The Meta Distribution of the SINR in mm-Wave D2D Networks

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***Abstract*—In this paper, the meta distribution of the signal-to-interference-plus-noise ratio (SINR) in millimeter wave (mm-wave) device-to-device (D2D) networks is studied, which is the distribution of the conditional SINR distribution given a realization of the point process modeling the network. This analysis provides much more fine-grained information on the performance of individual links than just the mean that is usually evaluated. Modeling the D2D transmitters as a Poisson point process (PPP) and considering the unique mm-wave features, moments of the conditional SINR distribution given the point process are derived in order to calculate analytical expression for the meta distribution. It turns out that when the size of the antenna array tends to infinity, the noise is totally suppressed and the node density becomes the dominating factor affecting the interference in the network. Closed-form approximations of the meta distribution with the beta distribution are also provided. Numerical results give interesting insights into the effects of mm-wave features on the performance of D2D communication.**

I. Introduction

Recently, enabling device-to-device (D2D) communications in millimeter wave (mm-wave) networks has been regarded as a promising way for the next-generation mobile networks to provide very high data rate (multi-gigabits-per-second) for mobile devices with reduced end-to-end latency [1]. For existing communication systems, the main challenge of implementing underlaid D2D links is the difficulty in interference management between the inter-D2D interference and the cross-tier interference from other systems using the same spectrum, especially for the autonomous case without base station (BS) involvement. However, since the mm-wave spectrum has several unique features such as spectrum availability, high propagation loss, directivity, and sensitivity to blockage, the situation will be different when the D2D communication occurs in the mm-wave band. Thus, it is important to study how to exploit the interesting properties of mm-wave communication for D2D networks and to analyze their performance. This is the motivation of our work.

The signal-to-interference-plus-noise ratio (SINR) is a fundamental metric for how a communication system/link performs. When stochastic geometry is used for the analysis, the SINR performance is most commonly characterized as the success probability relative to an SINR threshold and evaluated at the typical link [2]. However, the performance of the typical link represents an average over all the spatial realizations of the point process (or over all the links in a single realization if the point process is ergodic), which provides very limited information on the individual links. To overcome this limitation, [3] introduced the concept of the *meta distribution*, which is the distribution of the conditional success probability given the point process and thus provides a much sharper version of the "SINR performance". Although the benefits of combining mm-wave and D2D technologies have attracted considerable attention recently, the existing works were investigated either through pure simulations or theoretical analysis with an finite spatial extent [4–6]. Only [7] analyzed the downlink coverage probability of a D2D relay-assisted mm-wave cellular network using stochastic geometry. To our best knowledge, there is no prior work that studied the meta distribution to provide fine-grained analysis for mm-wave D2D networks.

In this regard, we focus on the meta distribution of the SINR in mm-wave D2D networks considering the unique channel characteristics and antenna features of mm-wave communications, expecting to get different insights from the previous D2D networks. We first derive the moments of the conditional success probability for mm-wave D2D networks considering the effects of blockage and large beamforming gain from directional antenna arrays. It turns out that as the antenna array size tends to infinity, the noise is totally suppressed. In this regime, the node density as well as the side lobe interference become the dominate factors affecting interference characteristics and hence the performance. Then, to show what fraction of users in the network achieve a target reliability if the SINR requirement is given, we provide analytical expressions for the exact meta distribution of the SINR. Closed-form approximations of the meta distribution with the beta distribution are also provided. The numerical results demonstrate the significant impact of the unique mm-wave features, the SINR threshold, and the density of users on the performance of mm-wave D2D networks.

From a broader perspective, the contribution of the paper lies in providing fine-grained information on the SINR performance for individual links rather than for the commonly studied typical link, which gives network operators a comprehensive understanding of the performance differences among individual users in a D2D network.

II. Network Model

We consider a mm-wave D2D communication network, where the D2D transmitters are distributed according to a homogeneous Poisson point process (PPP) Φ with density λ. Each transmitter is assumed to have a dedicated receiver at distance r_0 in a random orientation, i.e., the D2D users form a *Poisson bipolar network* [2, Def. 5.8]. We consider a receiver at the origin that attempts to receive from an additional transmitter located at $(r_0, 0)$. Due to Slivnyak's theorem [2, Thm. 8.10], this receiver becomes the typical receiver under expectation over the PPP. We assume that each receiver has a single antenna and its corresponding transmitter is equipped with a square antenna array composed of N elements. All transmitters operate at a constant power μ and apply analog beamforming to overcome the severe path loss in the mm-wave band. We also assume that the direction of arrival (DoA) between the transmitter-receiver pair is known at the transmitter and thus the beam direction is perfectly aligned to obtain the maximum power gain. The ALOHA channel access scheme is adopted, i.e., in each time slot, D2D transmitters in Φ independently transmit with probability p.

