A Transdimensional Poisson Model for Vehicular Networks

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***Abstract*—Poisson line processes (PLPs) describe a street system as a random collection of lines. Poisson point processes (PPPs) model vehicle locations on each line or street as random points. The characterization of vehicular networks as their combination, *i.e.,* the PLP-PPP model, is relevant and widely used in the literature. However, the analytical expression for even a simple performance metric such as the probability of successful message reception in the PLP-PPP is quite complex and provides little insights into the network behavior. Here, we propose a transdimensional Poisson model—superposition of the 1D and 2D PPPs—as an alternative. It considers the vehicles on the same line as the receiving vehicle as a 1D PPP and the vehicles on the other streets as random points on the 2D plane neglecting their street geometry. We show that the success probability in the proposed transdimensional model is tractable and provides a tight approximation to that of the PLP-PPP. Also, it is asymptotically exact on both the upper and lower tails of the success probability.**

***Index Terms*—Poisson line process, Poisson point process, vehicle-to-vehicle communication.**

I. Introduction

Stochastic geometry [1] provides the mathematical tools to characterize different street patterns and uncertainties in the vehicle locations on the streets. A prominent 2D street model is the Poisson line process (PLP) [2] which represents the street system using a countable collection of lines. PLPs allow modeling streets with different orientations, making them relevant in characterizing regular and irregular street patterns [3]. For example, a part of the Manhattan city can be modeled by setting the orientations to $\{0, \pi/2\}$, *i.e.,* the streets are orthogonal to each other. The vehicle locations on each street are generally modeled using Poisson point processes (PPPs), which are known for their analytical tractability. We refer to the street system formed by the PLP coupled with the PPP for vehicle locations on each street as PLP-PPP.

A. Related Work

In [4] the authors analyze the probability of a vehicle receiving a message successfully from another vehicle at a fixed distance in the PLP-PPP setup. The analytical expression for the success probability involves nested integrals with no closed-form expression and thus is intractable. On the other hand, vehicle locations cannot be simply modeled as random points as in a 2D PPP [5]. The random locations of the points in a 2D PPP defy the certainty of vehicles located on a line. Hence there is a need for an intermediate model that provides high tractability *and* good accuracy.

Another line of work involving the PLP-PPP focuses on a vehicle successfully receiving a message from its nearest neighbor, which can be either another vehicle or an infrastructure node such as a roadside unit or a cellular base station [6]–[10]. The transmitting and receiving vehicles on each street are modeled as independent 1D PPPs. The roadside units are generally placed on the lines following a 1D PPP. The base stations form an independent 2D PPP. Although the PPPs are tractable, coupling them with PLPs leads to unwieldy results.

B. Contribution

In this work, we take the middle route between the PLP-PPP and the PPP and introduce a model that provides a good trade-off between accuracy and tractability. We propose a transdimensional Poisson model which is a superposition of a 1D PPP and a 2D PPP. It treats the vehicles on the same street as the receiving vehicle as a 1D PPP and the remaining vehicles on the plane as a 2D PPP. By superimposing a 1D PPP and a 2D PPP, we account for the line or street passing through the receiver, and, at the same time, we obviate the need to incorporate the geometry of the remaining streets.

We compare the proposed transdimensional Poisson model with the PLP-PPP setup considered in [4], where a vehicle receives a message from another vehicle at a fixed distance. As the intersections are critical in street systems, we additionally focus on the success probability of vehicle at an intersection. We show that the success probability in the transdimensional Poisson model provides a tractable and tight lower bound to that of the PLP-PPP.

II. Vehicular Network Modeling

A. Streets and Vehicle Locations

Throughout the paper, we will use the definition, models, and notations presented in this section unless otherwise stated.

Definition 1. *A street system $\mathcal{S}$ is a random closed subset of $\mathbb{R}^2$ that contains no singletons or isolated points satisfying $|\mathcal{S}|_2 = 0$, and for some $\tau > 0$,*

$$\mathbb{E}|\mathcal{S} \cap \mathcal{B}|_1 = \tau|\mathcal{B}|_2 \quad \text{for Borel sets } \mathcal{B} \subset \mathbb{R}^2, \tag{1}$$

where $|\cdot|_d$ is the Lebesgue measure in d dimensions. τ is the mean total street length per unit area.

