the same row index, e.g., $(a_1 f_1 h_1)(b_2 d_2 i_2)(a_3 f_3 h_3)$.

C. Decompositions that contain illegal sub-products

So far in our example, the same-index decomposition of a permanent-product is equal to its standard decomposition. In the following section, we see that this is not always the case. For example, this following decomposition in $\overline{\alpha}_\kappa$ could also occur:

$$\left[\begin{array}{ccc|ccc|ccc} \boxed{1} & \textcircled{2} & 3 & & & & & & \\ \hline & & & \boxed{1} & \textcircled{2} & & & 3 & \\ \hline & & & & \boxed{1} & & \textcircled{2} & 3 & \end{array}\right],$$

yielding the following permanent-products of $\theta^{\uparrow \mathbf{P}}$:

$$\begin{aligned} a_1 a_2 a_3 \; e_1 e_2 f_3 \; h_1 i_2 i_3 &= (a_1 e_1 h_1)^\dagger (a_2 e_2 i_2)(a_3 f_3 i_3)^\dagger \\ &= (a_1 e_1 i_3)(a_2 e_2 i_2)(a_3 f_3 h_1). \end{aligned}$$

In this case, not all of the products of 3 entries of the same index correspond to permanent-products in the matrix θ; we marked with $\dagger$ the ones that do not, for example, $(a_1 e_1 h_1)^\dagger$ corresponds to aeh in θ which is not a permanent-product. We call such a product *illegal*. This illegal three-product needs to be grouped with another illegal three-product in the same grouping, in this case $(a_3 f_3 i_3)^\dagger$, and rearranged as $(a_1 e_1 i_3)(a_3 f_3 h_1)$ to obtain a standard decomposition, i.e., a product of permanent-products of θ. We call these sub-products that correspond to a permanent-product in θ *legal*.

D. Mapping illegal products into legal products

Next we show that we can always assume that all permanent-products in $\theta^{\uparrow \mathbf{P}}$ are products of θ-permanent-products by showing that any permanent-product of $\theta^{\uparrow \mathbf{P}}$ containing some illegal sub-products can be mapped uniquely into some product of M same-index permanent-products of θ. In addition, this product has the same exponent matrix as the original permanent-product but is not a permanent-product of $\theta^{\uparrow \mathbf{P}}$. This way, we establish a one-to-one correspondence between permanent-products of $\theta^{\uparrow \mathbf{P}}$ and products of M permanent-products in θ.

This correspondence illustrated in the previous example can be generalized to all permanent-products of $\theta^{\uparrow \mathbf{P}}$ with same-index decompositions that contain some illegal sub-products in the following way.

- Let θ be an $m \times m$ non-negative matrix and $\theta^{\uparrow \mathbf{P}}$ be a reduced matrix of degree M.
- Let τ be a permutation on $[mM]$ and A_τ be a permanent-product in $\theta^{\uparrow \mathbf{P}}$ that is not trivially zero. Let R_τ be its exponent matrix.
- Write A_τ as the same-index decomposition; A_τ can or not contain illegal same-index sub-products, i.e., products of m entries in θ of the same index that are not permanent-products in θ.
- List all distinct products of M *same-index permanent-products* in θ corresponding to all standard decompositions of R_τ that start with the entries in A_τ that are in the first M columns of $\theta^{\uparrow \mathbf{P}}$. Call them $A'_{\tau,1}, \dots, A'_{\tau,l}$ and reorder, if needed, the entries in the sub-products of A_τ and $A'_{\tau,1}, \dots, A'_{\tau,l}$ such that the entries from the first M columns are always first in the subproduct, followed by the entries ordered by the row index in θ increasingly from 1 to m and such that the indices of the θ-permanent-products are ordered increasingly from 1 to M.

This procedure, henceforth called *standard mapping*, is formalized in the following lemma.

Lemma 1 (Standard mapping)**.** *Initially, set* $\mathcal{L} := \{A'_{\tau,1}, \dots, A'_{\tau,l}\}$.

Start *Let* $0 \leqslant s \leqslant M$ *and* $1 \leqslant t < m$ *be such that*

- A_τ *and each* $A'_{\tau,j} \in \mathcal{L}$ *have their first* s θ*-permanent-products equal and*
- A_τ *and each* $A'_{\tau,j} \in \mathcal{L}$ *have their* $(s+1)$*th* θ*-permanent-products either equal in the first* t *entries or have all of the first* t *entries distinct except for the first entry and*
- A_τ *and* $A'_{\tau,i} \in \mathcal{L}$ *have their* $(s+1)$*th* θ*-permanent-product equal in the* $(t+1)$*th entry, while there exists* $A'_{\tau,j} \neq A'_{\tau,i}$*, such that* A_τ *and* $A'_{\tau,j}$ *have the* $(s+1)$*th* θ*-permanent-product distinct in the* $(t+1)$*th entry.*

Let $\{A'_{\tau,j_1}, \dots, A'_{\tau,j_k}\} \subset \{A'_{\tau,1}, \dots, A'_{\tau,l}\}, 1 \leqslant k < l$*, such that* A_τ *and each* A'_{τ,j_n}*,* $n \in [k]$*, have their* $(s+1)$*th* θ*-permanent-product equal in the* $(t+1)$*th entry.*

Map $A_\tau \mapsto A'_{\tau,i}$ *if* $k = 1$*, otherwise update* $\mathcal{L} := \mathcal{L}_k$ *and repeat the steps from* *Start*.