A. Blockage and Propagation Model

The generalized LOS ball model [8] is used to capture the blockage effect in mm-wave communication since it has been validated in [9] as a better fit with real-world scenarios than other blockage models adopted in previous works. Specifically, the LOS probability of the channel between two nodes with separation d in this model is

$$P_{\mathrm{LOS}}(d) = p_{\mathrm{L}} \mathbf{1}(d < R), \tag{1}$$

where $\mathbf{1}(\cdot)$ is the indicator function, R is the maximum length of a LOS channel, and the LOS fraction constant $p_{\mathrm{L}} \in [0,\ 1]$ represents the average fraction of the LOS area within a circular ball of radius R around the receiver under consideration. Thus, p_{L} is the LOS probability if the distance d is less than R. The blockage effect induces different path loss exponents, denoted as α_{L} and α_{N}, for LOS and NLOS channels, respectively. Typical values for mm-wave path loss exponents can be found in measurement results in [10] with approximated ranges of $\alpha_{\mathrm{L}} \in [1.9, 2.5]$ and $\alpha_{\mathrm{N}} \in [2.5, 4.7]$.

For the sake of mathematical tractability, the sectorized antenna model is adopted to approximate the actual antenna pattern, as in [8]. In particular, the array gains within the half-power beam width are assumed to be the maximum power gain (i.e., main lobe gain), and the gains of the other DoAs are approximated to be the first minor maximum gain (i.e., side lobe gain) of the actual antenna pattern, which can be formulated as

$$G(\varphi) = \begin{cases} G_{\mathrm{m}} & \text{if } |\varphi| \le w/2 \\ G_{\mathrm{s}} & \text{otherwise,} \end{cases} \tag{2}$$

where $w \in (0, 2\pi]$ is the half-power beam width and correlated with the size of antenna array, and $\varphi \in [-\pi, \pi)$ is the angle off the boresight direction. With the assumption of a $\sqrt{N} \times \sqrt{N}$ uniform planar square antenna array with half-wavelength antenna spacing, the half-power beamwidth w, main lobe gain G_{m}, and side lobe gain G_{s} are summarized in Table I.

TABLE I. Antenna parameters of a uniform planar square antenna array [7]

Parameters	Description	value
w	Half-power beamwidth	$\frac{\sqrt{3}}{\sqrt{N}}$
G_{m}	Main lobe gain	N
G_{s}	Side lobe gain	$1/\sin^2\left(\frac{3\pi}{2\sqrt{N}}\right)$

B. SINR Analysis

We assume that the link between the transmitter-receiver pair is in the LOS condition with deterministic path loss $r_0^{-\alpha_{\mathrm{L}}}$. In fact, if the receiver was associated with a NLOS transmitter, the link would quite likely be in outage due to the severe propagation loss and high noise power at mm-wave bands as well as the fact that the interferers can be arbitrarily close to the receiver. The power fading coefficient associated with node $x \in \Phi$ is denoted by h_x, which is an exponential random variable with $\mathbb{E}(h_x) = 1$ (Rayleigh fading)[1] for both LOS and NLOS to enhance the analytical tractability, and all h_x are mutually independent and also independent of the point process Φ. $\ell(x)$ is the random path loss function associated with the interfering transmitter location x, given by

$$\ell(x) = \begin{cases} |x|^{-\alpha_{\mathrm{L}}} & \text{w.p. } P_{\mathrm{LOS}}(|x|) \\ |x|^{-\alpha_{\mathrm{N}}} & \text{w.p. } 1 - P_{\mathrm{LOS}}(|x|) \end{cases} \tag{3}$$

where all $\ell(x)_{x\in\Phi}$ are independent. For the typical receiver, the interferers outside the LOS ball are NLOS and thus can be ignored due to the severe path loss over the large distance (at least R). As a result, the analysis for the network originally composed by the PPP Φ with density λ reduces to the analysis of a finite network region, namely, the disk of radius R centered at the origin. Due to the incorporation of the blockages, the LOS transmitters with LOS propagation to the typical receiver form a PPP Φ_{L} with density $p_{\mathrm{L}}\lambda$, while Φ_{N} with density $p_{\mathrm{N}}\lambda$ is the transmitter set with NLOS propagation, where $p_{\mathrm{L}} + p_{\mathrm{N}} = 1$ such that $\Phi = \Phi_{\mathrm{L}} \cup \Phi_{\mathrm{N}}$.