$|\mathcal{S}|_2 = 0$ implies that $\mathcal{S}$ is the union of 1D subsets and (1) is the condition for stationarity (translation invariance). Let

$$S(t, \varphi_t) = \{(x, y) \in \mathbb{R}^2 : x\cos\varphi_t + y\sin\varphi_t = t\} \quad (2)$$

denote a line (*i.e.,* an infinitely long street) in $\mathbb{R}^2$. $|t|$ is the distance of the perpendicular from the origin $(0, 0)$ to the line, and $\varphi_t \in [0, \pi]$ is the angle the perpendicular makes with the positive $x-$axis. A line-based street system $\mathcal{S}$ is a countable collection of lines defined by (2).

The locations of the transmitting vehicles on each street independently form a point process of intensity λ. A transmitter is active with probability p. Then the intensity of active transmitters on each street is λp. Each active transmitter transmits to its dedicated receiver on the same street at a distance R, forming a bipolar network [1]. We can also interpret the bipolar communication as *broadcast* mode of communication. If a vehicle can communicate to another vehicle at distance R, other vehicles within distance R are also highly likely to receive the message.

B. Types of Users and Success Probability

Our metric of interest is the success probability, the probability that the transmitter-receiver link can satisfy a certain target data rate. It is evaluated at the typical user. We condition a user (receiver) to be at the origin. On averaging over the point process, the user becomes the typical user. Note that the user can be conditioned to be at any location as the street system $\mathcal{S}$ is stationary. As vehicles are located on the streets, having a user at the origin implies that at least one street passes through the origin.

We consider two kinds of users: 1) The typical general user, where the origin is not an intersection. 2) The typical intersection user, where the origin is an intersection of two lines or streets. The term 'typical user' refers to both the kinds of users unless otherwise specified. We refer to the line(s) passing through the typical user as the *typical street(s)*.

The success probability of the typical user is defined as

$$p_{\mathrm{s}} = \mathbb{P}(\mathsf{SIR} > \theta), \quad (3)$$

where SIR is the signal-to-interference ratio and θ parametrizes the target rate. The SIR for the typical user at the origin with its transmitter at x is given by

$$\mathsf{SIR} = \frac{h_x\|x\|^{-\alpha}}{\sum_{z\in\mathcal{V}} h_z\|z\|^{-\alpha}}, \quad (4)$$

where $\|x\| = R$, $\mathcal{V}$ denotes the point process of active transmitters on $\mathcal{S}$, and $I = \sum_{z\in\mathcal{V}} h_z\|z\|^{-\alpha}$ is the total interference power at the origin. The channel power gain h is exponentially distributed with mean 1 (Rayleigh fading), and α is the path-loss exponent.

C. Preliminaries

Here, we list a few results for the PPP [1], which we will need in the rest of the paper. Let Φ_d denote a homogeneous d-dimensional PPP of intensity λ_d. Φ_d is stationary, and

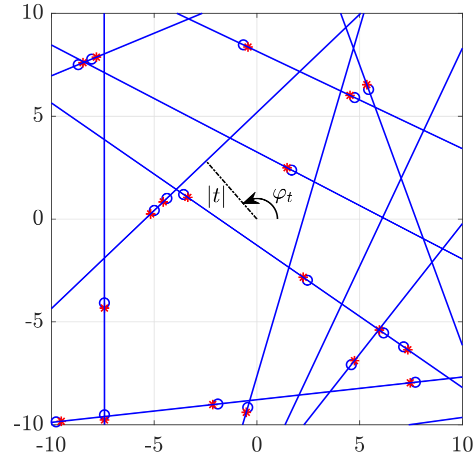

Figure 1: A snapshot of the PLP-PPP bipolar network. $\mu = \lambda = 0.1$, and $R = 0.25$. Lines represent the streets. '$\circ$' and '$*$' represent the transmitters and receivers, respectively. (t, φ_t) are the parameters of a line/street defined by (2).

isotropic (rotation invariant). Let c_d denote the volume of a unit $d-$dimensional ball. $c_1 = 2$, and $c_2 = \pi$.

Lemma 1. *The superposition of* n $d-$*dimensional PPPs with intensities* ν_i, $1 \leq i \leq n$, *is a* $d-$*dimensional PPP with intensity* $\sum_{i=1}^{n} \nu_i$.

Lemma 2. *The pair correlation function* $g^{\Phi_d}(r)$ *is identical to 1.*

The pair correlation function is a second-order statistic quantifying the dependence between the points. For an isotropic process, it can be expressed as a function of a scalar parameter and is related to the probability of finding two points in Φ_d separated by a distance r. $g^{\Phi_d}(r) = 1$ denotes a lack of correlation, which follows from the independence among the points in the PPP.