Then, this map is a well-defined one-to-one (injective) map from the set of all permanent products of $\theta^{\uparrow \mathbf{P}}$ *of a certain exponent matrix to the set of all products of* M θ*-permanent-products of the same exponent matrix. This gives a one-to-one map from the set of all permanent-products in* $\theta^{\uparrow \mathbf{P}}$ *to the set of all products of* M θ*-permanent-products.*

Proof: The fact that the map is well defined is easy to see since there can only be one matrix $A'_{\tau,i}$ satisfying the conditions, while the existence of this matrix is ensured by the decomposition algorithm presented in Section II-A. Indeed, the decomposition algorithm based on the exponent matrix guarantees the existence of the list of products of θ-permanent-products, which has cardinality at least one. It also guarantees the existence of a standard decomposition of the permanent-product into legal sub-products not necessarily of the same index. The standard decomposition can be mapped into a product of same-index θ-permanent-products, thus guaranteeing the existence of the map.

The fact that no two permanent-products can be mapped into the same $A'_{\tau,i}$ is also ensured by the conditions of the mapping; if two different permanent products A_τ and A_ν map into the same $A'_{\tau,i}$, then they must have a first entry in which they differ; this entry must be necessarily after the first s entries. This means, however, that there must exist an $A'_{\tau,j}$ that shares with A_ν that entry but not with $A'_{\tau,i}$. Therefore, A_ν cannot get mapped into the same $A'_{\tau,i}$ as A_τ, proving that the function is one-to-one. In addition, if A_τ contains illegal same-index sub-products, then $A'_{\tau,i}$ such that $A_\tau \mapsto A'_{\tau,i}$ cannot be a permanent-product in $\theta^{\uparrow \mathbf{P}}$. To see this, erase from $\theta^{\uparrow \mathbf{P}}$ all rows and columns corresponding to the entries that the two share. Suppose that there are k entries in which the two products are different, say, $x_1, x_2, \dots, x_k$ in A_τ and $x'_1, x'_2, \dots, x'_k$ in $A'_{\tau,i}$. Because the two products $A_\tau, A'_{\tau,i}$ have the same exponent matrix, so do the two products $x_1 x_2 \dots x_k$ and $x'_1 x'_2 \dots x'_k$. Therefore, in each block in which there exists some x_i, $i \in [k]$, there must exist also a $j \in [k]$ such that x'_j is also in that block. We can reorder $x'_1, x'_2, \dots, x'_k$ so that each x'_l is in the same block as x'_l. Note that there can be more entries in one block, but to each entry x_l corresponds a unique entry x'_l in the same block. Since there is only one column in the $k \times k$ submatrix crossing the term x_i and since $x'_j \notin \{x_1, \dots x_k\}$, we obtain that x_i and x'_j must be on the same column which contradicts the fact that the block is a weighted permutation matrix.

Therefore, if A_τ contains illegal same-index sub-products, then A_τ is mapped through the above mapping into a product $A'_{\tau,i}$ that is not a permanent-product in $\theta^{\uparrow \mathbf{P}}$. This also implies that an all-legal permanent-product A_τ and a permanent-product containing some illegal same-index sub-products A_κ do not map into the same product of M θ-permanent-products, which in this case would be A_τ. Indeed, if A_τ does not contain any illegal sub-products, i.e., it is a product of M θ-permanent-products, then $A_\tau = A'_{\tau,i}$, for some i, and the mapping corresponds to $A_\tau \mapsto A_\tau$ as expected.

Such a mapping can be defined for each exponent matrix, which proves the existence of the overall one-to-one map from the set of all permanent-products in $\theta^{\uparrow \mathbf{P}}$ to the set of all products of M θ-permanent-products. ∎

E. Upper bounding the permanent of a lifting of a matrix

The mapping in Section II-D allows us to compute, for a fixed exponent matrix $R = (r_{ij})$, the coefficient of $\prod_{j=1}^{m} \prod_{l=1}^{m} (\theta_{jl})^{r_{jl}}$ in $\operatorname{perm}(\theta^{\uparrow \mathbf{P}})$, or, equivalently, the maximum possible number of permutations $\tau \in \mathcal{S}_{mM}$ such that $A_\tau = \prod_{j=1}^{m} \prod_{l=1}^{m} (\theta_{jl})^{r_{jl}}$ is a permanent-product with exponent matrix R that is not trivially-zero, and, using this, to prove the upper bound $\operatorname{perm}(\theta^{\uparrow \mathbf{P}}) \leqslant \operatorname{perm}(\theta)^M$.