Thus, the interference at the origin is defined as

$$I \triangleq \sum_{x\in\Phi} \mu G(\varphi_x) h_x \ell(x) B_x, \tag{4}$$

where μ is the constant transmit power, $G(\varphi_x)$ is the directional antenna gain function with DoA φ_x, and B_x is a Bernoulli variable with parameter p to indicate whether x transmits a message to its receiver. Since all transmitters are oriented toward the corresponding receivers, the DoAs between the interferers and the typical receiver are uniformly random in $[-\pi, \pi)$. Hence, $G(\varphi_x)$ is equal to G_{m} with probability $\bar{w} = \frac{w}{2\pi}$ and G_{s} with probability $1 - \bar{w}$. Without

[1] Although the LOS dependent mm-wave scenarios may be more in line with Nakagami fading, simulation results have shown that Nakagami and Rayleigh fading present the same trends in terms of SINR performance [11].

loss of generality, the noise power is set to one, and the SINR at the typical receiver is then given by

$$\mathrm{SINR}=\frac{\mu G_{\mathrm{m}} h_{x_0} r_0^{-\alpha_{\mathrm{L}}}}{1+\sum_{x\in\Phi}\mu G(\varphi_x)h_x\ell(x)B_x}. \quad (5)$$

III. The Meta Distribution of the SINR

In this section, we focus on the meta distribution of the SINR, which is the distribution of the conditional success probability given a realization of the point process. The meta distribution of the SINR is a two-parameter distribution function defined as [3]

$$\bar{F}_{P_{\mathrm{s}}(\theta)}(x)\triangleq\mathbb{P}_o^!(P_{\mathrm{s}}(\theta)>x), \quad (6)$$

where $\theta\in\mathbb{R}^+, x\in[0,1]$. It represents the complementary cumulative distribution function (CCDF) of the link success probability $P_{\mathrm{s}}(\theta)$ conditioned on the point process. Here $\mathbb{P}_o^!$ denotes the reduced Palm probability conditioning on the typical receiver at the origin o and its corresponding transmitter to be active, and the link success probability $P_{\mathrm{s}}(\theta)$ is a random variable given as

$$P_{\mathrm{s}}(\theta)\triangleq\mathbb{P}(\mathrm{SINR}>\theta\mid\Phi), \quad (7)$$

where θ is the SINR threshold. Due to the ergodicity of the point process, the meta distribution can be interpreted as the fraction of links in each realization of the point process that have a SINR greater than θ with probability at least x. Hence, the standard success probability is the mean of $P_{\mathrm{s}}(\theta)$, obtained by integrating the meta distribution (6) over $x\in[0,1]$.

A. Moments of $P_{\mathrm{s}}(\theta)$

Since a direct calculation of the meta distribution seems infeasible, we will derive the exact analytical expression through the moments $M_b(\theta)\triangleq\mathbb{E}\left[P_{\mathrm{s}}(\theta)^b\right]$ first, and then approximate it with much simpler closed-form expressions.

Theorem 1. (Moments for mm-wave D2D network with ALOHA) *Given that the typical link is LOS and active, the moment M_b ($b\in\mathbb{C}$) of the conditional success probability in mm-wave D2D networks is*

$$M_b(\theta)=\exp\left(-\lambda\pi R^2(1-p_{\mathrm{L}}A_{\mathrm{L}}-p_{\mathrm{N}}A_{\mathrm{N}})-\frac{b\theta r_0^{\alpha_{\mathrm{L}}}}{\mu G_{\mathrm{m}}}\right), \quad (8)$$

where

$$A_i=\sum_{n=0}^{\infty}\binom{b}{n}(-p)^n\sum_{m=0}^{n}\binom{n}{m}\bar{w}^m(1-\bar{w})^{n-m}F_{m,n-m}(\alpha_i,\theta), \quad (9)$$

and

$$F_{m,n-m}(\alpha_i,\theta)=\tilde{F}_1\left(\delta_i,m,n-m,\delta_i+1,\frac{-R^{\alpha_i}}{\theta r_0^{\alpha_{\mathrm{L}}}},\frac{-G_{\mathrm{m}}R^{\alpha_i}}{G_{\mathrm{s}}\theta r_0^{\alpha_{\mathrm{L}}}}\right). \quad (10)$$

Here $\tilde{F}_1(\cdot)$ is the hypergeometric function of two variables[2] *[12, Chap. 9.18], and $\delta_i=2/\alpha_i$, $i\in\{\mathrm{L},\mathrm{N}\}$.*

Proof: See Appendix A.

[2]The $\tilde{F}_1$ function is also called the Appell function and is implemented in the Wolfram Language as **AppellF1**$[a,b_1,b_2,c,x,y]$.