Lemma 3. *The success probability* p_{s} *of the typical user in* Φ_d *is given by*

$$p_{\mathrm{s}}^{\Phi_d} = \exp(-c_d\lambda_d R^d\theta^{\delta'}\Gamma(1+\delta')\Gamma(1-\delta')), \quad (5)$$

where $\delta' = d/\alpha$.

Lemma 4. *The nearest-neighbor distance distribution in* Φ_d *is given by*

$$F_R^{\Phi_d}(r) = 1 - \exp(-c_d\lambda_d r^d). \quad (6)$$

Next, we analyze the success probability of the typical user in the PLP-based reference model.

III. PLP-BASED VEHICULAR NETWORK

Let $\mathcal{S} = \bigcup_{t\in\mathcal{P}} S(t, \varphi_t)$. When $\mathcal{P}$ is a 1D PPP of intensity μ and φ_t is i.i.d. on $[0, \pi]$, $\mathcal{S}$ forms a Poisson line process. The vehicles on each street independently form a 1D PPP of intensity λ. We refer to this network model as PLP-PPP. Fig. 1 depicts a realization of the PLP-PPP.

A. Properties

By Definition 1 and the homogeneity of the 1D PPP on the streets, the PLP-PPP is stationary. As φ_t is i.i.d. on $[0, \pi]$, the PLP-PPP is also isotropic. Below we list a few preliminaries for the PLP-PPP [2].

Lemma 5. *The mean total street length per unit area is $\tau = \pi\mu$. Then the 2D intensity of vehicles in the PLP-PPP is $\lambda\pi\mu$.*

Lemma 6. *The pair correlation function for the PLP-PPP is*

$$g^{\mathrm{PLP-PPP}}(r) = 1 + \frac{1}{\lambda\mu\pi^2 r}, \quad r > 0. \tag{7}$$

The pair correlation function $g^{\mathrm{PLP-PPP}}(r) \geq 1$ reflects the concentration of points on the lines. We observe from (7) that $g^{\mathrm{PLP-PPP}}(r) \to \infty$ as $r \to 0$. This implies that even for a small disk centered at a point, there exists at least one line—the line on which the point lies—of intensity λ intersecting the disk. At large distances, the correlation between the points vanishes. Hence $g^{\mathrm{PLP-PPP}}(r) \to 1$ as $r \to \infty$.

Lemma 7. *The nearest-neighbor distance distribution in the PLP-PPP is $F_R^{\mathrm{PLP-PPP}}(r) = 1 - \mathbb{P}(R > r)$, where*

$$\mathbb{P}(R > r) = \exp\left(- 2\lambda m r - 2\pi\mu \int_0^r (1 - e^{-2\lambda\sqrt{r^2 - u^2}})\,\mathrm{d}u \right), \tag{8}$$

with $m = 1$ for the typical general user and 2 for the typical intersection user.

Proof: The case of $m = 1$ is derived in [7]. The term $\exp(-2\lambda m r)$ in (8) with $m = 1$ is the probability of not finding a neighbor to the general user within a distance r on the typical street. The remainder in (8) is the contribution from the other streets, which is the same for both the general and intersection users as the distribution of the other streets does not vary with the type of the user. As two streets pass through the intersection and they are independent 1D PPPs, the probability that there is no neighbor on the typical streets within distance r of the typical intersection user is $\exp(-4\lambda r)$, *i.e.*, $m = 2$. ■

B. Success Probability

The success probability of the typical general user is derived in [4]. Here we give a slightly more general result that applies to the typical intersection user as well.

Lemma 8. *The success probability of the typical user in the PLP-PPP is given by*

$$p_{\mathrm{s}}^{\mathrm{PLP-PPP}} = \exp(-2m\lambda p R \theta^{\delta/2}\Gamma(1+\delta/2)\Gamma(1-\delta/2)) \times \exp\left(- 2\pi\mu \int_0^\infty (1 - \mathcal{L}_I(\theta R^\alpha \mid t))\,\mathrm{d}t \right), \tag{9}$$

where

$$\mathcal{L}_I(\theta R^\alpha \mid t) = \exp\left(- \lambda p R \theta^{\delta/2} \int_{v_0}^\infty \frac{1}{(1+v^{1/\delta})\sqrt{v - v_0}}\mathrm{d}v \right),$$