The following corollary is an immediate consequence of the one-to-one mapping.

Corollary 1. *Let* $R = (r_{ij})$ *be an exponent matrix of some permanent-product in* $\operatorname{perm}(\theta^{\uparrow \mathbf{P}})$*. For each* $\tau \in \mathcal{S}_{mM}$

with $A_\tau = \prod_{j=1}^{m}\prod_{l=1}^{m}(\theta_{jl})^{r_{jl}}$*, let* $A'_{\tau,1},\ldots,A'_{\tau,l}$ *be the possible products of* M θ*-permanent-products associated with* R*. For each* $j \in [l]$*, denote by* $N_{\tau,j}$ *the number of products of* M θ*-permanent-products that are equivalent to* $A'_{\tau,j}$*, i.e., they can be obtained from* $A'_{\tau,j}$ *by applying an* M*-permutation on the indices. Then, the coefficient of* $\prod_{j=1}^{m}\prod_{l=1}^{m}(\theta_{jl})^{r_{jl}}$ *in* $\mathrm{perm}(\theta^{\uparrow\mathbf{P}})$ *is upper bounded by* $\sum_{j=1}^{l} N_{\tau,j}$*.*

The following lemma determines $N_{\tau,j}$ for all $j \in [l]$.

Lemma 2. *For each* $j \in [l]$ *and* $\sigma \in \mathcal{S}_m$*, let* $0 \leqslant t_{j,\sigma} \leqslant M$ *such that* $\sum_{\sigma\in\mathcal{S}_m} t_{j,\sigma} = M$ *and* $A'_{\tau,j} = \prod_{\sigma\in\mathcal{S}_m}\left(\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)}\right)^{t_{j,\sigma}}$*. Then* $N_{\tau,j} = \binom{M}{\mathbf{t_j}}$ *where* $\binom{M}{\mathbf{t_j}}$ *is the multinomial coefficient associated with the vector* $\mathbf{t_j} \triangleq (t_{j,\sigma})_{\sigma\in\mathcal{S}_m}$*.*

Proof: The entries that lie in the first M columns of $\theta^{\uparrow\mathbf{P}}$ uniquely determine the way the products of θ-permanent-products $\left(\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)}\right)^{t_{j,\sigma}}$ are formed. We can choose these in $\binom{M}{\mathbf{t_j}}$ ways. ■

The main result of the paper now follows immediately.

Theorem 2. *Let* $\theta = (\theta_{ij})$ *be a non-negative matrix of size* $m \times m$ *and let* $\mathbf{P} = (P_{ij}) \in \mathcal{M}_m(\mathcal{P}_M)$*. Then*

$$\mathrm{perm}(\theta^{\uparrow\mathbf{P}}) \leqslant \mathrm{perm}(\theta)^M.$$

Proof: The upper bound follows immediately from Lemma 2 and the expansion of $\mathrm{perm}(\theta)^M$ as

$$\mathrm{perm}(\theta)^M = \left(\sum_{\sigma\in\mathcal{S}_m}\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)}\right)^M = \sum_{|\mathbf{t_j}|=M}\binom{M}{\mathbf{t_j}}\prod_{\sigma\in\mathcal{S}_m}\left(\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)}\right)^{t_{j,\sigma}}.$$

■

III. Conclusions

The consequences of the results in this paper are more than just purely theoretical. They provide new insight into the structure of the permanent of a $\mathbf{P}$-lifting of a matrix, which can be exploited algorithmically to decrease the computational complexity of the permanent of the $\mathbf{P}$-liftings. Such an algorithm can search for products of groups of entries formed according to the groupings we presented in this paper to check if they form valid permanent-products. In addition, the structure of the permanent-products of $\mathbf{P}$-liftings of a matrix may have some implications on the constant C in the inequality $\mathrm{perm}(\theta) \leqslant C \cdot \mathrm{perm}_{\mathrm{B}}(\theta)$.

Lastly, since a $\mathbf{P}$-lifting of a matrix θ corresponds to an M-graph cover of the protograph (base graph) described by θ, which, in turn, correspond to LDPC codes, these results may help explain the performance of these codes through the techniques presented in [17]–[21], which are based on explicit constructions of codewords and pseudo-codewords with components equal to determinants or permanents, of some $m \times m$ submatrices of $\mathbf{H}$ over the binary field or over the integers.

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