The first moment of the conditional success probability is the standard success probability for the mm-wave D2D network, denoted as $p_{\mathrm{s}}(\theta)$ or $M_1(\theta)$. It is given as

$$p_s(\theta)=M_1(\theta)=\exp\left(-\lambda p\pi R^2(p_{\mathrm{L}}\xi_1^{\mathrm{L}}+p_{\mathrm{N}}\xi_1^{\mathrm{N}})-\frac{\theta r_0^{\alpha_{\mathrm{L}}}}{\mu G_{\mathrm{m}}}\right), \quad (11)$$

where

$$\xi_1^i=\bar{w}\,{}_2F_1\left(1,\delta_i,\delta_i+1,\frac{-R^{\alpha_i}}{\theta r_0^{\alpha_{\mathrm{L}}}}\right)+(1-\bar{w})\times{}_2F_1\left(1,\delta_i,\delta_i+1,\frac{-G_{\mathrm{m}}R^{\alpha_i}}{G_{\mathrm{s}}\theta r_0^{\alpha_{\mathrm{L}}}}\right),\ i\in\{\mathrm{L},\mathrm{N}\}. \quad (12)$$

When $\alpha_{\mathrm{L}}=2$ and $\alpha_{\mathrm{N}}=4$, we have the simpler expressions

$$\xi_1^{\mathrm{L}}=\bar{w}\frac{\theta r_0^2}{R^2}\ln\left(1+\frac{R^2}{\theta r_0^2}\right)+(1-\bar{w})\frac{G_{\mathrm{s}}\theta r_0^2}{G_{\mathrm{m}}R^2}\ln\left(1+\frac{G_{\mathrm{m}}R^2}{G_{\mathrm{s}}\theta r_0^2}\right),$$

$$\xi_1^{\mathrm{N}}=\bar{w}\frac{\sqrt{\theta}r_0}{R^2}\arctan\left(\frac{R^2}{\sqrt{\theta}r_0}\right)+(1-\bar{w})\frac{\sqrt{G_{\mathrm{s}}\theta}r_0}{\sqrt{G_{\mathrm{m}}}R^2}\arctan\left(\frac{\sqrt{G_{\mathrm{m}}}R^2}{\sqrt{\theta G_{\mathrm{s}}}r_0}\right).$$

The second moment of $P_{\mathrm{s}}(\theta)$ is given as

$$M_2(\theta)=(M_1(\theta))^2\exp\left(\lambda p^2\pi R^2(p_{\mathrm{L}}\xi_2^{\mathrm{L}}+p_{\mathrm{N}}\xi_2^{\mathrm{N}})\right), \quad (13)$$

where

$$\xi_2^i=\bar{w}^2\,{}_2F_1\left(2,\delta_i,\delta_i+1,\frac{-R^{\alpha_i}}{\theta r_0^{\alpha_{\mathrm{L}}}}\right)+2\bar{w}(1-\bar{w})F_{1,1}(\alpha_i,\theta)+(1-\bar{w})^2\,{}_2F_1\left(2,\delta_i,\delta_i+1,\frac{-G_{\mathrm{m}}R^{\alpha_i}}{G_{\mathrm{s}}\theta r_0^{\alpha_{\mathrm{L}}}}\right),\ i\in\{\mathrm{L},\mathrm{N}\}. \quad (14)$$

When $\alpha_{\mathrm{L}}=2$ and $\alpha_{\mathrm{N}}=4$, we have

$$\xi_2^{\mathrm{L}}=\frac{\bar{w}^2\theta r_0^2}{\theta r_0^2+R^2}+\frac{(1-\bar{w})^2G_{\mathrm{s}}\theta r_0^2}{G_{\mathrm{s}}\theta r_0^2+G_{\mathrm{m}}R^2}+\frac{2\bar{w}(1-\bar{w})G_{\mathrm{s}}\theta r_0^2}{(G_{\mathrm{m}}-G_{\mathrm{s}})R^2}\ln\left(\frac{G_{\mathrm{s}}\theta r_0^2+G_{\mathrm{m}}R^2}{G_{\mathrm{s}}\theta r_0^2+G_{\mathrm{s}}R^2}\right),$$

$$\xi_2^{\mathrm{N}}=\frac{\bar{w}^2}{2}\left(\frac{\sqrt{\theta}r_0}{R^2}\arctan\frac{R^2}{\sqrt{\theta}r_0}+\frac{\theta r_0^2}{\theta r_0^2+R^4}\right)+\frac{(1-\bar{w})^2}{2}\left(\frac{\sqrt{G_{\mathrm{s}}\theta}r_0}{R^2}\arctan\frac{R^2}{\sqrt{\theta}r_0}+\frac{\theta r_0^2}{\theta r_0^2+R^4}\right)+\frac{2\bar{w}(1-\bar{w})G_{\mathrm{s}}\sqrt{\theta}r_0}{(G_{\mathrm{m}}-G_{\mathrm{s}})R^2}\left(\arctan\frac{\sqrt{G_{\mathrm{m}}}R^2}{\sqrt{G_{\mathrm{s}}\theta}r_0}-\arctan\frac{R^2}{\sqrt{\theta}r_0}\right).$$

The variance of the conditional success probability can be obtained as var $P_{\mathrm{s}}(\theta)=M_2(\theta)-M_1^2(\theta)$. As is seen in [3], we also find in the mm-wave D2D networks that the same λp leads to the same standard success probability but the variance depends on both λ and p, not just the product, which highlights the importance of the fine-grained analysis based on the meta distribution. Moreover, in mm-wave D2D networks, as in bipolar networks with standard path loss and single-antenna nodes, we also have the following concentration property.