$v_0 = \frac{t^2}{R^2\theta^\delta}$, $\delta = 2/\alpha$, $m = 1$ *for the typical general user and* $m = 2$ *for the typical intersection user.*

Proof: Comparing (9) with (5), we observe that the first term in the product expression (9) with $m = 1$ is the success probability of the typical user in a 1D PPP of intensity λp. The second term in (9) is the contribution from all but the typical streets. It remains the same for the typical intersection user as the distribution of the other streets with respect to both the users is the same. Similar to the proof of Lemma 7, to find the reduction in the success probability of the typical intersection user due to the typical streets, we exploit the independence between the 1D PPPs on them. It follows that the reduction in the success probability due to the two typical streets is just the square of that due to a single street, *i.e.*, $m = 2$ for the typical intersection user as shown in (9). ■

The success probability of the typical user in the PLP-PPP (9) does not have a closed-form expression. To gain insights into the network behavior, we next focus on the asymptotic regimes of θ.

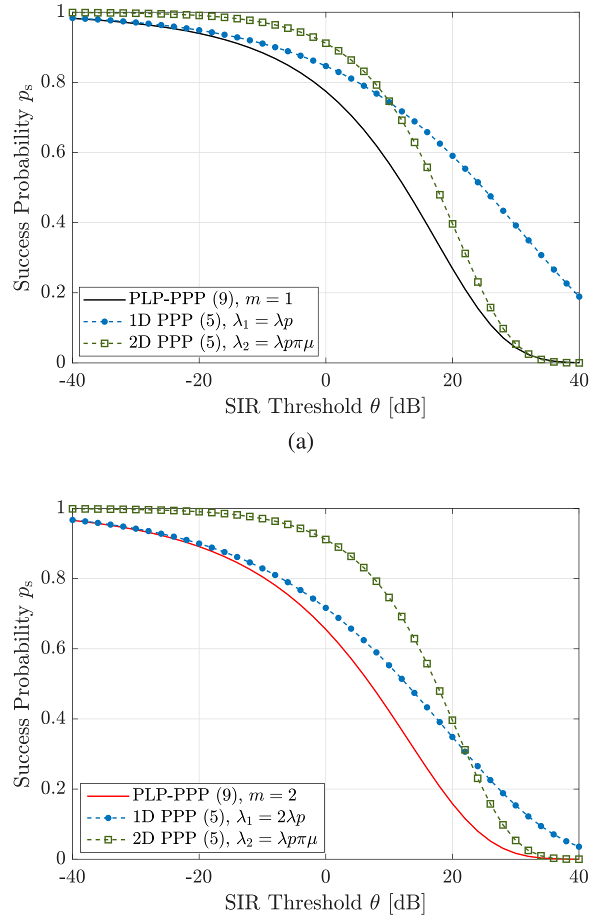

Figure 2: Comparison of success probabilities of the (a) typical general user and (b) typical intersection user in the PLP-PPP to that of the typical user in 1D PPP, and 2D PPP. $\mu = 1/\pi$, $\lambda = 1$, $p = 0.3$, $R = 0.25$, and $\alpha = 4$. The equation numbers are given in the parentheses in the legends.

C. Asymptotic Analysis

Theorem 1. *The PLP-PPP behaves like a 1D PPP as $\theta \to 0$, i.e.,*

$$1 - p_{\mathrm{s}}^{\mathrm{PLP-PPP}}(\theta) \sim 2m\lambda p R \theta^{\delta/2}\Gamma(1+\delta/2)\Gamma(1-\delta/2),$$

where $\delta = 2/\alpha$, $m = 1$ for the typical general user and 2 for the typical intersection user. In general, the success probability of the typical user in the PLP-PPP is upper bounded by that of the 1D PPP.

Proof: See Appendix A. ■

Theorem 2. *The PLP-PPP behaves like a 2D PPP as $\theta \to \infty$, i.e.,*

$$p_{\mathrm{s}}^{\mathrm{PLP-PPP}}(\theta) \sim \exp(-\pi\lambda_2 R^2 \theta^{\delta}\Gamma(1+\delta)\Gamma(1-\delta)), \quad (10)$$

where $\lambda_2 = \lambda p \pi \mu$, and $\delta = 2/\alpha$.

Proof: See Appendix B. ■

Fig. 2 shows the success probabilities of the typical user in the PLP-PPP, 1D PPP, and 2D PPP. The intensity of active transmitters in the 1D PPP is λp and that of the equivalent 2D PPP is $\lambda p \pi \mu$ (see Lemma 5). As two streets pass through the typical intersection user, the probability that the nearest interferer is closer, and the mean number of interferers within a certain distance r are higher than for the typical general user, resulting in lower success probability.

We observe from Fig. 2 that the success probability of the typical user is upper bounded by the minimum of the success probabilities of the typical user in the 1D and 2D PPPs. Furthermore, the success probability of the typical user in the PLP-PPP tends to that of the 1D and 2D PPPs in the asymptotic regimes as stated in Theorems 1 and 2. This raises the question of whether a simpler, purely PPP-based model exists that provides great tractability and good accuracy. The answer is affirmative, as shown in the next section.

IV. A Transdimensional Poisson Model

The first term in the success probability of the typical user in the PLP-PPP (9) denotes the contribution from the typical streets. It is independent of φ_t, *i.e.,* the orientations of the typical streets are irrelevant. Then, by Lemma 1, m streets of intensity λ passing through the typical user is equivalent to a single street of intensity $m\lambda$. We propose a transdimensional Poisson model that considers the vehicles on the street(s) passing through the typical user at the origin as a 1D PPP of intensity $\lambda_1 = m\lambda$, and the remaining vehicles on the plane as a 2D PPP neglecting the geometry of the other streets. The formal definition follows.

Definition 2. *Let $\Phi_1 = \{(t_1, 0), (t_2, 0), \dots\}$ where $\{t_k\}$, $k \in \mathbb{N}$, is a 1D PPP of intensity $m\lambda$ and Φ_2 denote a 2D PPP of intensity λ_2. The transdimensional Poisson point process (TPPP) is the superposition of Φ_1 and Φ_2, i.e., $\mathcal{T} \triangleq \Phi_1 \cup \Phi_2$.*

By Lemma 5, $\lambda_2 = \lambda \pi \mu$. Fig. 3 depicts a realization of the TPPP. The TPPP is non-stationary and thus $p_{\mathrm{s}}^{\mathrm{TPPP}}$ and $F_R^{\mathrm{TPPP}}(r)$ are location dependent.

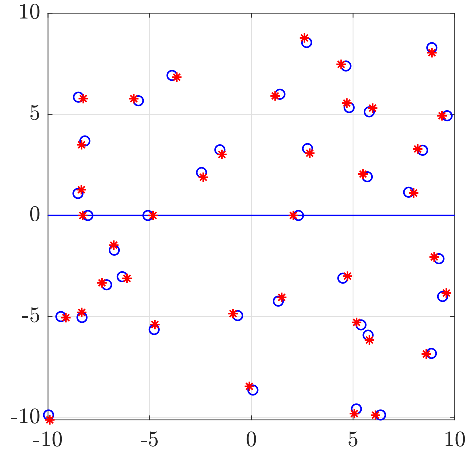

Figure 3: A TPPP-based realization of the vehicular network. $m\lambda = \lambda_2 = 0.1$, and $R = 0.25$. The 1D PPP is highlighted by the line. '$\circ$' and '$*$' represent the transmitters and receivers, respectively.

Lemma 9. *The success probability of the typical user at the origin in the TPPP is*

$$\begin{aligned} p_s^{\mathrm{TPPP}} &= \exp(-2m\lambda p R \theta^{\delta/2}\Gamma(1+\delta/2)\Gamma(1-\delta/2)) \\ &\times \exp(-\lambda p \mu \pi^2 R^2 \theta^{\delta}\Gamma(1+\delta)\Gamma(1-\delta)), \quad (11) \end{aligned}$$

where $\delta = 2/\alpha$.

Proof: As Φ_1 and Φ_2 are independent, the success probability of the typical user at the origin in the TPPP is the product of the success probabilities of the typical user in Φ_1 and Φ_2 given by (5). ■

Theorem 3. *The nearest-neighbor distance in the PLP-PPP is stochastically dominated by that of the typical user at the origin in the TPPP.*

Proof: Using $e^{-x} \geq 1 - x$, we can upper bound the nearest-neighbor distance distribution given in (8) as

$$\begin{aligned} F_R^{\mathrm{PLP-PPP}}(r) &\leq 1 - \exp\left(-2\lambda m r - 4\pi\mu\lambda \int_0^r \sqrt{r^2 - u^2}\,\mathrm{d}u\right) \\ &= 1 - \exp(-2\lambda m r - \lambda\mu\pi^2 r^2), \\ &\overset{(a)}{=} F_R^{\Phi_1 \cup \Phi_2}(r) \overset{(b)}{\equiv} F_R^{\mathrm{TPPP}}(r), \quad (12) \end{aligned}$$

where (a) follows from the nearest-neighbor distance distributions of the 1D and 2D PPPs given in (6) with $\lambda_1 = m\lambda$ and $\lambda_2 = \lambda\pi\mu$, and (b) follows from Definition 2. ■