Corollary 1. (Concentration as $p\to 0$) *Keeping the transmitter density $t\triangleq\lambda p$ fixed and letting $p\to 0$, we have*

$$\lim_{p\to 0,\lambda p=t}P_{\mathrm{s}}(\theta)=M_1(\theta)=p_{\mathrm{s}}(\theta) \quad (15)$$

in mean square (and probability and distribution).

Proof: From the expression of ξ_2^i in (13), when $p \to 0$, we have $\xi_2^i = 2\xi_1^i$ and thus $M_2(\theta) = M_1^2(\theta)$. In this case, the limiting variance is zero, i.e., $\lim\limits_{p\to 0, \lambda p = t} \operatorname{var} P_{\rm s}(\theta) = 0$. ■

The concentration property means that in the limit of a very dense network with very small p, all links in the network have exactly the same success probability (or reliability).

An interesting question is what happens when massive antenna arrays are used in mm-wave D2D networks, i.e., as $N \to \infty$. It is answered in the following corollary.

Corollary 2. (Limit as $N \to \infty$) *Letting $N \to \infty$, M_b ($b \in \mathbb{C}$) is given by*

$$\lim_{N\to\infty} M_b(\theta) = \exp\left(-\lambda\pi R^2(p_{\rm L}A_{\rm L} + p_{\rm N}A_{\rm N})\right), \quad (16)$$

where

$$A_i = \sum_{n=1}^{\infty}\binom{b}{n}(-1)^{n+1}p^n\, {}_2F_1\left(n, \delta_i, \delta_i+1, \frac{9\pi^2 R^{\alpha_i}}{4\theta r_0^{\alpha_{\rm L}}}\right). \quad (17)$$

Here ${}_2F_1(\cdot)$ is the Gaussian hypergeometric function [12, Chap. 9.11], and $\delta_i = 2/\alpha_i$, $i \in \{\rm L, N\}$.

Proof: When $N \to \infty$, we have $\bar{w} \to 0$, $G_{\rm m} \to \infty$ and $G_{\rm m}/G_{\rm s} \to 9\pi^2/4$. According to Theorem 1, we have

$$\begin{aligned}\lim_{N\to\infty} M_b &= e^{-\lambda\pi R^2}\exp\Bigg(\sum_{i\in\{\rm L,N\}}\lambda\pi R^2 p_i\delta_i \\ &\qquad \times \int_0^1\Bigg(1-\frac{p}{1+\frac{yR^{\alpha_i}9\pi^2}{\theta r_0^{\alpha_{\rm L}}4}}\Bigg)^b y^{\delta_i-1}{\rm d}y\Bigg) \\ &\overset{(a)}{=} \exp\Bigg(-\sum_{i\in\{\rm L,N\}}\lambda\pi R^2 p_i\sum_{n=1}^{\infty}\binom{b}{n}(-1)^{n+1}p^n \\ &\qquad \times {}_2F_1\Big(n,\delta_i,\delta+1,\frac{-9\pi^2R^{\alpha_i}}{4\theta r_0^{\alpha_{\rm L}}}\Big)\Bigg) \end{aligned} \quad (18)$$

where step (a) uses the general binomial theorem and follows from the definition of the Gaussian hypergeometric function in [12, Chap. 9.11]. ■

The limit (16) offers the following insights: (1) As $N\to\infty$, the effect of the noise is totally suppressed since $G_{\rm m}\to\infty$; (2) The side lobe interference gradually becomes the main performance-limiting factor as $N \to \infty$. In this regime, the node density plays a critical role in determining the interference power in the network. For instance, a sparse network makes the interference in a massive MIMO network arbitrarily small while a dense one is interference-limited.

B. Exact expression

The exact meta distribution can be obtained by the Gil-Pelaez theorem [13] with the imaginary moments M_{jt} of $P_{\rm s}(\theta)$, $t \in \mathbb{R}$, $j \triangleq \sqrt{-1}$.

Corollary 3. (Exact expression) *The meta distribution for the mm-wave D2D networks is given by*

$$\bar{F}_{P_s(\theta)}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{e^{-\lambda\pi R^2(1-\Re(\zeta))}}{t} \times \sin\Big(t\log x + t\frac{\theta r_0^{\alpha_{\rm L}}}{\mu G_{\rm m}} - \lambda\pi R^2\Im(\zeta)\Big){\rm d}t, \quad (19)$$

where $\zeta = p_{\rm L}A_{\rm L} + p_{\rm N}A_{\rm N}$ is given in (9) with $b = jt$ and $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of $z \in \mathbb{C}$, respectively.