Conjecture 1. *The distance to the nth nearest-neighbor in the PLP-PPP is stochastically dominated by that of for the typical user at the origin in the TPPP for all $n \in \mathbb{N}$.*

An important consequence of Conjecture 1 is that the success probability of the typical user at the origin in the PLP-PPP is lower bounded by that of the TPPP. Here, we give a heuristic argument for Conjecture 1. The 2D density of the vehicles in the TPPP is the same as that of the PLP-PPP. Then the comparison of the distances r_n can be based only on the vehicle placement with respect to the typical user.

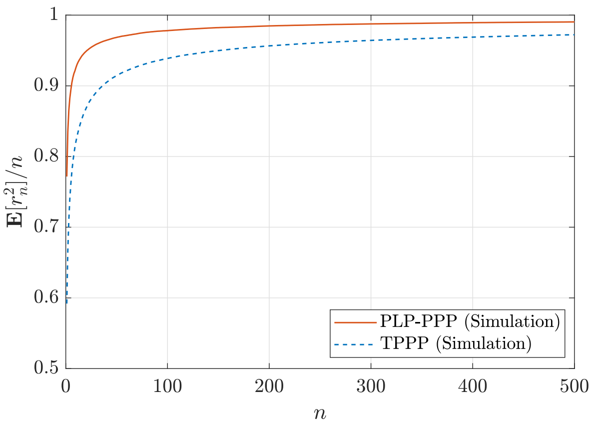

Figure 4: Comparison of normalized mean squared distances to the nth nearest neighbor from the typical general user at the origin in the PLP-PPP and the TPPP. $\lambda = \mu = 1/\pi$, and $p = 1$.

Fig. 4 compares the simulated values of $\mathbb{E}[r_n^2]/n$ in the PLP-PPP and the TPPP. We observe that the mean squared distance from the typical general user to the nth nearest neighbor is higher for the PLP-PPP than the TPPP. The case of $n = 1$ follows from Theorem 3. We presume that this observation can be extended to higher values of n. Since the TPPP includes points at random independent locations compared to the PLP-PPP with points only concentrated on the lines, the probability that the nth nearest neighbor is at a distance r_n is higher for the TPPP. It follows that the average distance to the nth nearest interferer from the origin is higher for the PLP-PPP, which in turn, leads to a higher success probability than for the TPPP.

Fig. 5 compares the success probabilities of the typical general user in the PLP-PPP and the TPPP for different network parameters. We observe that the outage probability of the typical user in the TPPP is a tight approximation of that of the PLP-PPP. The TPPP better approximates the PLP-PPP for small θ than just a 1D PPP. As $\theta \to 0$, for SIR $> \theta$, there should not be any interferers in a small disk $b(o, r)$ of some radius r centered at the origin. From Lemma 6, we learn that the pair correlation function for the PLP-PPP diverges as $r \to 0$, which indicates there definitely exists at least one line with vehicles of intensity λ intersecting $b(o, r)$. For infinitesimally small θ, only the typical street intersects $b(o, r)$. However, for non-vanishing values of θ, there may be more than one line intersecting $b(o, r)$, and the 1D PPP is not sufficient to capture the effect of the streets other than the typical street intersecting $b(o, r)$. Given that the PLP-PPP behaves like a 2D PPP as $\theta \to \infty$ with the street geometry becoming irrelevant, it immediately follows that the TPPP also behaves like a 2D PPP as $\theta \to \infty$.

Fig. 6 plots the difference in the success probabilities in the PLP-PPP and the TPPP for different combinations of λp and $R^2\theta^\delta$. The parameters are combined in the manner they appear in the results (9) and (11). If $R = 0.25$, and $\delta = 0.5$ or $\alpha = 4$, then $\theta \in (-\infty, 50]$ dB. The maximum difference between the success probabilities of the PLP-PPP and the TPPP is in the order of -14 dB. Therefore, the success probability of the

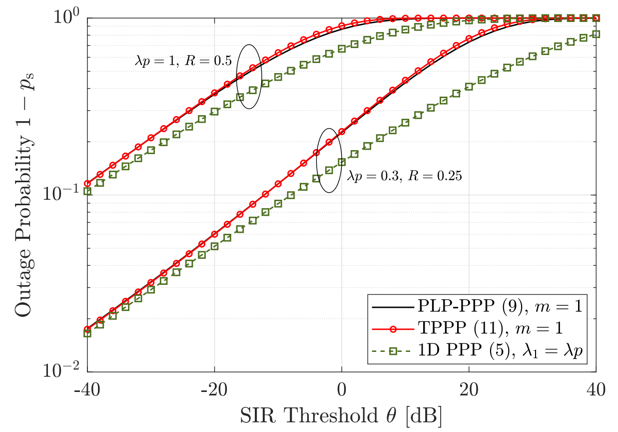

Figure 5: Comparison of outage probabilities of the typical general user in the PLP-PPP and the TPPP. $\mu = 1/\pi$, $\lambda = 1$, $p = 0.3$, $R = 0.25$, and $\alpha = 4$. The equation numbers corresponding to the success probability p_s are given in the parentheses in the legend.

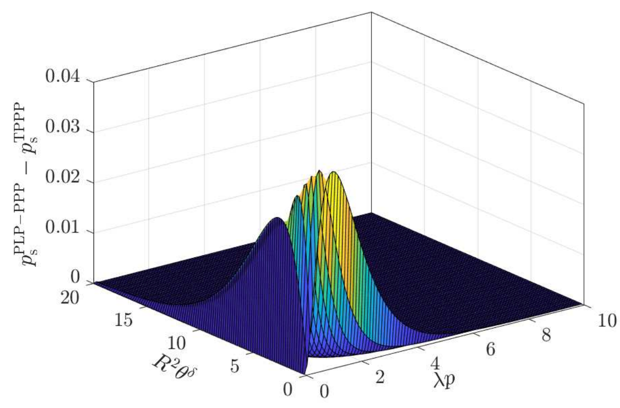

Figure 6: Difference between the success probabilities of the typical general user in the PLP-PPP and the TPPP. $\mu = 1/\pi$, and $m = 1$.

TPPP is a tight lower bound to that of the PLP-PPP. Note that the inferences obtained from the Figs. 4-6 also apply to the typical intersection user.

V. Conclusions

We introduced a simple and tractable approach to analyze the probability of successful message reception in a vehicular network whose streets follow a PLP, and vehicles on each street form a 1D PPP. We showed that the success probability in the proposed transdimensional Poisson model tightly approximates that of the PLP-PPP-based vehicular network. The key takeaway is that it is not essential to account for the geometry of every single street as in the PLP, *i.e.,* it suffices to consider the vehicles on the same street as the receiver as a 1D PPP and those on the different streets as a 2D PPP.

Appendix

A. Proof of Theorem 1

We note that the first term in the product (9) with $m = 1$ is the success probability of the typical user in the 1D PPP given by (5). For the typical intersection user, $m = 2$. Consequently, as $0 \le p_\mathrm{s} \le 1$, the success probability of the typical user in the PLP-PPP is upper bounded by that of the 1D PPP.

As $\theta \to 0$, the first term in (9) can be approximated using Taylor's series as $1 - 2m\lambda p R\theta^{\delta/2}\Gamma(1+\delta/2)\Gamma(1-\delta/2)$. Applying Taylor's series to the second term in (9), we get

$$\exp\bigg(-2\pi\mu\int_0^\infty (1-\mathcal{L}_I(\theta R^\alpha \mid t))\mathrm{d}t\bigg)$$
$$\approx 1 - 2\pi\mu\lambda p R\theta^{\delta/2}\int_0^\infty\int_{\frac{t^2}{R^2\theta^\delta}}^\infty \frac{1}{\left(1+v^{1/\delta}\right)\sqrt{v-\frac{t^2}{b^2\theta^\delta}}}\mathrm{d}v\,\mathrm{d}t$$
$$= 1 - o(\theta^{\delta/2}),$$

since $\frac{t^2}{b^2\theta^\delta} \to \infty$ as $\theta \to 0$. Then we can express p_s as

$$p_\mathrm{s} \sim 1 - 2m\lambda p R\theta^{\delta/2}\Gamma(1+\delta/2)\Gamma(1-\delta/2), \quad \theta \to 0. \quad (13)$$

Similarly, we can approximate (5) as

$$1 - p_\mathrm{s} \sim c_d\lambda_d R^d\theta^{\delta'}\Gamma(1+\delta')\Gamma(1-\delta'), \quad \theta \to 0. \quad (14)$$

Setting $\delta/2 = \delta' = 1/\alpha$ *i.e.,* $d = 1$ ($c_1 = 2$), $\lambda p = \lambda_1$, and $m = 1$ in (13), we obtain (14), which refers to a 1D PPP or a single street. For the typical intersection user, $m = 2$ as two streets pass through the origin.