Proof: According to the Gil-Pelaez theorem, the ccdf of $P_s(\theta)$ is given by

$$\bar{F}_{P_s(\theta)}(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\Im\left(e^{-jt\log x}M_{jt}\right)}{t}{\rm d}t, \quad (20)$$

where $M_{jt} = M_{b=jt}$, $t \in \mathbb{R}$ is given in (8), and $\Im(z)$ is the imaginary part of $z \in \mathbb{C}$. Letting $\zeta = p_{\rm L}A_{\rm L} + p_{\rm N}A_{\rm N}$, we have

$$\begin{aligned}&\bar{F}_{P_s(\theta)}(x) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Im\left(e^{-jt\log x}e^{-\lambda\pi R^2(1-\zeta)-jt\theta r_0^{\alpha_{\rm L}}/(\mu G_{\rm m})}\right)\frac{1}{t}{\rm d}t \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty e^{-\lambda\pi R^2(1-\Re(\zeta))}\sin\Big(t\log x + \frac{t\theta r_0^{\alpha_{\rm L}}}{\mu G_{\rm m}} - \lambda\pi R^2\Im(\zeta)\Big)\frac{1}{t}{\rm d}t\end{aligned}$$

■

C. Approximations of the Meta Distribution

Though the expression in Cor. 3 is exact and can be calculated via numerical integration techniques, it is difficult to gain insights directly and apply it to obtain analytical results. To approximate the meta distribution, we use the same approach adopted in [3] by matching the mean and variance of the beta distribution with $M_1(\theta)$ and $M_2(\theta)$ given in Theorem 1 to verify whether this approximation is also acceptable for mm-wave D2D networks, where the interference characteristics are different from those in microwave networks. The probability density function (PDF) of a beta distributed random variable X with parameters κ and β is

$$f_X(x) = x^{\kappa-1}(1-x)^{\beta-1}/B(\kappa,\beta), \quad (21)$$

where $B(\cdot,\cdot)$ is the beta function. The first- and second-moment of X are given as

$$\mathbb{E}X = \frac{\kappa}{\kappa+\beta}, \quad \mathbb{E}(X^2) = \frac{\kappa+1}{\kappa+\beta+1}\mathbb{E}X. \quad (22)$$

Letting $\mathbb{E}X = M_1(\theta)$ and $\mathbb{E}(X^2) = M_2(\theta)$, we have

$$\kappa = \frac{M_1M_2 - M_1^2}{M_1^2 - M_2}, \quad \beta = \frac{(1-M_1)(M_2-M_1)}{M_1^2 - M_2}. \quad (23)$$

Hence, the approximate meta distribution of the typical mm-wave D2D receiver follows as

$$\bar{F}_{P_s(\theta)}(x) \approx 1 - I_x(\kappa,\beta), \quad (24)$$

where $I_x(\kappa,\beta)$ is the regularized incomplete beta function.

IV. Numerical Results

In this section, we give some numerical results of the standard success probability and the meta distribution for mm-wave D2D networks, where $\lambda = 0.1$, $\mu = 20$, $\theta = 0$, $r_0 = 3$, $p = 0.5$, $p_{\rm L} = 0.2$, $\alpha_{\rm L} = 2.5$, $\alpha_{\rm N} = 4$, $R = 200$, and $N = 4$ are default values.

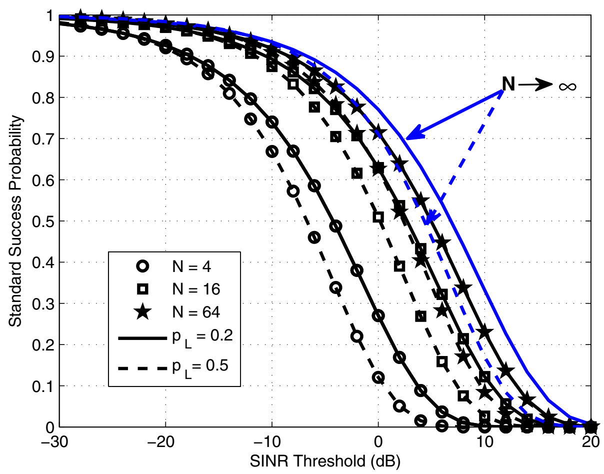

Fig. 1. The standard success probability M_1.