B. Proof of Theorem 2

Let G_u denote an orthogonal grid with the locations of the lines $u \in \mathbb{Z}$, *i.e.,* u is the x-intercept or y-intercept of the line. The vehicles on each line form a 1D PPP. One way of obtaining such a model is to start with a 2D PPP Φ_2 and quantize one of the coordinates of the points with equal probability [11, Def. 2]. Similarly, we can generate the PLP-PPP from a 2D PPP.

In the PLP-PPP, the locations of the streets are characterized by t, which follows a 1D PPP $\mathcal{P}$ of intensity μ. This implies that the distances between the points in $\mathcal{P}$ are exponentially distributed. With each t, there is an associated orientation φ_t i.i.d. on $[0,\pi]$. Then, by rotating the lines in the orthogonal grid with exponential spacing G_e, we can obtain the PLP-PPP. However, direct mapping from Φ_2 to G_e would result in an inhomogeneous distribution of points on the streets.

Let $\mathcal{M} : \Phi_2 \to \mathcal{V}_u \to \mathcal{V}$ define the mapping from a 2D PPP to the PLP-PPP, where $\mathcal{V}_u$ and $\mathcal{V}$ denote the locations of the transmitters in G_u and the PLP-PPP, respectively. We order the streets based on their perpendicular distances to the origin. By $\Phi_2 \to \mathcal{V}_u$, the points on Φ_2 are translated to G_u through the quantization process described above. The mapping $\mathcal{V}_u \to \mathcal{V}$ involves two steps: 1) The points on the ith street on the grid G_u are translated to the ith street on the grid G_e. 2) Each line is rotated independently such that φ_t is i.i.d. on $[0,\pi]$.

From (3) and (4), the success probability is given by

$$p_\mathrm{s}^{\mathrm{PLP-PPP}} = \mathbb{P}\bigg(h_x > R^\alpha\sum_{z\in\mathcal{V}} h_z\|\theta^{-1/\alpha}z\|^{-\alpha}\bigg). \quad (15)$$

Using the mapping function $\mathcal{M}$, we can express (15) as

$$p_\mathrm{s}^{\mathrm{PLP-PPP}} = \mathbb{P}\bigg(h_x > R^\alpha\sum_{z\in\Phi_2} h_z\|\theta^{-1/\alpha}\mathcal{M}(z)\|^{-\alpha}\bigg). \quad (16)$$

The success probability of the typical user in Φ_2 is

$$p_\mathrm{s}{}^{\Phi_2} = \mathbb{P}\bigg(h_x > b^\alpha\sum_{z\in\Phi_2} h_z\|\theta^{-1/\alpha}z\|^{-\alpha}\bigg). \quad (17)$$

The mapping $\Phi_2 \to \mathcal{V}_u$ displaces each point by at most $1/2$ as $u \in \mathbb{Z}$. Since the exponential distribution has a finite variance and the PLP is isotropic, the mapping $\mathcal{V}_u \to \mathcal{V}$ translates the points in $\mathcal{V}_u$ only by a finite distance. Then, by the Cauchy-Schwarz inequality, $|\|z\| - \|\mathcal{M}(z)\|| \le \|z - \mathcal{M}(z)\| < \infty$. Multiplying by $\theta^{-1/\alpha}$, we obtain

$$|\|\theta^{-1/\alpha}z\| - \|\theta^{-1/\alpha}\mathcal{M}(z)\|| \to 0 \quad \text{as} \quad \theta \to \infty. \quad (18)$$

Applying (18) to (16) and (17), we infer that the interference experienced by the typical general/intersection user in the PLP-PPP tends to that of in a 2D PPP as $\theta \to \infty$. Note that the type of user does not matter since the mapping does not affect the points of Φ_2 as $\theta \to \infty$, *i.e.*, there is no difference between a general user and an intersection user. Hence the success probability in the PLP-PPP tends to that in Φ_2.

Setting $d = 2$ ($c_2 = \pi$), $\delta' = \delta = 2/\alpha$, and $\lambda_2 = \lambda p\pi\mu$ (see Lemma 5) in (5), we obtain (10).

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