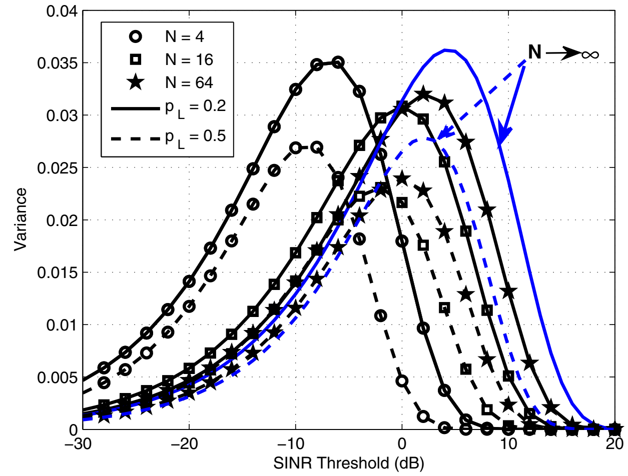

Fig. 2. The variance $M_2 - M_1^2$ of the conditional success probability.

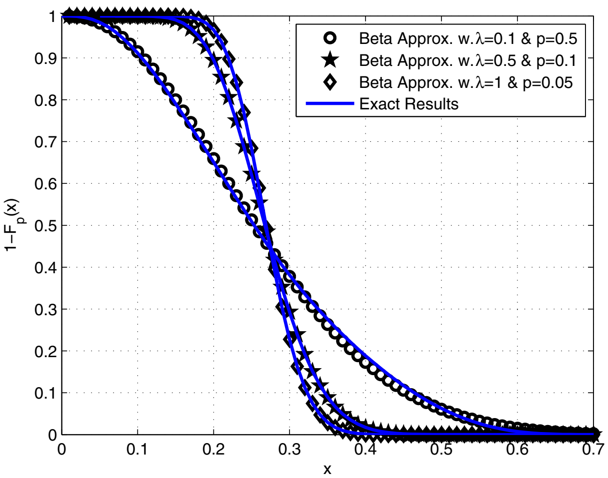

Fig. 3. Meta distribution for different λ and p with $N = 4$, $p_{\mathrm{L}} = 0.2$.

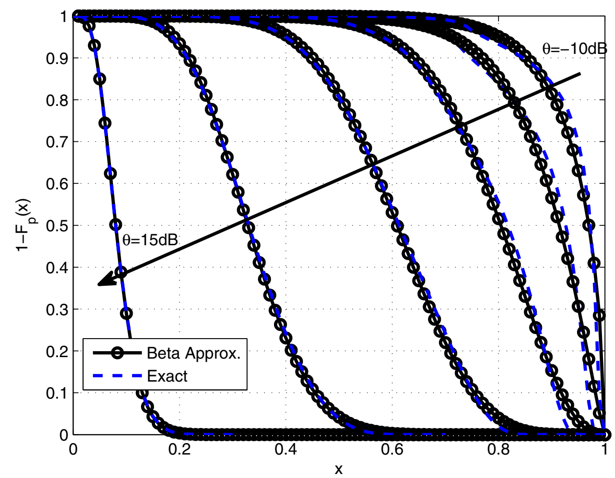

Fig. 4. Meta distribution for $\theta = -10, -5, 0, 5, 10, 15$ dB.

A. The Impact of mm-Wave Features

Directional transmission and sensitivity to blockage are two key features of mm-wave communication. In this subsection, we focus on the impacts of the LOS probability p_{L} and antenna array size N on the mean and variance of the conditional success probability $P_{\mathrm{s}}(\theta)$.

As shown in Fig. 1, increasing the number of antennas improves the standard success probability due to the fact that a larger antenna array can form a narrower beam, thus causing less interference. As the number of antennas tends to infinity, the standard success probability converges to an upper limit (see Cor. 2), since the side-lobe leakage restricts the performance improvement. However, for the LOS probability, a higher p_{L} means the number of LOS links is larger, leading to more severe interference. Thus, the standard success probability deteriorates with the increase of p_{L}.

Fig. 2 presents the variance of the conditional success probability as a function of θ for different p_{L} and N. Since the variance necessarily tends to zero for both $\theta \to 0$ and $\theta \to \infty$, it assumes a maximum at some finite value of θ. It can be seen that given a p_{L}, the θ with the maximum variance increases with N and converges to an upper limit, similar to M_1. Moreover, a larger p_{L} leads to smaller variance, because in LOS environment with smaller path loss exponent, the randomness of the spatial relative locations between the interferers and the receiver has a smaller effect on the variance of the interference than in the NLOS case.

B. Meta Distribution in mm-Wave Band

Fig. 3 compares the exact results and beta approximations for $\lambda p = 0.05$ with different λ and p, respectively. As seen from the plots, for the given system parameters, the approximations match the exact results extremely well. Moreover, the three curves have the same value of λp and hence the same standard success probability, but the corresponding meta distributions are rather different. This shows that the standard success probability provides only limited information on the network performance.

Fig. 4 shows the meta distributions for different SINR thresholds, which enables a precise statement about what fraction of links achieve an SINR threshold with a target reliability. For example, for $\theta = -5$ dB, about 90% of the links have a success probability of at least 80%; while for $\theta = 5$ dB, less than 10% of the links achieve the same reliability.

V. Conclusion

In this paper, we analyze the meta distribution of the SINR for mm-wave D2D networks and provide an exact expression as well as an approximation with the beta distribution, which turns out to be matching the exact distribution extremely well. Hence the complete distribution of the conditional link success probability in Poisson bipolar networks can be characterized, which provides more fine-grained information about the performance of the individual links than just the mean (i.e., the standard SINR distribution at the typical link). Moreover, the mm-wave D2D networks with directional antenna arrays exhibit the key insight that when extremely massive antenna arrays are used, i.e., $N \to \infty$, the noise is totally suppressed and the interference behavior depends on the density of concurrently active links. Numerical results show the significant impact of the unique mm-wave features on the performance and the advantages of the meta distribution, helping the network operator acquire a comprehensive knowledge to the performance of individual links or users.

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Appendix

A. Proof of Theorem 1

Proof: Letting $\theta' = \theta r_0^{\alpha_{\rm L}}/G_{\rm m}$, we have from (5)

$$\begin{aligned}P_{\rm s}(\theta) &= \mathbb{E}\left(\exp\left(-\theta'(I+1)/\mu\right) \mid \Phi\right)\\ &= e^{-\frac{\theta'}{\mu}}\mathbb{E}\prod_{x\in\Phi}\left(\frac{p}{1+\theta' G(\varphi_x)\ell(x)}+1-p\right)\\ &= e^{-\frac{\theta'}{\mu}}\prod_{i\in\{\rm L,N\}}\prod_{x\in\Phi_i}\left(\frac{p\bar{w}}{1+\theta' G_{\rm m}|x|^{-\alpha_i}}+\frac{p(1-\bar{w})}{1+\theta' G_{\rm s}|x|^{-\alpha_i}}+1-p\right)\end{aligned}$$

Letting $\delta_i = 2/\alpha_i$, $i \in \{\rm L, N\}$, we have

$$\begin{aligned}M_b &= \mathbb{E}\left[P_{\rm s}(\theta)^b\right]\\ &= e^{-\frac{b\theta'}{\mu}}\prod_{i\in\{\rm L,N\}}\mathbb{E}_{\Phi_i}\left[\prod_{x\in\Phi_i}\left(\frac{p\bar{w}}{1+\frac{\theta' G_{\rm m}}{|x|^{\alpha_i}}}+\frac{p(1-\bar{w})}{1+\frac{\theta' G_{\rm s}}{|x|^{\alpha_i}}}+1-p\right)^b\right]\\ &\overset{(a)}{=} e^{-\frac{b\theta'}{\mu}}\prod_{i\in\{\rm L,N\}}\exp\left(-2\pi\lambda p_i\int_0^R\left(1-\left(\frac{p\bar{w}}{1+\frac{\theta' G_{\rm m}}{r^{\alpha_i}}}+\frac{p(1-\bar{w})}{1+\frac{\theta' G_{\rm s}}{r^{\alpha_i}}}+1-p\right)^b\right)r{\rm d}r\right)\\ &\overset{(b)}{=} e^{-\frac{b\theta'}{\mu}}\exp\Bigg(-\sum_{i\in\{\rm L,N\}}\lambda\pi R^2 p_i\delta_i\sum_{n=1}^{\infty}\binom{b}{n}(-1)^{n+1}p^n\\ &\quad\times\sum_{m=0}^{n}\binom{n}{m}\bar{w}^m(1-\bar{w})^{n-m}\int_0^1\frac{y^{\delta_i-1}{\rm d}y}{\left(1+\frac{yR^{\alpha_i}}{\theta r_0^{\alpha_{\rm L}}}\right)^m\left(1+\frac{yG_{\rm m}R^{\alpha_i}}{\theta r_0^{\alpha_{\rm L}}G_{\rm s}}\right)^{n-m}}\Bigg)\\ &\overset{(c)}{=} e^{-\frac{b\theta r_0^{\alpha_{\rm L}}}{G_{\rm m}\mu}}\exp\Bigg(-\sum_{i\in\{\rm L,N\}}\lambda\pi R^2 p_i\sum_{n=1}^{\infty}\binom{b}{n}(-1)^{n+1}p^n\sum_{m=0}^{n}\binom{n}{m}\bar{w}^m\\ &\quad\times(1-\bar{w})^{n-m}\tilde{F}_1\left(\delta_i,m,n-m,\delta_i+1,\frac{-R^{\alpha_i}}{\theta r_0^{\alpha_{\rm L}}},\frac{-G_{\rm m}R^{\alpha_i}}{G_{\rm s}\theta r_0^{\alpha_{\rm L}}}\right)\Bigg)\end{aligned}$$

where step (a) uses the probability generating functional (PGFL) of the PPP [2], step (b) follows from the general binomial theorem and step (c) is obtained with the help of the formula in [12, Eq. 3.211]. ■

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